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COPS AND ROBBERS ON HYPERGRAPHS

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ปิ่นแก้ว ศิริวงศ์ : ตำรวจและโจรบนไฮเพอร์กราฟ, (COPS AND ROBBERS ON HYPERGRAPHS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก:รศ.ดร. รตินันท์ บุญเคลือบ, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม:ผศ.ดร. ศิริรัตน์ สิงหันต์, 64 หน้า.

ตำรวจและโจรเป็นเกมที่นิยมเล่นบนกราฟเชื่อมโยงจำกัดประกอบด้วยผู้เล่นสองคนได้แก่ ตำรวจ และ โจร ปัจจุบันนี้ มีการแนะนำตำรวจและโจรบนไฮเพอร์กราฟ การให้โอกาสที่ดีกว่าแก่ตำรวจโดยการอนุญาตให้มีตำรวจมากกว่าหนึ่งคน และตำรวจอย่างน้อยหนึ่งคนต้องเดิน ส่งผลให้เกิดการศึกษาจำนวนตำรวจซึ่งคือจำนวนตำรวจที่น้อยที่สุดที่รับประกันว่าพวกเขาจะชนะเกมนี้บนกราฟและไฮเพอร์กราฟ วิทยานิพนธ์นี้ให้ (i) ลักษณะเฉพาะของไฮเพอร์กราฟที่ตำรวจชนะ (ii) ผลของผลคูณของไฮเพอร์กราฟ และ (iii) จำนวนตำรวจของไฮเพอร์กราฟ k ส่วนบริบูรณ์และ n-ปริซึมของไฮเพอร์กราฟ ยิ่งไปกว่านั้นยังได้กำหนดจำนวนตำรวจของกราฟประเภทพิเศษ

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Cops and robbers game is a game usually played on a finite connected graph with two players, cop and robber. Recently, cops and robbers game played on hypergraphs was introduced. To give a better chance to a cop by allowing more than one cop and at least one cop has to move, the cop-number, the least number of cops to guarantee that they win the game, on graphs and hypergraphs is studied. This thesis provides (i) a characterization of a cop-win hypergraph (ii) some results on the products of hypergraphs and (iii) the cop-number of complete k-partite hypergraphs and n-prisms over a hypergraph. Moreover, the cop-number of a special class of graphs is determined.

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CHAPTER I
INTRODUCTION

Game structures and graph structures are sometimes associated. Known games and puzzles are simplified into graph theory problems; for examples, tic-tac-toe, crossword and sudoku, which are paper-and-pencil games. Besides paper-and-pencil games, several researchers are interested in pursuit-and-evasion games. They are combinatorial games with two players which are one pursuer and one evader. The game of cops and robbers on graphs [11, 13] is one of interesting pursuit-and-evasion games by following two simple rules which are (i) two players (a cop and a robber) choose their beginning vertices to stay, and (ii) they alternatively take moves from the present vertices to its neighbor or choose not to move. Later, researchers generalized the idea of the game on graphs into hypergraphs. Baird [2] was the first one who introduced the cops and robbers game on hypergraphs with slightly different rules.

In this thesis, we investigate the results on the cops and robbers game on hypergraphs and certain special class of graphs.

We collect the basic concepts in graph and hypergraph theory and the game of cops and robbers in Chapter II. First of all, we give some definitions and interesting hypergraph structures. In the second part of this chapter, we introduce useful definitions in graph theory and the structure of a special class of graphs. Finally, we provide the rule of a considered cops and robbers game on both graphs and hypergraphs and some studies on this game in the third part of this chapter.

In Chapter III, we characterize the cop-win hypergraphs and investigate their products.

In the case of robber-win hypergraphs, we give a better chance to a cop and allow more than one cop to play this game. The cop-number of certain hypergraphs can be seen in Chapter IV.

After that, in Chapter V, we consider the game on a special class of graphs which is a Kneser graph. Then, we study the cop-number of Kneser graphs and give an algorithm of choosing vertices to prove the desired results.

Finally, we give conclusions and discussions on the future researches in Chapter VI.

CHAPTER II

PRELIMINARIES

We separate this chapter into three sections. In the first section, we give some definitions and some properties of the related hypergraph structures. Then, we give the definition of a special class of graphs and some useful graph-theoretic properties in the second section. Next, we introduce the rule of cops and robbers game on graphs and hypergraphs with some relevant results in the last section.

2.1 Hypergraph Structures

Definition 2.1. [16] The pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is called a *hypergraph* including the *vertex set* $\mathcal{V}$ or $\mathcal{V}(\mathcal{H})$ which is a finite non-empty set and the (*hyper*)*edge set* $\mathcal{E}$ or $\mathcal{E}(\mathcal{H})$ which is a family of non-empty subsets of $\mathcal{V}$. A hypergraph in which all edges have the same size $r \geq 0$ is called r*-uniform*.

Definition 2.2. [2] A vertex which is contained in only one hyperedge is called an *internal vertex* and a vertex which is contained in two or more hyperedges is called an *external vertex*.

Definition 2.3. [16] In a hypergraph $\mathcal{H}$, two vertices are said to be *adjacent* if there is a (hyper)edge $E \in \mathcal{E}(\mathcal{H})$ that contains both vertices.

Definition 2.4. (i) [16] For a hypergraph $\mathcal{H}$, the adjacent vertices are sometimes called *neighbors* to each other, and the *neighborhood* of a vertex x of $\mathcal{H}$, denoted by $N_{\mathcal{H}}(x)$, is the set of all neighbors of x and the set $N_{\mathcal{H}}(x) \cup \{x\}$ is denoted by $N_{\mathcal{H}}[x]$.

(ii) [2] A vertex x of a hypergraph $\mathcal{H}$ is called a *corner* or a *pitfall* if x and all vertices connected to x are also adjacent to some other vertices; that is, there exists a vertex $y \neq x$ of $\mathcal{H}$ such that $N_{\mathcal{H}}[x] \subseteq N_{\mathcal{H}}[y]$.

(iii) [16] A *weak deletion* of a vertex x in a hypergraph $\mathcal{H}$ is a removal of x from $\mathcal{V}(\mathcal{H})$ and from each hyperedge containing x. We write this operation as $\mathcal{H} - x$ and also use $\mathcal{H} - x$ to represent the resulting hypergraph after a weak deletion.

Example 2.5. Consider a hypergraph $\mathcal{H}$ given in Figure 2.1. The vertex set $\mathcal{V}(\mathcal{H}) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the edge set $\mathcal{E}(\mathcal{H})$ consists of $\{x_1, x_2, x_3\}$, $\{x_4, x_5, x_6\}$, $\{x_2, x_5\}$, and $\{x_3, x_6\}$.

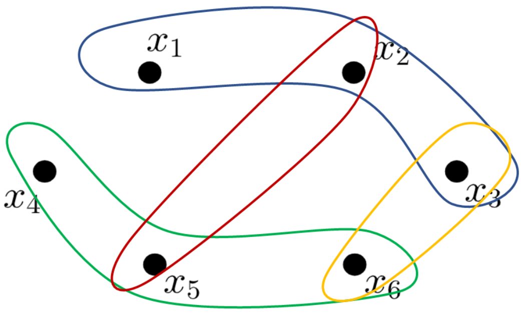

Figure 2.1: A hypergraph $\mathcal{H}$

From $\mathcal{H}$ shown in Figure 2.1, we have $N_{\mathcal{H}}[x_1] = \{x_1, x_2, x_3\}$, $N_{\mathcal{H}}[x_2] = \{x_1, x_2, x_3, x_5\}$, $N_{\mathcal{H}}[x_3] = \{x_1, x_2, x_3, x_6\}$, $N_{\mathcal{H}}[x_4] = \{x_4, x_5, x_6\}$, $N_{\mathcal{H}}[x_5] = \{x_2, x_4, x_5, x_6\}$ and $N_{\mathcal{H}}[x_6] = \{x_3, x_4, x_5, x_6\}$. Since $N_{\mathcal{H}}[x_1] = \{x_1, x_2, x_3\} \subseteq \{x_1, x_2, x_3, x_5\} = N_{\mathcal{H}}[x_2]$ and $N_{\mathcal{H}}[x_4] = \{x_4, x_5, x_6\} \subseteq \{x_2, x_4, x_5, x_6\} = N_{\mathcal{H}}[x_5]$. By Definition 2.4 (ii), vertices x_1 and x_4 are corners of $\mathcal{H}$. Furthermore, if we delete a vertex x_3 weakly from $\mathcal{H}$, we obtain the hypergraph shown in Figure 2.2.

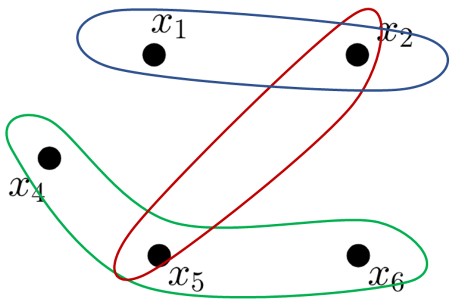

Figure 2.2: A hypergraph $\mathcal{H} - x_3$

After weak deleting vertex x_3, we see that the hyperedge $\{x_1, x_2, x_3\} \in \mathcal{E}(\mathcal{H})$ of size 3 becomes a hyperedge $\{x_1, x_2\} \in \mathcal{E}(\mathcal{H} - x_3)$ of size 2 connecting the remaining two vertices.

We would like to give the definitions of interesting hypergraph structures. First of all, we introduce two types of k-partite hypergraphs.

Definition 2.6. [10] Let $k \geq 2$ be a positive integer. A *k-uniform k-partite hypergraph* has a vertex set $\mathcal{V}$ partitioned into k subsets $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots, \mathcal{V}_k$, and E is a hyperedge if $E = \{v_1, v_2, v_3, \ldots, v_k\}$ where $v_j \in \mathcal{V}_j$ for all $1 \leq j \leq k$. A k-uniform k-partite hypergraph with partite sets $\mathcal{V}_j = \{v_j^1, v_j^2, v_j^3, \ldots, v_j^{|\mathcal{V}_j|}\}$ for all $1 \leq j \leq k$ is said to be *complete* if $\mathcal{E} = \{\{v_1^{i_1}, v_2^{i_2}, v_3^{i_3}, \ldots, v_k^{i_k}\} \mid v_j^{i_j} \in \mathcal{V}_j$ for all $1 \leq j \leq k$ and $1 \leq i_j \leq |\mathcal{V}_j|\}$.

Example 2.7. Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ where $\mathcal{V}_1 = \{a, b, c\}$, $\mathcal{V}_2 = \{a', b', c'\}$ and $\mathcal{V}_3 = \{\overline{a}, \overline{b}, \overline{c}\}$ are mutually disjoint. The complete 3-uniform 3-partite hypergraph from Definition 2.6 constructed from $\mathcal{V}$ has edges as follows.

$$\{a,a',\overline{a}\}, \{a,a',\overline{b}\}, \{a,a',\overline{c}\}, \{a,b',\overline{a}\}, \{a,b',\overline{b}\}, \{a,b',\overline{c}\},$$
$$\{a,c',\overline{a}\}, \{a,c',\overline{b}\}, \{a,c',\overline{c}\}, \{b,a',\overline{a}\}, \{b,a',\overline{b}\}, \{b,a',\overline{c}\},$$
$$\{b,b',\overline{a}\}, \{b,b',\overline{b}\}, \{b,b',\overline{c}\}, \{b,c',\overline{a}\}, \{b,c',\overline{b}\}, \{b,c',\overline{c}\},$$
$$\{c,a',\overline{a}\}, \{c,a',\overline{b}\}, \{c,a',\overline{c}\}, \{c,b',\overline{a}\}, \{c,b',\overline{b}\}, \{c,b',\overline{c}\},$$
$$\{c,c',\overline{a}\}, \{c,c',\overline{b}\}, \{c,c',\overline{c}\}.$$

The second definition is modified from the definition given by Jirimutu and Wang [9]

Definition 2.8. [9] Let $k \geq 2$, $\mathcal{V}$ be partitioned into k subsets $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots, \mathcal{V}_k$ and σ be an integer with $|\mathcal{V}| \geq \sigma$. A *$\sigma$-uniform k-partite hypergraph* has the vertex set $\mathcal{V}$ and $E \subseteq \mathcal{V}$ is an edge if $|E| = \sigma$ and $E \nsubseteq \mathcal{V}_i$ for all $1 \leq i \leq k$. A σ-uniform k-partite hypergraph is said to be *complete* if the edge set $\mathcal{E}$ contains all edges satisfying the above property.

Example 2.9. Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_1 = \{a, b, c\}$ and $\mathcal{V}_2 = \{a', b', c'\}$ are mutually disjoint. The 3-uniform 2-partite hypergraph from Definition 2.8 has edges as follows.

$$\{a,b,a'\},\{a,c,a'\},\{b,c,a'\},\{a,b,b'\},$$
$$\{a,c,b'\},\{b,c,b'\},\{a,b,c'\},\{a,c,c'\},\{b,c,c'\}.$$

After that, we consider the operations between two hypergraphs which leads us to study the their products, namely, (i) the Cartesian product (ii) the minimal and maximal rank preserving direct products and (iii) the normal and standard strong products.

Definition 2.10. [8] Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_2)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be hypergraphs. The *Cartesian product* $\mathcal{H} = \mathcal{H}_1 \,\square\, \mathcal{H}_2$ of two hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ has the vertex set $\mathcal{V}(\mathcal{H}) = \mathcal{V}_1 \times \mathcal{V}_2$ and the edge set $\mathcal{E}(\mathcal{H}) = \{\{x_1\} \times e_2 \mid x_1 \in \mathcal{V}_1, e_2 \in \mathcal{E}_2\} \cup \{e_1 \times \{x_2\} \mid e_1 \in \mathcal{E}_1, x_2 \in \mathcal{V}_2\}$.

Example 2.11. Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ where $\mathcal{V}_1 = \{x_1, x_2, x_3\}$ and $\mathcal{E}_1 = \{\{x_1, x_2, x_3\}\}$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ where $\mathcal{V}_2 = \{y_1, y_2\}$ and $\mathcal{E}_1 = \{\{y_1, y_2\}\}$ shown in Figure 2.3 and 2.4, respectively.

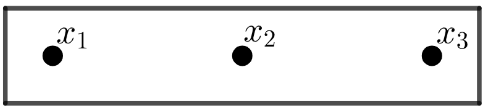

Figure 2.3: A hypergraph $\mathcal{H}_1$

Figure 2.4: A hypergraph $\mathcal{H}_2$

We use hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ to construct the Cartesian product $\mathcal{H}_1 \,\square\, \mathcal{H}_2$. The vertex set $\mathcal{V}_1 \times \mathcal{V}_2 = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\}$ where the edge set $\mathcal{E}(\mathcal{H}_1 \,\square\, \mathcal{H}_2) = \{\{(x_1, y_1), (x_2, y_1), (x_3, y_1)\}, \{(x_1, y_2), (x_2, y_2), (x_3, y_2)\}, \{(x_1, y_1), (x_1, y_2)\}, \{(x_2, y_1), (x_2, y_2)\}, \{(x_3, y_1), (x_3, y_2)\}\}$. We use ij instead of (i, j) in $\mathcal{H}_1 \,\square\, \mathcal{H}_2$ shown in Figure 2.5.

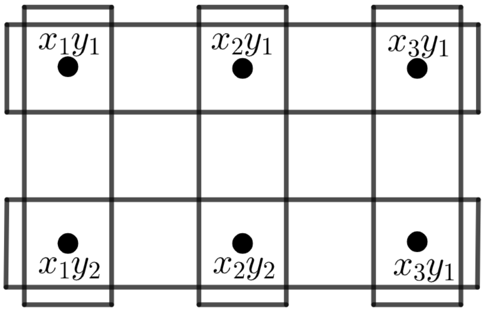

Figure 2.5: The Cartesian product $\mathcal{H}_1 \square \mathcal{H}_2$

Definition 2.12. [8] For two hypergraphs $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, their *minimal rank preserving direct product* $\mathcal{H}_1 \times_1 \mathcal{H}_2$ has the vertex set $\mathcal{V}_1 \times \mathcal{V}_2$. A subset of $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_r, y_r)\}$ of $\mathcal{V}_1 \times \mathcal{V}_2$ is an edge in $\mathcal{H}_1 \times_1 \mathcal{H}_2$ if and only if

(i) $\{x_1, x_2, x_3, \ldots, x_r\}$ is an edge in $\mathcal{H}_1$ and $\{y_1, y_2, y_3, \ldots, y_r\}$ is a subset of an edge in $\mathcal{H}_2$ where x_i's are distinct vertices in $\mathcal{H}_1$ and y_i's are distinct vertices in $\mathcal{H}_2$ for all $1 \leq i \leq r$, or

(ii) $\{x_1, x_2, x_3, \ldots, x_r\}$ is a subset of an edge in $\mathcal{H}_1$ and $\{y_1, y_2, y_3, \ldots, y_r\}$ is an edge in $\mathcal{H}_2$ where x_i's are distinct vertices in $\mathcal{H}_1$ and y_i's are distinct vertices in $\mathcal{H}_2$ for all $1 \leq i \leq r$.

Example 2.13. We use hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ in Example 2.11 to construct the minimal rank preserving direct product $\mathcal{H}_1 \times_1 \mathcal{H}_2$. The vertex set $\mathcal{V}_1 \times \mathcal{V}_2 = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\}$ where the edge set $\mathcal{E}(\mathcal{H}_1 \times_1 \mathcal{H}_2) = \{\{(x_1, y_1), (x_2, y_2)\}, \{(x_1, y_1), (x_3, y_2)\}, \{(x_1, y_2), (x_2, y_1)\}, \{(x_2, y_1), (x_3, y_2)\}, \{(x_1, y_2), (x_3, y_1)\}, \{(x_2, y_2), (x_3, y_1)\}\}$. We use ij instead of (i, j) in $\mathcal{H}_1 \times_1 \mathcal{H}_2$ shown in Figure 2.6.

Definition 2.14. [8] For two hypergraphs $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, their *maximal rank preserving direct product* $\mathcal{H}_1 \times_2 \mathcal{H}_2$ has the vertex set $\mathcal{V}_1 \times \mathcal{V}_2$. A subset of $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_r, y_r)\}$ of $\mathcal{V}_1 \times \mathcal{V}_2$ is an edge in $\mathcal{H}_1 \times_2 \mathcal{H}_2$ if and only if

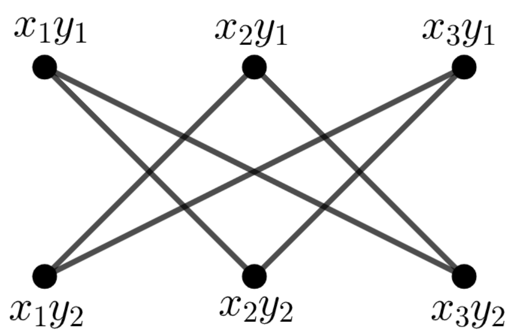

Figure 2.6: The minimal rank preserving direct product $\mathcal{H}_1 \times_1 \mathcal{H}_2$

(i) $\{x_1, x_2, x_3, \ldots, x_r\}$ is an edge of size r in $\mathcal{H}_1$ and there is an edge e_2 in $\mathcal{E}_2$ such that $\{y_1, y_2, y_3, \ldots, y_r\}$ is an r-multiset of elements of e_2 and $e_2 \subseteq \{y_1, y_2, y_3, \ldots, y_r\}$, or

(ii) $\{y_1, y_2, y_3, \ldots, y_r\}$ is an edge of size r in $\mathcal{H}_2$ and there is an edge e_1 in $\mathcal{E}_1$ such that $\{x_1, x_2, x_3, \ldots, x_r\}$ is an r-multiset of elements of e_1 and $e_1 \subseteq \{x_1, x_2, x_3, \ldots, x_r\}$.

Example 2.15. We use hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ in Example 2.11 to construct the maximal rank preserving direct product $\mathcal{H}_1 \times_2 \mathcal{H}_2$. The vertex set $\mathcal{V}_1 \times \mathcal{V}_2 = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\}$ where the edge set $\mathcal{E}(\mathcal{H}_1 \times_2 \mathcal{H}_2) = \{\{(x_1, y_1), (x_2, y_1), (x_3, y_2)\}, \{(x_1, y_1), (x_2, y_2), (x_3, y_1)\}, \{(x_1, y_1), (x_2, y_2), (x_3, y_2)\}, \{(x_1, y_2), (x_2, y_1), (x_3, y_1)\}, \{(x_1, y_2), (x_2, y_1), (x_3, y_2)\}, \{(x_1, y_2), (x_2, y_2), (x_3, y_1)\}\}$. We use ij instead of (i, j) in $\mathcal{H}_1 \times_2 \mathcal{H}_2$ shown on Figure 2.7.

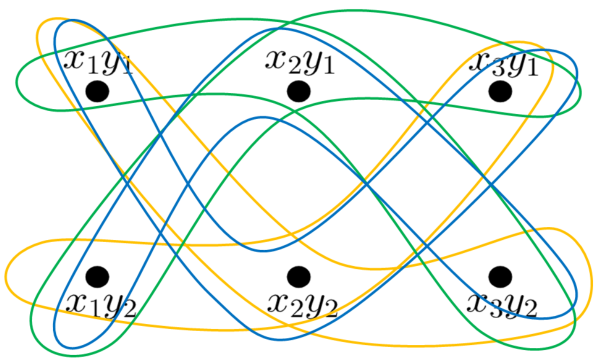

Figure 2.7: The maximal rank preserving direct product $\mathcal{H}_1 \times_2 \mathcal{H}_2$

Notice that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are r-uniform hypergraphs, then $\mathcal{H}_1 \times_1 \mathcal{H}_2 = \mathcal{H}_1 \times_2 \mathcal{H}_2$.

Definition 2.16. [8] For two hypergraphs $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, their *normal strong product* $\mathcal{H}_1 \boxtimes_1 \mathcal{H}_2$ has the vertex set $\mathcal{V}_1 \times \mathcal{V}_2$ and the edge set $\mathcal{E}(\mathcal{H}_1 \boxtimes_1 \mathcal{H}_2) = \mathcal{E}(\mathcal{H}_1 \,\Box\, \mathcal{H}_2) \cup \mathcal{E}(\mathcal{H}_1 \times_1 \mathcal{H}_2)$.

That is, a subset $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_r, y_r)\}$ of $\mathcal{V}_1 \times \mathcal{V}_2$ is an edge in $\mathcal{H}_1 \boxtimes_1 \mathcal{H}_2$ if and only if

(i) $\{x_1, x_2, x_3, \dots, x_r\} \in \mathcal{E}_1$ where x_i's are distinct vertices in $\mathcal{H}_1$ for all $1 \le i \le r$ and $y_1 = y_2 = y_3 = \cdots = y_r \in \mathcal{V}_2$, or

(ii) $\{y_1, y_2, y_3, \dots, y_r\} \in \mathcal{E}_2$ where y_i's are distinct vertices in $\mathcal{H}_2$ for all $1 \le i \le r$ and $x_1 = x_2 = x_3 = \cdots = x_r \in \mathcal{V}_1$, or

(iii) $\{x_1, x_2, x_3, \dots, x_r\} \in \mathcal{E}_1$ and $\{y_1, y_2, y_3, \dots, y_r\}$ is a subset of an edge in $\mathcal{H}_2$ where x_i's are distinct vertices in $\mathcal{H}_1$ and y_i's are distinct vertices in $\mathcal{H}_2$ for all $1 \le i \le r$, or

(iv) $\{y_1, y_2, y_3, \dots, y_r\} \in \mathcal{E}_2$ and $\{x_1, x_2, x_3, \dots, x_r\}$ is a subset of an edge in $\mathcal{H}_1$ where x_i's are distinct vertices in $\mathcal{H}_1$ and y_i's are distinct vertices in $\mathcal{H}_2$ for all $1 \le i \le r$.

Example 2.17. We use hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ in Example 2.11 to construct the normal strong product $\mathcal{H}_1 \boxtimes_1 \mathcal{H}_2$. The vertex set $\mathcal{V}_1 \times \mathcal{V}_2 = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\}$ where the edge set $\mathcal{E}(\mathcal{H}_1 \boxtimes_1 \mathcal{H}_2) = \mathcal{E}(\mathcal{H}_1 \,\Box\, \mathcal{H}_2) \cup \mathcal{E}(\mathcal{H}_1 \times_1 \mathcal{H}_2)$ in Example 2.11 and 2.13. We use ij instead of (i, j) in $\mathcal{H}_1 \boxtimes_1 \mathcal{H}_2$ shown in Figure 2.8.

Definition 2.18. [8] For two hypergraphs $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, their *standard strong product* $\mathcal{H}_1 \boxtimes_2 \mathcal{H}_2$ has the vertex set $\mathcal{V}_1 \times \mathcal{V}_2$ and the edge set $\mathcal{E}(\mathcal{H}_1 \boxtimes_2 \mathcal{H}_2) = \mathcal{E}(\mathcal{H}_1 \,\Box\, \mathcal{H}_2) \cup \mathcal{E}(\mathcal{H}_1 \times_2 \mathcal{H}_2)$.

That is, a subset $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_r, y_r)\}$ of $\mathcal{V}_1 \times \mathcal{V}_2$ is an edge in $\mathcal{H}_1 \boxtimes_2 \mathcal{H}_2$ if and only if

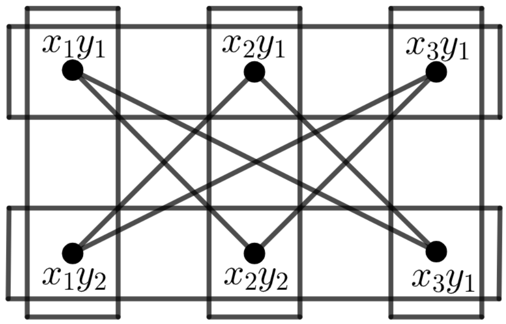

Figure 2.8: The normal strong product $\mathcal{H}_1 \boxtimes_1 \mathcal{H}_2$

(i) $\{x_1, x_2, x_3, \ldots, x_r\} \in \mathcal{E}_1$ where x_i's are distinct vertices in $\mathcal{H}_1$ for all $1 \leq i \leq r$ and $y_1 = y_2 = y_3 = \cdots = y_r \in \mathcal{V}_2$, or

(ii) $\{y_1, y_2, y_3, \ldots, y_r\} \in \mathcal{E}_2$ where y_i's are distinct vertices in $\mathcal{H}_2$ for all $1 \leq i \leq r$ and $x_1 = x_2 = x_3 = \cdots = x_r \in \mathcal{V}_1$, or

(iii) $\{x_1, x_2, x_3, \ldots, x_r\} \in \mathcal{E}_1$ where x_i's are distinct vertices in $\mathcal{H}_1$ for all $1 \leq i \leq r$ and there is an edge e_2 in $\mathcal{E}_2$ such that $\{y_1, y_2, y_3, \ldots, y_r\}$ is an r-multiset of elements of e_2 and $e_2 \subseteq \{y_1, y_2, y_3, \ldots, y_r\}$, or

(iv) $\{y_1, y_2, y_3, \ldots, y_r\} \in \mathcal{E}_2$ where y_i's are distinct vertices in $\mathcal{H}_2$ for all $1 \leq i \leq r$ and there is an edge e_1 in $\mathcal{E}_1$ such that $\{x_1, x_2, x_3, \ldots, x_r\}$ is an r-multiset of elements of e_1 and $e_1 \subseteq \{x_1, x_2, x_3, \ldots, x_r\}$.

Example 2.19. We use hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ in Example 2.11 to construct the standard strong product $\mathcal{H}_1 \boxtimes_2 \mathcal{H}_2$. The vertex set $\mathcal{V}_1 \times \mathcal{V}_2 = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\}$ where the edge set $\mathcal{E}(\mathcal{H}_1 \boxtimes_2 \mathcal{H}_2) = \mathcal{E}(\mathcal{H}_1 \square \mathcal{H}_2) \cup \mathcal{E}(\mathcal{H}_1 \times_2 \mathcal{H}_2)$ in Example 2.11 and 2.15. We use ij instead of (i, j) in $\mathcal{H}_1 \boxtimes_2 \mathcal{H}_2$ shown in Figure 2.9.

Besides the products of two hypergraphs, we are interested in the n-prisms over a hypergraph.

Definition 2.20. Let $n \geq 1$ and $k \geq 2$ be integers. Let $\mathcal{H}_0^{(k)} = (\mathcal{V}_0, \mathcal{E}_0)$ be a k-uniform hypergraph where $|\mathcal{V}_0| \geq k$ and let $\mathcal{H}_i^{(k)} = (\mathcal{V}_i, \mathcal{E}_i)$ be an i^{th} copy of $\mathcal{H}_0^{(k)}$

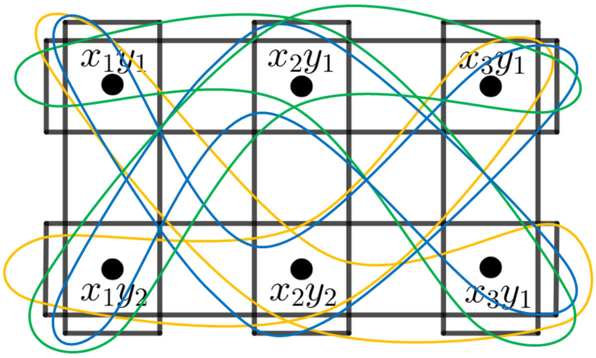

Figure 2.9: The standard strong product $\mathcal{H}_1 \boxtimes_2 \mathcal{H}_2$

for all $1 \leq i \leq n$. Let $v_i^\alpha \in \mathcal{V}_i$ denote an i^{th} clone of $v_0^\alpha \in \mathcal{V}_0$ and $E_i^\beta \in \mathcal{E}_i$ denote an i^{th} clone of $E_0^\beta \in \mathcal{E}_0$.

The n-*Prisms* over $\mathcal{H}_0^{(k)}$, denoted by $\mathrm{Prism}^n(\mathcal{H}_0^{(k)})$, is a hypergraph which consists of all vertices obtained from collection of vertices of $\mathcal{H}_0^{(k)}, \mathcal{H}_1^{(k)}, \mathcal{H}_2^{(k)}, \ldots, \mathcal{H}_n^{(k)}$. The hyperedge set of the n-prisms contains two types where the first type is the collection of all hyperedges obtained from the collection of all hyperedges of $\mathcal{H}_0^{(k)}, \mathcal{H}_1^{(k)}$, $\mathcal{H}_2^{(k)}, \ldots, \mathcal{H}_n^{(k)}$. The second type is a collection $\mathcal{B}$ such that for all $1 \leq j \leq n$, $\{v_{j-1}^\alpha, v_j^\alpha, u_1^\alpha, u_2^\alpha, u_3^\alpha, \ldots, u_{k-2}^\alpha\} \in \mathcal{B}$, where $u_l^\alpha \in (E_{j-1}^\beta \cup E_j^\beta) - \{v_{j-1}^\alpha, v_j^\alpha\}$, $v_{j-1}^\alpha \in V_{j-1}$, $v_j^\alpha \in V_j$ and $E_{j-1}^\beta \in \mathcal{E}_{j-1}$ with $v_{j-1}^\alpha \in E_{j-1}^\beta$.

A vertex v_s^α is called an s^{th} *descendant* of a vertex v_t^α for all $0 \leq t \leq s-1$ and a vertex v_r^α is called an r^{th} *ancestor* of a vertex v_w^α for all $r+1 \leq w \leq n$. A hyperedge E_s^β is called an s^{th} *descendant* of a hyperedge E_t^β for all $0 \leq t \leq s-1$ and a hyperedge E_r^β is called an r^{th} *ancestor* of a hyperedge E_w^β for all $r+1 \leq w \leq n$.

Example 2.21. Let $\mathcal{H}_0^{(3)}$ denote the 3-uniform 2-partite hypergraph from Definition 2.8 using $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_1 = \{a, b, c\}$ and $\mathcal{V}_2 = \{a', b', c'\}$ are mutually disjoint. $\mathcal{H}_0^{(3)}$ has edges as follows.

$$\{a, b, a'\}, \{a, c, a'\}, \{b, c, a'\}, \{a, b, b'\},$$
$$\{a, c, b'\}, \{b, c, b'\}, \{a, b, c'\}, \{a, c, c'\}, \{b, c, c'\}.$$

The following list shows all edges of $\mathrm{Prism}^1(\mathcal{H}_0^{(3)})$, where the first two lines are edges of $\mathcal{H}_0^{(3)}$, the next two lines are edges of $\mathcal{H}_1^{(3)}$ and the remaining edges are elements in the collection $\mathcal{B}$.

$$
\begin{gathered}
\{a_0, b_0, a'_0\}, \{a_0, c_0, a'_0\}, \{b_0, c_0, a'_0\}, \{a_0, b_0, b'_0\}, \{a_0, c_0, b'_0\}, \\
\{b_0, c_0, b'_0\}, \{a_0, b_0, c'_0\}, \{a_0, c_0, c'_0\}, \{b_0, c_0, c'_0\} \\
\{a_1, b_1, a'_1\}, \{a_1, c_1, a'_1\}, \{b_1, c_1, a'_1\}, \{a_1, b_1, b'_1\}, \{a_1, c_1, b'_1\}, \\
\{b_1, c_1, b'_1\}, \{a_1, b_1, c'_1\}, \{a_1, c_1, c'_1\}, \{b_1, c_1, c'_1\} \\
\{a_0, a_1, b_0\}, \{a_0, a_1, a'_0\}, \{a_0, a_1, c_0\}, \{a_0, a_1, b_1\}, \{a_0, a_1, a'_1\}, \{a_0, a_1, c_1\}, \\
\{b_0, b_1, a_0\}, \{b_0, b_1, a'_0\}, \{b_0, b_1, a_1\}, \{b_0, b_1, a'_1\}, \{b_0, b_1, c_0\}, \{b_0, b_1, c_1\} \\
\{c_0, c_1, a_0\}, \{c_0, c_1, a'_0\}, \{c_0, c_1, a_1\}, \{c_0, c_1, a'_1\}, \{c_0, c_1, b_0\}, \{c_0, c_1, b_1\} \\
\{a'_0, a'_1, a_0\}, \{a'_0, a'_1, b_0\}, \{a'_0, a'_1, c_0\}, \{a'_0, a'_1, a_1\}, \{a'_0, a'_1, b_1\}, \{a'_0, a'_1, c_1\}, \\
\{a_0, a_1, b'_0\}, \{a_0, a_1, b'_1\}, \{b_0, b_1, b'_0\}, \{b_0, b_1, b'_1\}, \{c_0, c_1, b'_0\}, \{c_0, c_1, b'_1\}, \\
\{b'_0, b'_1, a_0\}, \{b'_0, b'_1, a_1\}, \{b'_0, b'_1, b_0\}, \{b'_0, b'_1, b_1\}, \{b'_0, b'_1, c_0\}, \{b'_0, b'_1, c_1\}, \\
\{a_0, a_1, c'_0\}, \{a_0, a_1, c'_1\}, \{b_0, b_1, c'_0\}, \{b_0, b_1, c'_1\}, \{c_0, c_1, c'_0\}, \{c_0, c_1, c'_1\}, \\
\{c'_0, c'_1, a_0\}, \{c'_0, c'_1, a_1\}, \{c'_0, c'_1, b_0\}, \{c'_0, c'_1, b_1\}, \{c'_0, c'_1, c_0\}, \{c'_0, c'_1, c_1\}.
\end{gathered}
$$

Next, Baird [2] also gave a graph representation of a given hypergraph $\mathcal{H}$ as follows.

Definition 2.22. Let $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ be a hypergraph. A *graph of a hypergraph*, denoted by $G(\mathcal{H})$, consists of a vertex set $V(G(\mathcal{H}))$ and an edge set $E(G(\mathcal{H}))$ such that $V(G(\mathcal{H})) = \mathcal{V}(\mathcal{H})$ and for $u, v \in V(G(\mathcal{H}))$, $uv \in E(G(\mathcal{H}))$ if $\{u, v\} \subseteq e$ for some $e \in \mathcal{E}(\mathcal{H})$.

Example 2.23. From a hypergraph $\mathcal{H}$ given in Example 2.5, its $G(\mathcal{H})$ is shown in Figure 2.10.

2.2 Graph Structures

First of all, let us introduce some important definitions involving a graph G analogous to a hypergraph $\mathcal{H}$ in Definition 2.4.

Definition 2.24. (i) [16] For a graph G, the adjacent vertices are sometimes called *neighbors* of each other, and all neighbors of a given vertex x are called

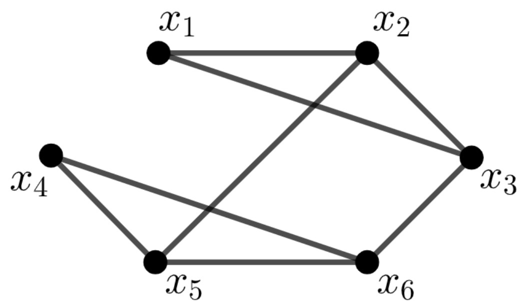

Figure 2.10: $G(\mathcal{H})$, where $\mathcal{H}$ is a hypergraph given in Example 2.5

the *neighborhood* of x. The neighborhood of x is denoted by $N_G(x)$ and the set $N_G(x) \cup \{x\}$ is denoted by $N_G[x]$.

(ii) [1] We say that a vertex x in a graph G is a *corner* (or a pitfall or irreducible) in G if for some vertex $y \neq x$ of G, $N_G[x] \subseteq N_G[y]$.

(iii) [16] Let V and E be the vertex set and the edge set of a graph G, respectively. A *deletion* of $x \in V$ from G is removing of x from V together with all edges of G incident to x from E, denoted by $G - x$.

Example 2.25. Consider a graph G shown in Figure 2.11.

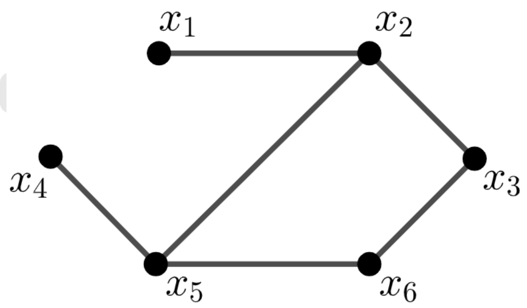

Figure 2.11: A graph G

We obtain that $N_G(x_1) = \{x_2\}$, $N_G(x_2) = \{x_1, x_3, x_5\}$, $N_G(x_3) = \{x_2, x_6\}$, $N_G(x_4) = \{x_5\}$, $N_G(x_5) = \{x_2, x_4, x_6\}$ and $N_G(x_6) = \{x_3, x_5\}$. Since $N_G[x_1] = \{x_1, x_2\} \subseteq \{x_1, x_2, x_3, x_5\} = N_G[x_2]$ and $N_G[x_4] = \{x_4, x_5\} \subseteq \{x_2, x_4, x_5, x_6\} =$

$N_G[x_5]$. By Definition 2.24, vertices x_1 and x_4 are corners of G. Finally, if we delete the vertex x_3 in a graph G, we obtain the graph shown in Figure 2.12.

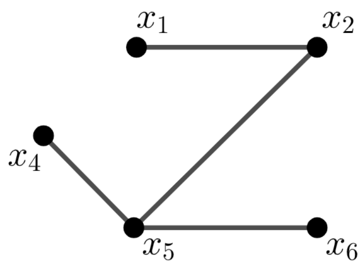

Figure 2.12: A graph $G - x_3$

Next, we provide the special class of graphs and its basic properties.

Definition 2.26. Let n and k be positive integers and $[n] = \{1, 2, 3, \ldots, n\}$. Let $[n]^{(k)}$ denote the family of all k-subsets of $[n]$. Let $k \in [n]$. The *Kneser graph* $KG(n, k)$ is a graph whose vertex set is $V(KG(n, k)) = [n]^{(k)}$ and the edge set $E(KG(n, k))$, where $UV \in E(KG(n, k))$ if and only if $U \cap V = \varnothing$.

Example 2.27. Consider $n = 5$ and $k = 2$. The vertex set $V(KG(5, 2)) = [5]^{(2)} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$. We obtain the Kneser graph $KG(5, 2)$ shown in Figure 2.13.

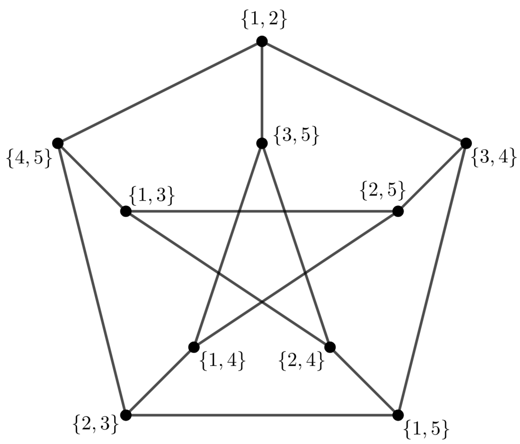

Figure 2.13: A Kneser graph $KG(5, 2)$

Many researches have studied several properties on Kneser graphs [4, 5, 7, 12, 15]. We see that $KG(n,k)$ has $\binom{n}{k}$ vertices and $\frac{1}{2}\binom{n}{k}\binom{n-k}{k}$ edges. We know that $KG(n,k)$ is a regular graph. Indeed, each vertex of $KG(n,k)$ has exactly $\binom{n-k}{k}$ neighbors, which is the degree of each vertex in $KG(n,k)$.

Note that if $n < 2k$, then $KG(n,k)$ is an empty graph and $KG(n,k)$ is usually studied for $n > 2k$. Furthermore, if $n = 2k$, then $KG(n,k)$ is a perfect matching with $\frac{1}{2}\binom{n}{k}$ edges, which is a disconnected graph. It is easy to see that $KG(n,k)$ is a connected graph for $n \geq 2k+1$. Indeed, the diameter of $KG(n,k)$ has been determined by Valencia-Pabon and Vera [15], as follows.

Theorem 2.28. *[15] For $k \geq 1$ and $n \geq 2k+1$, $diam(KG(n,k)) = \lceil \frac{k-1}{n-2k} \rceil + 1$.*

Definition 2.29. [12] A *dominating set* in a graph G is a subset S of the vertex set $V(G)$ such that each vertex in $V(G) - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of G is the minimum size of a dominating set in G.

It is easy to see that if $k \geq 2$ and n is suffciently large, then the smallest dominating set of $KG(n,k)$ is obtained by taking $k+1$ disjoint k-sets.

Proposition 2.30. *[12] Let $k \geq 2$ and $n \geq k^2 + k$. Then, the domination number $\gamma(KG(n,k)) = k+1$.*

2.3 Game Structures

Let G be a finite connected graph. A vertex-pursuit game of two players, *a cop and a robber*, played on a graph G was first introduced by Quilliot [13] and Nowakowski and Winkler [11]. The rules of the game are defined as follows:

(i) First, the cop selects a vertex to begin and the robber then selects another vertex to begin.

(ii) In each round, the cop and the robber take altenatively moving from their present vertex to other vertices along edges. However, they can also choose not to move from their positions at each of their turns as well.

There are two ways to finish the game such as the cop can catch the robber by occupying the same vertex as the robber after finite number of moves, or the robber can run away. The graph which the cop has the winning strategy is called a *cop-win graph*; otherwise, a *robber-win graph*.

In [1] and [11], the characterizations of cop-win graphs are shown. Nowakowski and Winkler [11] characterized by using the graph structure called dismantlable, where a graph G is *dismantlable* if there is $\{v_1, v_2, v_3, \ldots, v_n\}$ of the vertices of G such that for each $i < n$, v_i is a corner in the subgraph induced by $\{v_1, v_2, v_3, \ldots, v_n\}$ and stated the results on a finite path and an n-cycle where $n \geq 4$.

Theorem 2.31. *[11] A finite path is a cop-win graph and an n-cycle where $n \geq 4$ is a robber-win graph.*

In 1984, Aigner and Fromme [1] proved the statement that every cop-win graph has at least one corner and also gave a characterization of cop-win graphs by using a deletion of a vertex from a graph G and successively removing corners (in any order) from a graph G.

Theorem 2.32. *[1] Let x be a corner of a graph G and $\bar{G} = G - x$. Then, G is a cop-win graph if and only if $\bar{G}$ is a cop-win graph.*

By the previous theorem, they obtain the following theorem.

Theorem 2.33. *[1] G is a cop-win graph if and only if by successively removing corners (in any order), G can be reduced to a single vertex.*

Besides the characterization, Aigner and Fromme [1] gave a better chance to a cop by allowing more than one cops and at least one cop has to move on their turn which leads to investigate on the least number of cops guarateed their winning.

Definition 2.34. [1] For a finite connected graph G, $c(G)$ denotes the minimum number of cops needed for the cops to win and it is called the *cop-number*.

They investigated the bound of cop-number of given graphs.

Theorem 2.35. *[1] For a graph G, $c(G) \leq \gamma(G)$.*

Theorem 2.36. *[1] Let G be a graph with minimum degree $\delta(G) \geq n$ which contains no 3- or 4-cycles. Then, $c(G) \geq n$.*

Moreover, Tošić [14] proved the bounded of the cop-number of the Cartesian product of any two graphs as follows.

Theorem 2.37. *[14] Let G and H be graphs with the cop-numbers $c(G)$ and $c(H)$, respectively. Then, $c(G \,\Box\, H) \leq c(G) + c(H)$.*

According to Kneser graphs, we can see that $KG(n, 1)$ is the complete graph K_n which is obviously a cop-win graph whose cop-number is 1 and we know that the Petersen graph is the smallest Kneser graph $KG(5, 2)$ whose cop-number is 3.

Theorem 2.38. *[3] The Petersen graph with 10 vertices has cop-number 3. Moreover, for any connected graph G having at most 10 vertices and G is not isomorphic to the Petersen graph, we have $c(G) \leq 2$.*

Besides playing on graphs, Baird [2] introduced the game of cops and robbers played on hypergraphs in 2011. A cop and a robber can move from their present vertex x to any vertex y belonging to the same hyperedge as vertex x, which is slightly changed from the game played on graphs, or choose not to move. A hypergraph on which cop wins is called a *cop-win hypergraph* and a hypergraph on which robber wins is called a *robber-win hypergraph.* Then, he gave a result on winning strategy for a cop and considered a hyperpath and a hypercycle.

Lemma 2.39. *[2] The cop can play a winning strategy by remaining on external vertices until, perhaps, the final move of the game.*

Theorem 2.40. *[2] A hyperpath is a cop-win hypergraph and a hypercycle is a robber-win hypergraph.*

Then, he determined the cop-number in this game on hypergraphs in the following definition.

Definition 2.41. [2] Let $\mathcal{H}$ be a finite connected hypergraph. The *cop-number* $c(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the minimum number of cops needed to guarantee that cops win.

It is easy to see that $\mathcal{H}$ is a cop-win hypergraph if and only if its cop-number is 1. Baird [2] showed that the cop-number of hyperpath and hypercycle are 1 and 2, respectively.

CHAPTER III
COP-WIN HYPERGRAPHS

In this chapter, we focus on cop-win hypergraphs. We divide this chapter into two sections. In the first part, we provide a characterization of cop-win hypergraphs. Next, we show the results on each product, given in Chapter II, of cop-win hypergraphs by using its structure.

3.1 A Characterization of Cop-win Hypergraphs

To characterize the cop-win hypergraphs, we first give a necessary condition in Lemma 3.1. Then, we analyze a graph of a hypergraph instead of considering the hypergraph directly, which is the same idea as Baird [2].

Lemma 3.1. *If $\mathcal{H}$ is a cop-win hypergraph, then $\mathcal{H}$ has at least one corner.*

Proof. Assume that $\mathcal{H}$ is a cop-win hypergraph. Since the cop must catch the robber with his last move, they must stay in the same hyperedges at the end. Since the robber cannot run away, we have that, at the last move, all neighbors of the present vertex R of the robber must be a neighbor of the present vertex C of the cop, i.e., $N_{\mathcal{H}}[R] \subseteq N_{\mathcal{H}}[C]$. Hence, a vertex R is a corner in a hypergraph $\mathcal{H}$. □

By Lemma 3.1, if a hypergraph $\mathcal{H}$ has no corner, then $\mathcal{H}$ is a robber-win hypergraph. Conversely, it is not true. We can find an example of a robber-win hypergraph having a corner given in Figure 2.1 of Example 2.5.

After that, we give the relationship between a hypergraph $\mathcal{H}$ and a graph $G(\mathcal{H})$ of a hypergraph $\mathcal{H}$.

Lemma 3.2. *If $x \in \mathcal{V}(\mathcal{H})$ is a corner in a hypergraph $\mathcal{H}$, then x is a corner in a graph $G(\mathcal{H})$ of a hypergraph $\mathcal{H}$.*

Proof. Let x be a corner in a hypergraph $\mathcal{H}$. We claim that x is a corner in a graph $G(\mathcal{H})$ of a hypergraph $\mathcal{H}$; that is, there exists $y \in V(G(\mathcal{H}))$ such that $N_{G(\mathcal{H})}[x] \subseteq N_{G(\mathcal{H})}[y]$ for some $y \in V(G(\mathcal{H}))$. Let $a \in N_{G(\mathcal{H})}[x]$. By the definition of $G(\mathcal{H})$, we have $\{a, x\} \subseteq e$ for some $e \in \mathcal{E}(\mathcal{H})$. This means $a \in N_{\mathcal{H}}[x]$. Since x is a corner in $\mathcal{H}$, there exists a vertex $y \in \mathcal{V}(\mathcal{H})$ such that $N_{\mathcal{H}}[x] \subseteq N_{\mathcal{H}}[y]$. Thus, $a \in N_{\mathcal{H}}[y]$. If $a = y$, then $a \in N_{G(\mathcal{H})}[y]$. If $a \neq y$, then $\{a, y\} \in e'$ for some $e' \in \mathcal{E}(\mathcal{H})$. Thus, there is an edge connecting a and y in $G(\mathcal{H})$; that is, a is adjacent to y in $G(\mathcal{H})$. Therefore, $a \in N_{G(\mathcal{H})}(y) \subseteq N_{G(\mathcal{H})}[y]$. Hence, x is a corner in the graph $G(\mathcal{H})$ of the hypergraph $\mathcal{H}$. □

Theorem 3.3. *Cops and robbers game on a hypergraph $\mathcal{H}$ is equivalent to cops and robbers game on the graph $G(\mathcal{H})$. Therefore, a hypergraph $\mathcal{H}$ is a cop-win hypergraph if and only if its $G(\mathcal{H})$ is a cop-win graph.*

Proof. We consider a corresponding movement of a cop or a robber on hyperedges of a hypergraph $\mathcal{H}$ and on edges of its $G(\mathcal{H})$. For $1 \leq i \leq |\mathcal{E}(\mathcal{H})|$, we label each edge in $G(\mathcal{H})$ by $E_{i,j}$ when $E_{i,j}$ is an edge in $G(\mathcal{H})$ representing a hyperedge E_i of size r in $\mathcal{H}$ and $1 \leq j \leq \binom{r}{2}$. When a cop or a robber chooses to move from a vertex u to a vertex v along some hyperedge E_k of $\mathcal{H}$, there exists an edge $E_{k,l}$ of $G(\mathcal{H})$ for some $1 \leq l \leq \binom{r}{2}$ connecting a vertex u and a vertex v of $G(\mathcal{H})$. Similarly, when a cop or a robber moves from a vertex u to a vertex v along some edge $E_{k,l}$ of $G(\mathcal{H})$ for some $1 \leq l \leq \binom{r}{2}$, there exists a hyperedge E_k of $\mathcal{H}$ connecting a vertex u to a vertex v of $\mathcal{H}$. □

We see that Theorem 3.3 leads us to consider a graph $G(\mathcal{H})$ of a hypergraph $\mathcal{H}$ instead of $\mathcal{H}$ itself.

Lemma 3.4. *Let x be a vertex in a hypergraph $\mathcal{H}$. Then, $G(\mathcal{H} - x) = G(\mathcal{H}) - x$ where $\mathcal{H} - x$ is a weak deletion of a vertex x in a hypergraph $\mathcal{H}$ and $G(\mathcal{H}) - x$ is a deletion of a vertex x in a graph $G(\mathcal{H})$ of a hypergraph $\mathcal{H}$.*

Proof. First, we show that $V(G(\mathcal{H} - x)) = V(G(\mathcal{H}) - x)$. By the Definition 2.4 (iii) and Definition 2.22, $V(G(\mathcal{H} - x)) = \mathcal{V}(\mathcal{H}) - x = V(G(\mathcal{H}) - x)$.

Next, we prove that $E(G(\mathcal{H}-x)) = E(G(\mathcal{H})-x)$. Let $uv \in E(G(\mathcal{H})-x)$. Then, both u and v is not x. Thus, u is adjacent to v in $G(\mathcal{H})-x$; that is, there exists a hyperedge e in $\mathcal{H}$ such that $\{u,v\} \subseteq e$. Then, u is adjacent to v in $\mathcal{H}-x$. Therefore, $uv \in E(G(\mathcal{H}-x))$. Hence, $E(G(\mathcal{H})-x) \subseteq E(G(\mathcal{H}-x))$.

Conversely, let $uv \notin E(G(\mathcal{H})-x)$. Thus, u is not adjacent to v in $G(\mathcal{H})-x$. Then, there is no a hyperedge e in $\mathcal{H}-x$ such that $\{u,v\} \subseteq e$. This means that u is not adjacent to v in $\mathcal{H}-x$. Therefore, $uv \notin E(G(\mathcal{H}-x))$. Hence, $E(G(\mathcal{H}-x)) \subseteq E(G(\mathcal{H})-x)$. Thus, we can conclude that $G(\mathcal{H}-x) = G(\mathcal{H})-x$. □

By Theorem 3.3 and Lemma 3.4, we can characterize a cop-win hypergraph by weak deletion as shown in the following theorem.

Theorem 3.5. *Let $x \in \mathcal{V}(\mathcal{H})$ be a corner in a hypergraph $\mathcal{H}$. $\mathcal{H}$ is a cop-win hypergraph if and only if a weak deletion $\mathcal{H}-x$ is a cop-win hypergraph.*

Proof. Let x be a corner in a hypergraph $\mathcal{H}$. By Lemma 3.2, we have x is also a corner of a graph $G(\mathcal{H})$ of a hypergraph $\mathcal{H}$. Assume that $\mathcal{H}$ is a cop-win hypergraph. By Theorem 3.3, we have $G(\mathcal{H})$ is a cop-win graph. By Theorem 2.32, $G(\mathcal{H})-x$ is also a cop-win graph. By Lemma 3.4, $G(\mathcal{H})-x = G(\mathcal{H}-x)$. Since $G(\mathcal{H}-x) = G(\mathcal{H})-x$ is a cop-win graph, by Theorem 3.3, $\mathcal{H}-x$ is a cop-win hypergraph.

Conversely, we assume that $\mathcal{H}$ is a robber-win hypergraph. By Theorem 3.3, we have $G(\mathcal{H})$ is a robber-win graph. By Theorem 2.32, $G(\mathcal{H})-x$ is also a robber-win graph. Since $G(\mathcal{H}-x) = G(\mathcal{H})-x$ is a robber-win graph, by Theorem 3.3, $\mathcal{H}-x$ is also a robber-win hypergraph. □

Theorem 3.6. *A hypergraph $\mathcal{H}$ is a cop-win hypergraph if and only if by successively weak deletion corners (in any order), $\mathcal{H}$ can be reduced to a trivial hypergraph.*

Proof. Suppose that $x_1, x_2, x_3, \ldots, x_{|\mathcal{V}(\mathcal{H})|}$ are vertices of a hypergraph $\mathcal{H}$. Let $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_i = \mathcal{H}_{i-1} - x_{i-1}$ for all $2 \leq i \leq |\mathcal{V}(\mathcal{H})|$.

Let $n \in \mathbb{N}$ and let $|\mathcal{V}(\mathcal{H})| = n$. Assume that $\mathcal{H}$ is a cop-win hypergraph. By Lemma 3.1, $\mathcal{H}$ has at least one corner, say x_1. By Theorem 3.5, we obtain that

$\mathcal{H}_2 = \mathcal{H} - x_1$ is a cop-win hypergraph. By Lemma 3.1, $\mathcal{H}_2$ has at least one corner, say x_2. By Theorem 3.5, we have $\mathcal{H}_3 = \mathcal{H}_2 - x_2$ is a cop-win hypergraph. By Lemma 3.1, $\mathcal{H}_3$ has at least one corner, say x_3. Continue this process, by Lemma 3.1 and Theorem 3.5, we obtain that $\mathcal{H}_i$ is a cop-win hypergraph with at least one corner, say x_i for all $2 \leq i \leq n$. Therefore, we have a sequence of a weak deletion of corners $(x_1, x_2, x_3, \ldots, x_n)$. Before weak deleting x_n, a hypergraph $\mathcal{H}_n$ has only one vertex. This means $\mathcal{H}$ can be reduced to a trivial hypergraph.

Conversely, we assume that by successively weak deletion corners (in any order), a hypergraph $\mathcal{H}$ can be reduced to a trivial hypergraph. Then, we have a sequence of successively weak deletion of corners $(x_1, x_2, x_3, \ldots, x_n)$. We claim that $\mathcal{H}$ is a cop-win hypergraph. Since a trivial hypergraph or $\mathcal{H}_n$ is a cop-win hypergraph, by Theorem 3.5, $\mathcal{H}_{n-1}$ is a cop-win hypergraph. By Theorem 3.5 again, $\mathcal{H}_{n-2}$ is a cop-win hypergraph. Continue this process, by Theorem 3.5, $\mathcal{H} = \mathcal{H}_1$ is also a cop-win hypergraph. $\square$

3.2 Products of Hypergraphs

In this part, we investigate three types of the product of hypergraphs which is the Cartesian product, the minimal (maximal) rank preserving direct product and the normal (standard) strong product. First, we start with the Cartesian product of any hypergraphs.

According to the definition of the Cartesian product, we see that a subset $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_r, y_r)\}$ of $\mathcal{V}_1 \times \mathcal{V}_2$ is an edge in $\mathcal{H}_1 \,\square\, \mathcal{H}_2$ if and only if

(i) $\{x_1, x_2, x_3, \ldots x_r\} \in \mathcal{E}_1$ where x_i's are distinct vertices in $\mathcal{H}_1$ for all $1 \leq i \leq r$ and $y_1 = y_2 = y_3 = \cdots = y_r \in \mathcal{V}_2$, or

(ii) $\{y_1, y_2, y_3, \ldots y_r\} \in \mathcal{E}_2$ where y_i's are distinct vertices in $\mathcal{H}_2$ for all $1 \leq i \leq r$ and $x_1 = x_2 = x_3 = \cdots = x_r \in \mathcal{V}_1$.

Theorem 3.7. *The Cartesian product $\mathcal{H}_1 \,\square\, \mathcal{H}_2$ of any hypergraphs is a robber-win hypergraph where $|\mathcal{V}(\mathcal{H}_1)| \geq 2$ and $|\mathcal{V}(\mathcal{H}_2)| \geq 2$.*

Proof. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be hypergraphs where $|\mathcal{V}(\mathcal{H}_1)| \geq 2$ and $|\mathcal{V}(\mathcal{H}_2)| \geq 2$.

We would like to give a winning strategy to a robber. At some point, assume that the cop stays at a vertex (x_{i_1}, y_{j_1}) and the robber stays at a vertex (x_{i_2}, y_{j_2}), which is not a neighbor of (x_{i_1}, y_{j_1}). Without loss of generality, assume that $1 \leq i_1 \neq i_2 \leq |\mathcal{V}(\mathcal{H}_1)|$.

Case 1. $1 \leq j_1 = j_2 \leq |\mathcal{V}(\mathcal{H}_2)|$. Since we assume that the robber stays at a vertex (x_{i_2}, y_{j_1}) is not a neighbor of a vertex (x_{i_1}, y_{j_1}) of the cop, we consider the following cases.

Case 1.1. There is a common neighbor of vertices (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}), says (x_{i_3}, y_{j_2}) where $i_3 \neq i_2$, then the cop moves from a vertex (x_{i_1}, y_{j_1}) to a vertex (x_{i_3}, y_{j_1}). Therefore, the robber can move from a vertex (x_{i_2}, y_{j_1}) to a vertex (x_{i_1}, y_{j_3}) where $j_3 \neq j_1$.

Case 1.2. There are no common neighbors of vertices (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_1}). We see that the cop moves to some vertices which is not a neighbor of (x_{i_2}, y_{j_1}). Then, the robber chooses not to move.

Case 2. $1 \leq j_1 \neq j_2 \leq |\mathcal{V}(\mathcal{H}_2)|$.

Case 2.1. There is a common neighbor of vertices (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}), says (x_{i_1}, y_{j_2}). Then, the cop moves from a vertex (x_{i_1}, y_{j_1}) to a vertex (x_{i_1}, y_{j_2}). Therefore, the robber can move from a vertex (x_{i_2}, y_{j_2}) to a vertex (x_{i_2}, y_{j_1}).

Case 2.2. There is a common neighbor of vertices (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}), says (x_{i_2}, y_{j_1}). Then, the cop moves from a vertex (x_{i_1}, y_{j_1}) to a vertex (x_{i_2}, y_{j_1}). Therefore, the robber can move from a vertex (x_{i_2}, y_{j_2}) to a vertex (x_{i_1}, y_{j_2}).

Case 2.3. There are no common neighbors of vertices (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}). We see that the cop moves to some vertices which is not a neighbor of (x_{i_2}, y_{j_2}). Then, the robber chooses not to move.

Hence, the robber always find a free neighbor to run away. $\square$

Remark 1. (i) From Definition 2.12, we observe that for each vertex (x_i, y_j) in

$\mathcal{H}_1 \times_1 \mathcal{H}_2$, there are at least one vertex of the form $(x_i, y_{j'})$ and at least one vertex of the form $(x_{i'}, y_j)$ which are not neighbors of (x_i, y_j) where $i \neq i'$ and $j \neq j'$.

(ii) From Definition 2.14, if $\mathcal{H}_1$ is a k-uniform hypergraph and $\mathcal{H}_2$ is an l-uniform hypergraph where $k, l \geq 2$, we observe that for each vertex (x_i, y_j) in $\mathcal{H}_1 \times_2 \mathcal{H}_2$, there are either at least one vertex of the form $(x_i, y_{j'})$ or at least one vertex of the form $(x_{i'}, y_j)$ which are not neighbors of (x_i, y_j) where $i \neq i'$ and $j \neq j'$.

Next, we give the result on the minimal rank preserving direct product of any hypergraphs. We see that if at least one vertex of $\mathcal{H}_1$ or $\mathcal{H}_2$ has more than one neighbor, then there exists a vertex in $\mathcal{H}_1 \times_1 \mathcal{H}_2$ having more than one neighbor.

Theorem 3.8. *Assume that at least one vertex of $\mathcal{H}_1$ or $\mathcal{H}_2$ has more than one neighbor. The minimal rank preserving direct product $\mathcal{H}_1 \times_1 \mathcal{H}_2$ of any hypergraphs is a robber-win hypergraph where $|\mathcal{V}(\mathcal{H}_1)| \geq 2$ and $|\mathcal{V}(\mathcal{H}_2)| \geq 2$.*

Proof. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be hypergraphs where $|\mathcal{V}(\mathcal{H}_1)| \geq 2$ and $|\mathcal{V}(\mathcal{H}_2)| \geq 2$.

Let $(x^{(i)}, y^{(i)})$ denote a vertex of a cop at the i^{th} turn. At the 0^{th} turn, the cop starts at a vertex $(x^{(0)}, y^{(0)})$. If both $x^{(0)}$ and $y^{(0)}$ have no other neighbors, then $\mathcal{H}_1 \times_1 \mathcal{H}_2$ is disconnected. Without loss of generality, assume that $y^{(0)}$ has at least one neighbor in $\mathcal{H}_2$, says y. By Remark 1 (i), there is at least one vertex of the form $(x^{(0)}, v)$ which is not a neighbor of $(x^{(0)}, y^{(0)})$ where $v \neq y^{(0)}$, namely $(x^{(0)}, y)$. Then, the robber chooses a vertex $(x^{(0)}, y)$ to start.

Let $m \geq 1$ be a positive integer. We define the *stage* m to be the stage where the robber stays at $(x^{(m)}, y)$ where $y \neq y^{(m)}$ and y is a neighbor of $y^{(m)}$ in $\mathcal{H}_2$.

We see that $(x^{(i)}, y)$, where $y \neq y^{(i)}$ and y is a neighbor of $y^{(i)}$ in $\mathcal{H}_2$, is not a neighbor of $(x^{(i)}, y^{(i)})$.

We use the mathematical induction on m to provide a winning strategy for a robber. At the 1^{st} turn, the cop moves from $(x^{(0)}, y^{(0)})$ to $(x^{(1)}, y^{(1)})$. We see that $x^{(1)}$ is a neighbor of $x^{(0)}$ in $\mathcal{H}_1$ and $y^{(1)}$ is a neighbor of $y^{(0)}$ in $\mathcal{H}_2$. Then, the robber can move from $(x^{(0)}, y)$ to $(x^{(1)}, y^{(0)})$, which is not a neighbor of $(x^{(1)}, y^{(1)})$. Hence, the robber enters stage 1 and we complete the basis step.

For the induction step, assume that the robber enters stage $m \geq 2$. Then, the robber stays at $(x^{(m)}, y')$ where $y' \neq y^{(m)}$ and y' is a neighbor of $y^{(m)}$ in $\mathcal{H}_2$. Next, it's the cop's turn.

At the $(m+1)^{th}$ turn, the cop moves from $(x^{(m)}, y^{(m)})$ to $(x^{(m+1)}, y^{(m+1)})$. We see that $x^{(m+1)}$ is a neighbor of $x^{(m)}$ in $\mathcal{H}_1$ and $y^{(m+1)}$ is a neighbor of $y^{(m)}$ in $\mathcal{H}_2$. Then, the robber moves from $(x^{(m)}, y')$ to $(x^{(m+1)}, y^{(m)})$, which is not a neighbor of $(x^{(m)}, y^{(m)})$.

Therefore, the robber enters stage $m+1$. Hence, the robber always find a free neighbor to run away. $\square$

After that, under some condition on $\mathcal{H}_1$ and $\mathcal{H}_2$, we would like to show that the maximal rank preserving direct product $\mathcal{H}_1 \times_2 \mathcal{H}_2$ is a robber-win hypergraph. However, in our consideration, we need the uniformity of each hypergraph.

Theorem 3.9. *Let $\mathcal{H}_1$ be a k-uniform hypergraph and $\mathcal{H}_2$ an l-uniform hypergraph where $2 \leq k \leq l$. The maximal rank preserving direct product $\mathcal{H}_1 \times_2 \mathcal{H}_2$ is a robber-win hypergraph where $|\mathcal{V}(\mathcal{H}_1)| \geq 2$ and $|\mathcal{V}(\mathcal{H}_2)| \geq 2$.*

Proof. Let $k \geq 2$ and $l \geq 2$ be positive integers. Let $\mathcal{H}_1$ be a k-uniform hypergraph and $\mathcal{H}_2$ an l-uniform hypergraph where $k \leq l$.

Let $(x^{(i)}, y^{(i)})$ denote a vertex of a cop at the i^{th} turn. At the 0^{th} turn, the cop starts at a vertex $(x^{(0)}, y^{(0)})$. We see that $x^{(0)}$ has at least one neighbor in $\mathcal{H}_1$, says x and $y^{(0)}$ has at least one neighbor in $\mathcal{H}_2$, says y. By Remark 1 (ii), there is at least one vertex of the form $(u, y^{(0)})$ which is not a neighbor of $(x^{(0)}, y^{(0)})$ where $u \neq x^{(0)}$, namely $(x, y^{(0)})$. Then, the robber starts at a vertex $(x, y^{(0)})$ which is not a neighbor of $(x^{(0)}, y^{(0)})$.

Let $m \geq 1$ be a positive integer. We define the *stage* m to be the stage where the robber stays at $(x, y^{(m)})$ where $x \neq x^{(m)}$ and x is a neighbor of $x^{(m)}$ in $\mathcal{H}_1$. We see that $(x, y^{(i)})$, where $x \neq x^{(i)}$ and x is a neighbor of $x^{(i)}$ in $\mathcal{H}_1$, is not a neighbor of $(x^{(i)}, y^{(i)})$.

We use the mathematical induction on m to provide a winning strategy for a robber. At the 1^{st} turn, the cop moves from $(x^{(0)}, y^{(0)})$ to $(x^{(1)}, y^{(1)})$. Let $\bar{v}^{(i)}$

denote a vertex of a robber in the i^{th} turn. We consider a chosen $\bar{v}^{(1)}$ according to the following cases.

Case 1. $x^{(1)} = x^{(0)}$ and $y^{(1)} = y^{(0)}$. Then, the robber chooses not to move; that is, $\bar{v}^{(1)} = (x, y^{(0)}) = (x, y^{(1)})$, which is not a neighbor of $(x^{(1)}, y^{(1)})$. We see that $x \neq x^{(1)}$ and x is a neighbor of $x^{(1)}$ in $\mathcal{H}_1$.

Case 2. $x^{(1)} = x^{(0)}$ and $y^{(1)} \neq y^{(0)}$. Then, the robber can move from $(x, y^{(0)})$ to $(x, y^{(1)})$. Therefore, $\bar{v}^{(1)} = (x, y^{(1)})$, which is not a neighbor of $(x^{(1)}, y^{(1)})$. We see that $x \neq x^{(1)}$ and x is a neighbor of $x^{(1)}$ in $\mathcal{H}_1$.

Case 3. $x^{(1)} \neq x^{(0)}$ and $y^{(1)} \neq y^{(0)}$. Then, the robber can move from $(x, y^{(0)})$ to $(x^{(0)}, y^{(1)})$. Therefore, $\bar{v}^{(1)} = (x^{(0)}, y^{(1)})$, which is not a neighbor of $(x^{(1)}, y^{(1)})$.

Hence, the robber enters stage 1 and we complete the basis step.

For the induction step, assume that the robber enters stage $m \geq 2$. Then, the robber stays at $(x', y^{(m)})$ where $x' \neq x^{(m)}$ and x' is a neighbor of $x^{(m)}$ in $\mathcal{H}_1$.

At the $(m+1)^{th}$ turn, the cop moves from $(x^{(m)}, y^{(m)})$ to $(x^{(m+1)}, y^{(m+1)})$. To win the game, the robber chooses to stay at a vertex $\bar{v}^{(m+1)}$, which can be chosen according to the following cases.

Case 1. $x^{(m+1)} = x^{(m)}$ and $y^{(m+1)} = y^{(m)}$. Then, the robber chooses not to move; that is, $\bar{v}^{(m+1)} = (x', y^{(m)}) = (x', y^{(m+1)})$, which is not a neighbor of $(x^{(m+1)}, y^{(m+1)})$. We see that $x' \neq x^{(m+1)}$ and x' is a neighbor of $x^{(m+1)}$ in $\mathcal{H}_1$.

Case 2. $x^{(m+1)} = x^{(m)}$ and $y^{(m+1)} \neq y^{(m)}$. Then, the robber can move from $(x', y^{(m)})$ to $(x', y^{(m+1)})$. Therefore, $\bar{v}^{(m+1)} = (x', y^{(m+1)})$, which is not a neighbor of $(x^{(m+1)}, y^{(m+1)})$. We see that $x' \neq x^{(m+1)}$ and x' is a neighbor of $x^{(m+1)}$ in $\mathcal{H}_1$.

Case 3. $x^{(m+1)} \neq x^{(m)}$ and $y^{(m+1)} \neq y^{(m)}$. Then, the robber can move from $(x', y^{(m)})$ to $(x^{(m)}, y^{(m+1)})$. Therefore, $\bar{v}^{(m+1)} = (x^{(m)}, y^{(m+1)})$, which is not a neighbor of $(x^{(m+1)}, y^{(m+1)})$.

Therefore, the robber enters stage $m+1$. Hence, the robber always find a free neighbor to run away. $\square$

Corollary 3.10. *Let $m \geq 2$ be a positive integer. If $\mathcal{H}$ is a collection of m hypergraphs, then both Cartesian product and minimal (maximal) rank preserving direct product of such m hypergraphs are robber-win hypergraphs.*

Proof. We prove by the mathematical induction on m. For $m = 2$, the corollary is proved by Theorem 3.7, 3.8 and 3.9. Let $m > 2$. Assume that both Cartesian product and minimal (maximal) rank preserving direct product of $m-1$ hypergraphs are robber-win hypergraphs. By induction hypothesis and Theorem 3.7, 3.8 and 3.9, we obtain that both Cartesian product and minimal (maximal) rank preserving direct product of m hypergraphs are also robber-win hypergraphs. $\square$

By Corollary 3.10, we obtain that both Cartesian product and minimal (maximal) rank preserving direct product of m hypergraphs are robber-win hypergraphs, but the normal (standard) strong product of m cop-win hypergraphs is not a robber-win hypergraph.

Theorem 3.11. *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are cop-win hypergraphs, then $\mathcal{H}_1 \boxtimes_* \mathcal{H}_2$ is also a cop-win hypergraph where $* = 1$ or 2.*

Proof. Let k and l be positive integers. Assume that $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ are cop-win hypergraphs, where $\mathcal{V}_1 = \{x_1, x_2, x_3, \ldots, x_k\}$ and $\mathcal{V}_2 = \{y_1, y_2, y_3, \ldots, y_l\}$.

To consider $\mathcal{H}_1 \boxtimes_* \mathcal{H}_2$, for $1 \le i \le k$ and $1 \le j \le l$, let $S_i = \{x_i\} \times \mathcal{E}_2$ and $T_j = \mathcal{E}_1 \times \{y_j\}$. We consider three possible cases of the present vertex of a cop and the present vertex of a robber.

Case 1. The cop chooses (x_i, y_{j_1}) to stay and the robber chooses (x_i, y_{j_2}) to stay where $j_1 \neq j_2$. To catch the robber, the cop moves along some edges in S_i. If y_{j_1} and y_{j_2} are in the same edge in $\mathcal{H}_2$, then the cop can occupy the same vertex as the robber in $\mathcal{H}_1 \boxtimes_* \mathcal{H}_2$. Otherwise, there are two different edges of $\mathcal{H}_2$, one containing y_{j_1} and the other containing y_{j_2}, the cop moves to the vertex (x_i, y_{j_3}) where y_{j_3} is the vertex which the cop chooses in the next turn in his strategy in the game on $\mathcal{H}_2$.

Case 2. The cop chooses (x_{i_1}, y_j) to stay and the robber chooses (x_{i_2}, y_j) to stay where $i_1 \neq i_2$. To catch the robber, the cop moves along some edges in T_j. If x_{i_1} and x_{i_2} are in the same edge in $\mathcal{H}_1$, then the cop can occupy the same vertex as the robber in $\mathcal{H}_1 \boxtimes_* \mathcal{H}_2$. Otherwise, there are two different edges of $\mathcal{H}_1$, one containing x_{i_1} and the other containing x_{i_2}, the cop moves to the vertex (x_{i_3}, y_j) where x_{i_3} is the vertex which the cop chooses in the next turn in his strategy in the game on $\mathcal{H}_1$.

Case 3. The cop chooses (x_{i_1}, y_{j_1}) to stay and the robber chooses (x_{i_2}, y_{j_2}) to stay where $i_1 \neq i_2$ and $j_1 \neq j_2$. To catch the robber, the cop moves along some edges in $E(\mathcal{H}_1 \times_* \mathcal{H}_2)$. If both x_{i_1} and x_{i_2} are in the same edge in $\mathcal{H}_1$, and both y_{j_1} and y_{j_2} are in the same edge in $\mathcal{H}_2$, then the cop can occupy the same vertex as the robber in $\mathcal{H}_1 \boxtimes_* \mathcal{H}_2$. Otherwise, there are two different edges of $\mathcal{H}_1$, one containing x_{i_1} and the other containing x_{i_2}, and there are two different edges of $\mathcal{H}_2$, one containing y_{j_1} and the other containing y_{j_2}, the cop moves to the vertex (x_{i_3}, y_{j_3}) where x_{i_3} is the vertex which the cop chooses in the next turn in his strategy in the game on $\mathcal{H}_1$ and y_{j_3} is the vertex which the cop chooses in the next turn in his strategy in the game on $\mathcal{H}_2$.

Following the three cases after finite number of moves, at some point, the cop can stay in the same hyperedge in both $\mathcal{H}_1$ and $\mathcal{H}_2$ as the robber does. Then, the cop use the winning strategy on $\mathcal{H}_1$ and $\mathcal{H}_2$, before the final turn, to chase the robber and finally, he catches the robber. □

Corollary 3.12. *Let $m \geq 2$ be a positive integer. If $\mathcal{H}$ is a collection of m cop-win hypergraphs, then the normal (standard) strong product of such m cop-win hypergraphs is a cop-win hypergraph.*

Proof. We prove by the mathematical induction on m. For $m = 2$, the corollary is proved by Theorem 3.11. Let $m > 2$. Assume that the normal (standard) strong product of $m - 1$ cop-win hypergraphs is a cop-win hypergraph. By induction hypothesis and Theorem 3.11, we obtain that the normal (standard) strong product of m cop-win hypergraphs is also a cop-win hypergraph. □

Besides proving the products of hypergraphs by their structures, we would like to use some theorems to show that the Cartesian product of any hypergraphs are robber-win hypergraphs.

Lemma 3.13. *The Cartesian product of any two hypergraphs has no corner.*

Proof. Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be hypergraphs. Suppose that the Cartesian product $\mathcal{H} = \mathcal{H}_1 \,\square\, \mathcal{H}_2$ has at least one corner. Let (x_1, y_1) be a corner of $\mathcal{H}$. Then, there exists $(x, y) \in \mathcal{V}(\mathcal{H}) - \{(x_1, y_1)\}$ such that $N_{\mathcal{H}}[(x_1, y_1)] \subseteq N_{\mathcal{H}}[(x, y)]$. We see that $(x_1, y_1) \in N_{\mathcal{H}}[(x, y)]$. Thus, either $x = x_1$ or $y = y_1$.

Case 1. If $x = x_1$ and $y \neq y_1$, then there exists an edge $\{\{x_1\} \times e_2 \mid e_2 \in \mathcal{E}_2\}$ connecting (x_1, y) and its neighbors. We know that some neighbors of (x_1, y_1) are in the form (u, y_1) where $u \neq x_1$ which do not connect to (x_1, y), which is a contradiction.

Case 2. If $x \neq x_1$ and $y = y_1$, then, there exists an edge $\{e_1 \times \{y_1\} \mid e_1 \in \mathcal{E}_1\}$ connecting (x, y_1) and its neighbors. We know that some neighbors of (x_1, y_1) are in the form (x_1, v) where $v \neq y_1$ which do not connect to (x, y_1), which is a contradiction.

Hence, the Cartesian product of any two hypergraphs has no corner. $\square$

By Lemma 3.13, we can conclude the following theorem.

Theorem 3.14. *Cartesian product of any two hypergraphs are robber-win hypergraphs.*

By Corollary 3.10, we also have that Cartesian product of m hypergraphs are robber-win hypergraphs.

According to minimal (maximal) preserving direct product, there exists an example of a hypergraph $\mathcal{H}_1$ and a hypergraph $\mathcal{H}_2$ such that both $\mathcal{H}_1 \times_1 \mathcal{H}_2$ and $\mathcal{H}_1 \times_2 \mathcal{H}_2$ have a corner.

Example 3.15. We use hypergraphs $\mathcal{H}_1^{'}$ and $\mathcal{H}_2^{'}$, shown in Figure 3.1 and 3.2, respectively, to construct the minimal (maximal) rank preserving direct product $\mathcal{H}'_1 \times_* \mathcal{H}'_2$ where $* = 1$ or 2. The vertex set $\mathcal{V}_1 \times \mathcal{V}_2 = \{(x_1, y_1), (x_2, y_1), (x_1, y_2), (x_2, y_2), (x_1, y_3), (x_2, y_3), (x_1, y_4), (x_2, y_4)\}$ where the edge set $\mathcal{E}(\mathcal{H}_1^{'} \times_1 \mathcal{H}_2^{'}) = \{\{(x_1, y_1), (x_2, y_2)\}, \{(x_1, y_2), (x_2, y_1)\}, \{(x_1, y_2), (x_2, y_3)\}, \{(x_1, y_2), (x_2, y_4)\}, \{(x_1, y_3), (x_2, y_2)\}, \{(x_1, y_3), (x_2, y_4)\}, \{(x_1, y_4), (x_2, y_2)\}, \{(x_1, y_4), (x_2, y_3)\}\}$ and the edge set $\mathcal{E}(\mathcal{H}_1 \times_2 \mathcal{H}_2) = \{\{(x_1, y_1), (x_2, y_1), (x_1, y_2)\}, \{(x_1, y_1), (x_2, y_1), (x_2, y_2)\}, \{(x_1, y_1), (x_1, y_2), (x_2, y_2)\}, \{(x_1, y_2), (x_2, y_1), (x_2, y_2)\}, \{(x_1, y_2), (x_1, y_3), (x_2, y_4)\}, \{(x_1, y_2), (x_2, y_3), (x_1, y_4)\}, \{(x_1, y_2), (x_2, y_3), (x_2, y_4)\}, \{(x_2, y_2), (x_1, y_3), (x_1, y_4)\}, \{(x_2, y_2), (x_1, y_3), (x_2, y_4)\}, \{(x_2, y_2), (x_2, y_3), (x_1, y_4)\}\}$. We use ij instead of (i, j) in $\mathcal{H}_1^{'} \times_1 \mathcal{H}_2^{'}$ and $\mathcal{H}_1^{'} \times_2 \mathcal{H}_2^{'}$ shown in Figure 3.3 and 3.4, respectively.

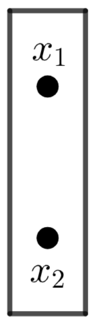

Figure 3.1: A hypergraph $\mathcal{H}_1^{'}$

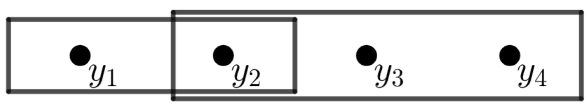

Figure 3.2: A hypergraph $\mathcal{H}_2^{'}$

We see that a vertex (x_1, y_1) is a corner of both minimal rank preserving direct product $\mathcal{H}_1^{'} \times_1 \mathcal{H}_2^{'}$ and maximal rank preserving direct product $\mathcal{H}_1^{'} \times_2 \mathcal{H}_2^{'}$.

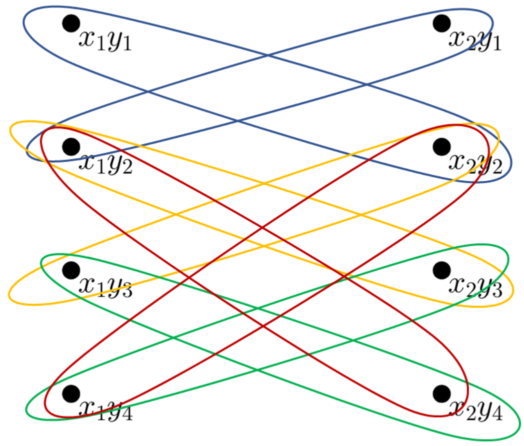

Figure 3.3: The minimal rank preserving direct product $\mathcal{H}_1' \times_1 \mathcal{H}_2'$

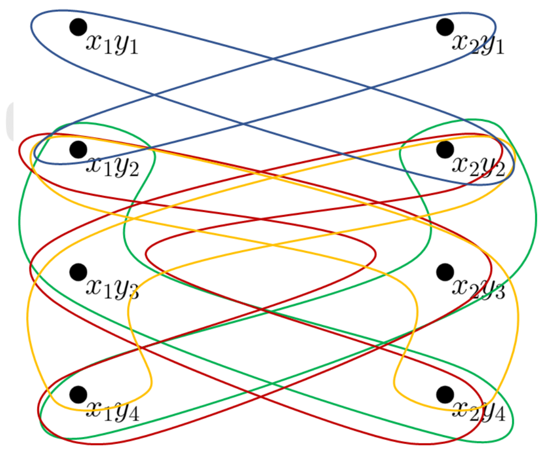

Figure 3.4: The maximal rank preserving direct product $\mathcal{H}_1' \times_2 \mathcal{H}_2'$

CHAPTER IV

COP-NUMBER OF CERTAIN HYPERGRAPHS

We consider the cop-number of two hypergraph structures, namely, k-partite hypergraphs and n-prisms over a hypergraph which slightly differ from such graph structures.

4.1 Cop-Number of Complete k-Partite Hypergraphs

There are two definitions of the k-partite hypergraph. The first definition given in Definition 2.6 is modified from the definition given by Kuhl and Schroeder [10] and the second definition given in Definition 2.8 is modified from the definition given by Jirimutu and Wang [9]. To investigate the cop-number of complete k-partite hypergraphs, we compare the position of a robber with the partite set that each cop stays. First, we consider the complete k-uniform k-partite hypergraphs from Definition 2.6.

Theorem 4.1. *A complete k-uniform k-partite hypergraph $\mathcal{H}$ from Definition 2.6 where each partite set has size at least 2 is a robber-win hypergraph and $c(\mathcal{H}) = 2$.*

Proof. First, we claim that $\mathcal{H}$ is a robber-win hypergraph. No matter where the cop is in $\mathcal{H}$, by the completeness of $\mathcal{H}$, the robber always moves to other vertices in the same partite set as the cop stays. Then, the robber has a winning strategy. Thus, we have $c(\mathcal{H}) \geq 2$.

Next, we claim that two cops have a winning strategy. In the first turn of cops, two cops choose some vertices which are in different partite sets. The robber then chooses one vertex in $\mathcal{H}$.

Case 1. The robber stays in the same partite set as one of two cops. Then, by the definition of hyperedge, there exists a hyperedge connecting the robber and the other cop. Thus, such a cop can catch the robber in his next move.

Case 2. The robber stays in the different partite set from both cops. Then, by the definition of hyperedge, there exists a hyperedge connecting each cop and the robber. Thus, one cop can catch the robber in his next move.

Therefore, two cops win and $c(\mathcal{H}) = 2$. □

By Theorem 4.1, we conclude that a complete bipartite graph is also a robber-win graph and two cops are minimum needed. Next, we consider the complete σ-uniform k-partite hypergraphs from Definition 2.8.

Theorem 4.2. *Let $\sigma \geq 3$. A complete σ-uniform k-partite hypergraph $\mathcal{H}$ from Definition 2.8 is a cop-win hypergraph.*

Proof. By the definition of hyperedge and the completeness of this hypergraph, there exists a hyperedge connecting the robber and the cop. Then, the robber is caught in the next move of the cop. □

In the case that $\sigma = 2$, the 2-uniform k-partite hypergraph $\mathcal{H}$ from Definition 2.8 is a k-partite graph and we can conclude the following.

Theorem 4.3. *Let $k \geq 3$. A complete 2-uniform k-partite hypergraph $\mathcal{H}$ from Definition 2.8 (or a complete k-partite graph) is a robber-win hypergraph (or a robber-win graph) having cop-number $c(\mathcal{H}) = 2$ where $|\mathcal{V}_i| \geq 2$ for all $1 \leq i \leq k$.*

Proof. First, we show that $\mathcal{H}$ is a robber-win hypergraph. The robber always moves to other vertices in the same partite set as the position of the cop. Then, the robber has a winning strategy. Now, we have $c(\mathcal{H}) \geq 2$.

By modifying the proof of Theorem 4.1, we consider the position of the robber.

Case 1. The robber stays in the same partite set as one of two cops. Then, there exists an edge connecting the robber and the other cop. Thus, such a cop can catch the robber in his next move.

Case 2. The robber stays in the different partite set from both cops. Then, there exists an edge connecting each cop and the robber. Thus, one cop can catch the robber in his next move.

We obtain that two cops are enough to catch the robber. Hence, $c(\mathcal{H}) = 2$. $\square$

4.2 Cop-Number of n-Prisms over a Hypergraph

The definition of this family of hypergraphs which is given in Definition 2.20 is modified from the definition given by Boonklurb et.al. [6]. To consider the cop-number of n-prisms over a k-uniform hypergraph $\mathcal{H}$, we separate into two subsections which are (i) $\mathcal{H}_0^{(k)}$ is a cop-win hypergraph and (ii) $\mathcal{H}_0^{(k)}$ is a robber-win hypergraph.

4.2.1 n-Prisms over a Cop-win Hypergraph

In this section, the cop-number of n-prisms over a cop-win k-uniform hypergraph when $k \geq 3$ and a cop-win 2-uniform hypergraph are considered separately.

Theorem 4.4. *Let $k \geq 3$ and $n \geq 2$ be integers. Assume that $\mathcal{H}_0^{(k)}$ is a cop-win k-uniform hypergraph. Then, $c(\mathrm{Prism}^n(\mathcal{H}_0^{(k)})) = 1$.*

Proof. Let $m \geq 0$ be a non-negative integer. We define the *stage m* to the stage where

- the robber stays on a vertex R of hypergraph $\mathcal{H}_p^{(k)}$ for some $p \geq m + 1$;
- the cop stays on an m^{th} ancestor of a vertex R;
- next, it is the robber's turn.

We use the mathematical induction on m to provide a winning strategy for one cop. First, the cop places himself anywhere on $\mathcal{H}_0^{(k)}$. Then, the robber places himself anywhere on $\mathrm{Prism}^n(\mathcal{H}_0^{(k)})$, say R. Since $\mathcal{H}_0^{(k)}$ is a cop-win hypergraph, the cop has a winning strategy on $\mathcal{H}_0^{(k)}$. By applying the winning strategy for the cop, he can stay on a 0^{th} ancestor of a vertex R. If the robber stays in a hypergraph $\mathcal{H}_0^{(k)}$ same as the cop does, then he is caught. Thus, the robber needs to stay on a hypergraph $\mathcal{H}_p^{(k)}$ for some $p \geq 1$ at the beginning. Next, it is the robber's turn. Therefore, the cop enters the stage 0 and we complete the basis step.

For the induction step, assume that the cop enters the stage m. That is, the cop stays on an m^{th} ancestor of a vertex R and then, the robber moves. Next, we consider how the robber moves. Since $\mathcal{H}_m^{(k)}$ is a copy of a hypergraph $\mathcal{H}_0^{(k)}$, $\mathcal{H}_m^{(k)}$ is also a cop-win hypergraph. If the robber stays on a hypergraph $\mathcal{H}_m^{(k)}$ same as the cop does, then he is caught. Therefore, the robber must stay on a hypergraph $\mathcal{H}_q^{(k)}$ for some $q \geq m+1$. We see that if the robber still stays on a hypergraph $\mathcal{H}_q^{(k)}$ where $q = m+1$, he will be caught in the next turn of a cop.

Case 1. Assume that the robber still stay vertex R in $\mathcal{H}_q^{(k)}$ where $q \geq m+2$. Then, the cop climbs up to stay on the $(m+1)^{th}$ descendant of his previous vertex, which means an $(m+1)^{th}$ ancestor of a vertex R.

Case 2. Assume that the robber move from vertex R to vertex R' in $\mathcal{H}_q^{(k)}$ where $q \geq m+2$. Then, the cop moves and climbs up from an m^{th} ancestor of a vertex R to an $(m+1)^{th}$ ancestor of a vertex R'.

Case 3. Assume that the robber move from vertex R in $\mathcal{H}_q^{(k)}$ to a $(q+1)^{th}$ descendent of vertex R. Then, the cop climbs up to stay on the $(m+1)^{th}$ descendant of his previous vertex, which means an $(m+1)^{th}$ ancestor of a vertex R.

Case 4. Assume that the robber move from vertex R in $\mathcal{H}_q^{(k)}$ to a $(q+1)^{th}$ descendent of vertex R'. Then, the cop moves and climbs up from an m^{th} ancestor of a vertex R to an $(m+1)^{th}$ ancestor of a vertex R'.

By every cases considered above, the cop stays on an $(m+1)^{th}$ ancestor of a vertex R. Then, the cop enters the stage $m+1$. Now, we complete the induction step. After the cop enters the stage $n-1$, the cop use the winning strategy in hypergraph $\mathcal{H}_{n-1}^{(k)}$ to stay on the $(n-1)^{th}$ ancestor of a vertex of the robber. Next, it is the robber's turn. No matter how the robber moves, the cop can climb up and catch the robber, which is the stage n. Hence, $\text{Prism}^n(\mathcal{H}_0^{(k)})$ is a cop-win hypergraph and $c(\text{Prism}^n(\mathcal{H}_0^{(k)})) = 1$. □

Actually, if $k=2$, $\text{Prism}^n(\mathcal{H}_0^{(2)})$ can be regarded as the Cartesian product of $\mathcal{H}_0^{(2)}$ and the path graph P_{n+1} in ordinary graph. By Theorem 2.37, the cop-number of the Cartesian product of two graphs can be bounded above by the sum of the cop-number of each graph. Since $\mathcal{H}_0^{(2)}$ is a cop-win graph and by Theorem 2.31,

$c(P_{n+1}) = 1$, we have $c(\text{Prism}^n(\mathcal{H}_0^{(2)})) \leq c(\mathcal{H}_0^{(2)}) + c(P_{n+1}) = 2$. Next, we claim that one cop cannot catch the robber. Then, we consider how the robber escapes.

Case 1. Assume that the robber stays in the same hypergraph as the cop. When the robber is threatened, the robber goes to the other hypergraphs.

Case 2. Assume that the robber stays in the different hypergraph from the cop. Then, the robber can choose not to move.

Therefore, no matter how the cop moves, the robber always runs away. Thus, $c(\text{Prism}^n(\mathcal{H}_0^{(2)})) \geq 2$ and we can conclude the following.

Theorem 4.5. *Let $\mathcal{H}_0^{(2)}$ be a 2-uniform hypergraph. Assume that $\mathcal{H}_0^{(2)}$ is a cop-win hypergraph. Then, $c(\text{Prism}^n(\mathcal{H}_0^{(2)})) = 2$.*

4.2.2 n-Prisms over a Robber-win Hypergraph

To find the cop-number of n-prisms over a robber-win hypergraph, we apply the minimum number of cops needed to win for the hypergraph $\mathcal{H}_0^{(k)}$ to obtain the following

Theorem 4.6. *Let $k \geq 3$ and $n \geq 2$ be integers. Assume that $\mathcal{H}_0^{(k)}$ is a robber-win k-uniform hypergraph having cop-number $c(\mathcal{H}_0^{(k)})$. Then, $c(\text{Prism}^n(\mathcal{H}_0^{(k)})) = c(\mathcal{H}_0^{(k)})$.*

Proof. First, we claim that $c(\mathcal{H}_0^{(k)}) - 1$ cops cannot catch the robber. Without loss of generality, assume that the robber stays on the hypergraph $\mathcal{H}_0^{(k)}$.

Case 1. All cops stay on the hypergraph $\mathcal{H}_0^{(k)}$. Since we need $c(\mathcal{H}_0^{(k)})$ cops to win on the hypergraph $\mathcal{H}_0^{(k)}$, $c(\mathcal{H}_0^{(k)}) - 1$ cops are not enough.

Case 2. Some cops stay on the hypergraph $\mathcal{H}_0^{(k)}$. Since a robber has a winning strategy on the hypergraph $\mathcal{H}_0^{(k)}$ when there are $c(\mathcal{H}_0^{(k)}) - 1$ cops , the robber uses such a strategy to play with the cops in $\mathcal{H}_0^{(k)}$ and the 0^{th} ancester of positions of the remaining cops. Therefore, $c(\mathcal{H}_0^{(k)}) - 1$ cops cannot catch the robber.

By both cases, $c(\mathcal{H}_0^{(k)}) - 1$ cops cannot catch the robber. Thus, $c(\text{Prism}^n(\mathcal{H}_0^{(k)})) \geq c(\mathcal{H}_0^{(k)})$. Next, we claim that $c(\mathcal{H}_0^{(k)})$ cops always catch the robber. Let $m \geq 0$ be a non-negative integer. We define the *stage* m to be the stage where

- $c(\mathcal{H}_0^{(k)})$ cops stay on a hypergraph $\mathcal{H}_m^{(k)}$;
- the robber stays on a vertex R of hypergraph $\mathcal{H}_p^{(k)}$ for some $p \geq m+1$;
- one cop stays on an m^{th} ancestor of a vertex R, say the good cop;
- next, it is the robber's turn.

We use the mathematical induction on m to provide a winning strategy for $c(\mathcal{H}_0^{(k)})$ cops. Starting with the basis step, $c(\mathcal{H}_0^{(k)})$ cops place themselves anywhere on $\mathcal{H}_0^{(k)}$ and the robber place himself anywhere on $\text{Prism}^n(\mathcal{H}_0^{(k)})$, say R. Then, the cops use the winning strategy of $\mathcal{H}_0^{(k)}$, to send one cop stays on a 0^{th} ancestor of a vertex R. Thus, the robber cannot stay in $\mathcal{H}_0^{(k)}$, which causes the robber must stay on $\mathcal{H}_p^{(k)}$ where $p \geq 1$ at the beginning. Now, it is the robber's turn. Therefore, the cops enter the stage 0.

For the induction step, we assume that the cops enter the stage m, that is, $c(\mathcal{H}_0^{(k)})$ cops stay on a hypergraph $\mathcal{H}_m^{(k)}$ and a good cop stays on an m^{th} ancestor of a vertex R and then, it is the robber's turn. Next, we consider how the robber moves. Since $\mathcal{H}_m^{(k)}$ is a copy of a hypergraph $\mathcal{H}_0^{(k)}$, $c(\mathcal{H}_m^{(k)}) = c(\mathcal{H}_0^{(k)})$. If the robber stays on a hypergraph $\mathcal{H}_m^{(k)}$ as same as the cops do, then he is caught. Thus, the robber must stay on a hypergraph $\mathcal{H}_q^{(k)}$ for some $q \geq m+1$. We see that if the robber still stays on a hypergraph $\mathcal{H}_q^{(k)}$ where $q = m+1$, he will be caught in the next turn of a group of cops.

Case 1. Assume that the robber still stay vertex R in $\mathcal{H}_q^{(k)}$ where $q \geq m+2$. Then, a good cop climbs up to stay on the $(m+1)^{th}$ descendant of his previous vertex, which means an $(m+1)^{th}$ ancestor of a vertex R and the rest of the cops also climb up to stay on the $(m+1)^{th}$ descendant of their previous vertices.

Case 2. Assume that the robber move from vertex R to vertex R' in $\mathcal{H}_q^{(k)}$ where $q \geq m+2$. Then, a good cop moves and climbs up from an m^{th} ancestor of a vertex R to an $(m+1)^{th}$ ancestor of a vertex R' and the rest of the cops also climb up to stay on the $(m+1)^{th}$ descendant of their previous vertices.

Case 3. Assume that the robber move from vertex R in $\mathcal{H}_q^{(k)}$ to a $(q+1)^{th}$ descendent of vertex R. Then, a good cop climbs up to stay on the $(m+1)^{th}$

descendant of his previous vertex, which means an $(m+1)^{th}$ ancestor of a vertex R and the rest of the cops also climb up to stay on the $(m+1)^{th}$ descendant of their previous vertices.

Case 4. Assume that the robber move from vertex R in $\mathcal{H}_q^{(k)}$ to a $(q+1)^{th}$ descendent of vertex R'. Then, a good cop moves and climbs up from an m^{th} ancestor of a vertex R to an $(m+1)^{th}$ ancestor of a vertex R' and the rest of the cops also climb up to stay on the $(m+1)^{th}$ descendant of their previous vertices.

Now, the cops enter the stage $m+1$. Thus, the cop can enter the stage $n-1$. We see that the cops need to follow the winning strategy in hypergraph $\mathcal{H}_{n-1}^{(k)}$ for placing one cop stays on the $(n-1)^{th}$ ancestor of the present vertex of the robber. Now, it is the robber's turn. After that, the cops climb up and catch the robber in hypergraph $\mathcal{H}_n^{(k)}$. Hence, $c(\text{Prism}^n(\mathcal{H}_0^{(k)})) = c(\mathcal{H}_0^{(k)})$. $\square$

CHAPTER V

COP-NUMBER OF KNESER GRAPHS

In this chapter, we study the cop-number of Kneser graphs $KG(n,k)$ where n and k are positive integers. This work collaborates with Associate Professor Henry Liu (Sun Yat-sen University, China). We divide into two parts which are (i) $n \geq k^2+k$ and (ii) $2k+1 \leq n \leq k^2+k-1$. Associate Professor Henry Liu showed that $c(KG(n,k)) = k+1$ in the first part and the result in the second part is my study.

5.1 Cop-Number of Kneser Graphs $KG(n,k)$ where $n \geq k^2+k$

A pitfall or a corner vertex in Definition 2.24 plays an important role by giving a good chance to a group of cops when the robber stays at such a vertex. Since each vertex in Kneser graphs is in the form of a set, we would like to extend this idea to determine a similar definition as follows.

Definition 5.1. Let m be a positive integer and $A_1, A_2, A_3, \ldots, A_m \in [n]^{(k)}$. If $X = \cup_{i=1}^{m} A_i$, then X is called a *pitfall cover*.

Next, we can show that there exists another k-set which is disjoint from only one k-set from a given collection of k-sets under some prescribed assumptions and intersects the other k-sets from this collection.

Lemma 5.2. *Let $k \geq 2$, $n \geq k^2+k$ and $l \in [k]$. Let $A_1, A_2, A_3, \ldots, A_l, B \in [n]^{(k)}$ be distinct k-sets such that $B \cap A_1 = \varnothing$. Then, there exists a k-set $C \in [n]^{(k)}$ such that $C \neq A_i$ and $C \cap A_i \neq \varnothing$ for all $i \in [l]$, and $C \cap B = \varnothing$.*

Proof. First, suppose that $l = 1$. Since $|[n] - (A_1 \cup B)| = n - 2k \geq k^2 - k \geq 2$, we may choose $x \in [n] - (A_1 \cup B)$. Choose $y \in A_1$ and set $C = (A_1 - \{y\}) \cup \{x\}$.

Now, suppose that $l \geq 2$. Let $A'_i = A_i - B$ for $i \in [l]$, and note that $A'_1 = A_1$, and $A'_i \neq \varnothing$ for $i \in [l]$. Let $X = \{x_1, x_2, x_3, \ldots, x_p\}$ be a pitfall cover for $\{A'_1, A'_2, A'_3, \ldots, A'_l\}$. Since $A'_1 = A_1$, $p \geq k \geq l$.

Case 1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A'_j \neq \varnothing$ for all $2 \leq j \leq l$. Since $\cup_{j=2}^{l} A'_j \subseteq A_1$ and $|B| = k$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$ and see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $i \in l$ and $C \cap B = \varnothing$.

Case 2. $X \neq A_1$. Let $x_1 \in A_1$. Assume that $x'_i \in A'_i - A_1$ for all $2 \leq i \leq l$. Let M be a pitfall cover for $\{A'_2 - A_1, A'_3 - A_1, A'_4 - A_1, \ldots, A'_l - A_1\}$.

If $|M| = 1$, then $x'_{j_1} = x'_{j_2}$ where $2 \leq j_1 \neq j_2 \leq l$. Thus, we set $C = \{x_1, x'_2, y_1, y_2, \ldots, y_{k-3}, z\}$ where $y_1, y_2, y_3, \ldots, y_{k-3} \in A_1 - \{x_1\}$ and $z \in ([n] - \{x'_2\}) - (A_1 \cup B)$.

If $2 \leq |M| = m < l - 1$, then without loss of generality, assume that $M = \{x'_2, x'_3, x'_4, \ \ldots, x'_{m+1}\}$, we set $C = \{x_1, x'_2, x'_3, \ldots, x'_{m+1}, y_1, \ldots, y_{k-(m+1)}\}$ where $y_1, y_2, y_3, \ldots, y_{k-(m+1)} \in A_1 - \{x_1\}$.

If $|M| = l - 1$, then we set $C = \{x_1, x'_2, x'_3, \ldots, x'_l, y_1, \ldots, y_{k-l}\}$ where $y_1, y_2, y_3, \ldots, y_{k-l} \in A_1 - \{x_1\}$. We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $i \in l$ and $C \cap B = \varnothing$. $\square$

By Lemma 5.2, we obtain that a group of cops stay at vertices $\{A_i\}$, the robber stays at a vertex B and then the robber moves from a vertex B to a vertex C in his next turn. Therefore, we obtain the following theorem.

Theorem 5.3. *Let $k \geq 2$ and $n \geq k^2 + k$. Then, $c(KG(n, k)) = k + 1$.*

Proof. By Proposition 2.30 and Theorem 2.35, we have $c(KG(n, k)) \leq \gamma(KG(n, k))$ $= k + 1$. To prove the lower bound $c(KG(n, k)) \geq k + 1$, we show that if there are only k cops, then the robber has a winning strategy. To evade the capture, the robber maintains a distance of two from every cops. Note that by Theorem 2.28, we have $\operatorname{diam}(KG(n, k)) = 2$.

Initially, suppose that k cops occupy the distinct vertices $A_1, A_2, A_3, \ldots, A_r$ for some $r \in [k]$ (i.e., some vertex A_i may contain more than one cop). The robber chooses a vertex B as follows. Let $X \in V(KG(n,k)) - \{A_1, A_2, A_3, \ldots, A_r\}$. If $X \cap A_i \neq \varnothing$ for all $i \in [r]$, then let $B = X$. Otherwise, if say $X \cap A_1 = \varnothing$, then by Lemma 5.2, there exists a vertex $Y \in V(KG(n,k)) - \{A_1, A_2, A_3, \ldots, A_r\}$ such that $Y \cap A_i \neq \varnothing$ for all $i \in [r]$ and we let $B = Y$. Then, the distance between a vertex B and a vertex A_i, $d(B, A_i)$, is 2 for all $i \in [r]$.

Subsequently, the pursuit proceeds similarly. Suppose that at some stage, the robber is at distance two from every cops. The cops then make their move and they cannot capture the robber. Suppose that k cops occupy the distinct vertices $U_1, U_2, U_3, \ldots, U_s$ for some $s \in [k]$ and the robber is at another vertex V. If $d(V, U_i) = 2$, then the robber remains at V and the pursuit continues. Otherwise, say $d(V, U_1) = 1$. Thus, $V \cap U_1 = \varnothing$. By Lemma 5.2, there exists a vertex W such that $W \neq U_i$, $W \cap U_i \neq \varnothing$ for all $i \in [s]$ and $W \cap V = \varnothing$. Then, the robber may move from V to W and $d(W, U_i) = 2$ for all $i \in [s]$. This process then continues and the robber can always evade the capture. □

5.2 Cop-Number of Kneser Graphs $KG(n,k)$ where $2k+1 \leq n \leq k^2+k-1$

To investigate the cop-number of $KG(n,k)$, there are two steps in the proof. First, we would like to show that $k+1$ cops are enough to win this game for $k \geq 3$ is a positive integer. Second, we prove that the robber can escape when it has exactly k cops for $k \geq 3$ is a positive integer.

We need some lemmas to determine the existence of a k-set that is disjoint from two distinct given k-sets as follows.

Lemma 5.4. *Let $k \geq 3$, $n \geq 2k+2$. Let $A, B \in [n]^{(k)}$ be two distinct k-sets such that $|A \cap B| \geq k-2$. Then, there exists a k-set $C \in [n]^{(k)}$ such that $C \cap A = \varnothing$ and $C \cap B = \varnothing$.*

Proof. Let $k \geq 3$, $n \geq 2k+2$. Let $A, B \in [n]^{(k)}$ be two distinct k-sets such that

$|A \cap B| \geq k-2$. We have $|A \cup B| \leq k+2$. Since $n \geq 2k+2$, $|[n]-(A \cup B)| = n - |A \cup B| \geq k$. Then, we can find a k-set $C \in [n]^{(k)}$ such that $C \cap A = \varnothing$ and $C \cap B = \varnothing$. $\square$

Lemma 5.5. *Let $k \geq 3$, $n = 2k+1$. Let $A, B \in [n]^{(k)}$ be two distinct k-sets such that $|A \cap B| \geq k-1$. Then, there exists a k-set $C \in [n]^{(k)}$ such that $C \cap A = \varnothing$ and $C \cap B = \varnothing$.*

Proof. Let $k \geq 3$, $n = 2k+1$. Let $A, B \in [n]^{(k)}$ be two distinct k-sets such that $|A \cap B| \geq k-1$. We have $|A \cup B| \leq k+1$. Since $n = 2k+1$, $|[n]-(A \cup B)| = n - |A \cup B| \geq k$. Then, we can find a k-set $C \in [n]^{(k)}$ such that $C \cap A = \varnothing$ and $C \cap B = \varnothing$. $\square$

By Lemmas 5.4 and 5.5, we let one of the cops occupies at the vertex A and a robber stays at the vertex B. We see that the robber needs to take a move every turn.

For a given collection of $k+2$ distinct k-sets under some prescribed assumptions, we show that it has no k-sets in which they intersect the $k+1$ k-sets from this collection and are disjoint from the remaining one.

Lemma 5.6. *Let $k \geq 3$ and $2k+1 \leq n \leq k^2+k-1$. Let $\{A_i\}_{i=1}^{k+1} \subseteq [n]^{(k)}$ and $B \in [n]^{(k)}$ be distinct k-sets. If $\{A_i\}_{i=1}^{k+1}$ has properties that $A_1 \cap B = \varnothing$, $|A_j \cap B| = k-1$, $|A_j \cap A_1| = 1$ for all $2 \leq j \leq k+1$ and $A_{j_1} \cap A_1 \neq A_{j_2} \cap A_1$ where $j_1 \neq j_2$ and $2 \leq j_1, j_2 \leq k+1$, then there are no k-sets $C \neq A_i$ such that $C \cap B = \varnothing$ and $C \cap A_i \neq \varnothing$ for all $i \in [k+1]$.*

Proof. Let $k \geq 3$ and $2k+1 \leq n \leq k^2+k-1$. Let $\{A_i\}_{i=1}^{k+1} \subseteq [n]^{(k)}$ and $B \in [n]^{(k)}$ be distinct k-sets. Assume that there exist $k+1$ A_i's such that $A_1 \cap B = \varnothing$, $|A_j \cap B| = k-1$, $|A_j \cap A_1| = 1$ for all $j \neq 1$ and $A_{j_1} \cap A_1 \neq A_{j_2} \cap A_1$ where $j_1 \neq j_2$.

Suppose that there is a k-set $C \neq A_i$ such that $C \cap B = \varnothing$ and $C \cap A_i \neq \varnothing$ for all $i \in [k+1]$. Since $|A_j \cap B| = k-1$, we have $|A_j - B| = 1$. Then, without loss of generality, assume that $A_j - B = \{j\}$ for all $2 \leq j \leq k+1$. Since $A_j - B = \{j\}$ for all $2 \leq j \leq k+1$, we have $C = \cup_{j=2}^{k+1}\{j\}$. Since $A_1 \cap B = \varnothing$, $|A_j \cap A_1| = 1$ for all $j \neq 1$ and $A_{j_1} \cap A_1 \neq A_{j_2} \cap A_1$ where $j_1 \neq j_2$, $A_1 = C$, which is a contradiction. $\square$

Therefore, we let the $k+1$ k-sets $\{A_i\}_{i=1}^{k+1}$ to be positions of each $k+1$ cops and the other k-set B to be the robber's position. Then, the nonexistence of a k-set C from Lemma 5.6 guarantees that the robber has no free neighbors. Now, we would like to show that $k+1$ cops is enough to win the game. First of all, we need to define the notation and one important cop using in the algorithm of choosing vertices in the next turn for each cop.

Definition 5.7. Let i and j be integers such that $i \in [k+1]$ and $j \geq 0$. Let $A_i^{(j)}$ denote a vertex which the i^{th} cop stays in the j^{th} turn and $B^{(j)}$ denote a vertex which a robber stays in the j^{th} turn.

Definition 5.8. On the graph $KG(n,k)$, let the i^{th} cop stays at A_i for each $i \in [k+1]$ and the robber stays at B. If $|A_i \cap B|$ is maximized, then we call the i^{th} cop a *guarding cop*.

Next, we determine an algorithm of choosing vertices in the next turn for each cop.

Our Inductive Choosing. Let $A_i^{(0)}$ denote a vertex for which the i^{th} cop stays at the beginning of the game. Without loss of generality, we let $A_1^{(0)}$ be a vertex of a guarding cop. Next, we define $A_2^{(0)}, A_3^{(0)}, A_4^{(0)}, \ldots, A_{k+1}^{(0)}$ such that $|A_l^{(0)} \cap B^{(0)}| \leq |A_{l+1}^{(0)} \cap B^{(0)}|$ for all $2 \leq l \leq k+1$. We let **MIN** be a condition that for a fixed $B^{(r)}$, $\sum_{l=2}^{k+1}\sum_{l'=l+1}^{k+1} |(A_l^{(r+1)} - B^{(r)}) \cap (A_{l'}^{(r+1)} - B^{(r)})|$ is minimized.

Basis Step. Let $2 \leq l \leq k+1$ be an integer. In the first turn, we let the l^{th} cop moves from $A_l^{(0)}$ to $A_l^{(1)}$ such that

(i) $A_l^{(1)} = (B^{(0)} - A_l^{(0)}) \cup C$ where $C \subseteq [n] - (B^{(0)} \cup A_l^{(0)})$ and $|C| = k - |B^{(0)} - A_l^{(0)}|$; and

(ii) $A_l^{(1)}$ satisfies $A_l^{(1)} \neq A_l^{(0)}$, $A_l^{(1)} \cap A_l^{(0)} = \varnothing$ and $(A_l^{(1)} - B^{(0)}) \cap A_1^{(0)} = \varnothing$; and

(iii) Check **MIN**.

By Lemma 5.4, we always let the 1^{st} cop moves from $A_1^{(0)}$ to $A_1^{(1)}$ such that $A_1^{(1)} \neq A_1^{(0)}$, $A_1^{(1)} \cap A_1^{(0)} = \varnothing$, $A_1^{(1)} \cap B^{(0)} = \varnothing$ and $|A_1^{(1)} \cap A_l^{(1)}| \geq 1$ for all $l \neq 1$. Then, the robber can move from $B^{(0)}$ to $B^{(1)}$, where $B^{(1)} \cap B^{(0)} = \varnothing$ and $B^{(1)} \cap A_i^{(1)} \neq \varnothing$ for all $i \in [k+1]$.

Remark 2. We give the example of how we choose the vertices when $k = 3$ and $n = 8$, we can choose vertices $\{1,2,3\}, \{1,2,5\}, \{3,4,5\}$ and $\{6,7,8\}$ as the four cops' beginning positions.

If the robber chooses $\{1,3,8\}$ as his beginning vertex, then the 1^{st} cop starts with a vertex $A_1^{(0)} = \{1,2,3\}$. We see that $\{1,2,3\}$ and $\{1,3,8\}$ satisfy Lemma 5.4. Next, the $2^{nd}, 3^{rd}$ and 4^{th} cops start with vertices $A_2^{(0)} = \{6,7,8\}, A_3^{(0)} = \{3,4,5\}$ and $A_4^{(0)} = \{1,2,5\}$, respectively.

In the first turn, we begin with a choosing vertex for the 2^{nd} cop by following the basis step. We let $A_l^{(1)} = (B^{(0)} - A_l^{(0)}) \cup C$ where $C \subseteq [n] - (B^{(0)} \cup A_l^{(0)})$ and $|C| = k - |B^{(0)} - A_l^{(0)}|$.

Since $A_2^{(0)} = \{6,7,8\}$ and $B^{(0)} = \{1,3,8\}$, we have $B^{(0)} - A_2^{(0)} = \{1,3\}$. Next, we choose 5 from $[n] - (B^{(0)} \cup A_2^{(0)}) = \{2,4,5\}$. Therefore, the 2^{nd} cop moves to the vertex $\{1,3,5\}$.

Next, we consider how to choose vertices for 3^{rd} and 4^{th} cops by the same idea as choosing for the 2^{nd} cop.

Since $A_3^{(0)} = \{3,4,5\}$ and $B^{(0)} = \{1,3,8\}$, we have $B^{(0)} - A_3^{(0)} = \{1,8\}$. Next, we choose 6 from $[n] - (B^{(0)} \cup A_3^{(0)}) = \{2,6,7\}$. Therefore, the 3^{rd} cop moves to the vertex $\{1,8,6\}$.

Since $A_4^{(0)} = \{1,2,5\}$ and $B^{(0)} = \{1,3,8\}$, we have $B^{(0)} - A_4^{(0)} = \{3,8\}$. Next, we choose 4 from $[n] - (B^{(0)} \cup A_4^{(0)}) = \{4,6,7\}$. Therefore, the 4^{th} cop moves to the vertex $\{3,8,4\}$.

We see that $A_l^{(1)}$ satisfies $A_l^{(1)} \neq A_l^{(0)}$, $A_l^{(1)} \cap A_l^{(0)} = \varnothing$ and $(A_l^{(1)} - B^{(0)}) \cap A_1^{(0)} = \varnothing$ for all $2 \leq l \leq 4$ and $\{A_l^{(1)}\}_{l=2}^{4}$ satisfies **MIN**.

We obtain that the $2^{nd}, 3^{rd}$ and 4^{th} cops move from vertices $A_2^{(0)} = \{6,7,8\}$, $A_3^{(0)} = \{3,4,5\}$ and $A_4^{(0)} = \{1,2,5\}$ to vertices $A_2^{(1)} = \{1,3,5\}, A_3^{(1)} = \{1,8,6\}$ and $A_4^{(1)} = \{3,8,4\}$, respectively. Next, we let the 1^{st} cop moves from $A_1^{(0)} = \{1,2,3\}$ to $A_1^{(1)} = \{4,5,6\}$.

We see that $A_1^{(1)} \neq A_1^{(0)}$, $A_1^{(1)} \cap A_1^{(0)} = \varnothing$, $A_1^{(1)} \cap B^{(0)} = \varnothing$ and $|A_1^{(1)} \cap A_l^{(1)}| \geq 1$ for all $l \neq 1$.

By Remark 2, we can guarantee that there exist positions for each four cops in the first turn of the game on $KG(8,3)$. Then, we continue this process for the other cases depending on n and k. Note that actually, the chosen numbers in this remark can be changed. However, one need to check that they satisfy **MIN**.

For $r \geq 2$, we let **GC** be a condition that $A_1^{(r)} \neq A_1^{(r-1)}$, $A_1^{(r)} \cap A_1^{(r-1)} = \varnothing$, $A_1^{(r)} \cap B^{(r-1)} = \varnothing$ and $|A_1^{(r)} \cap A_l^{(r)}| \geq 1$ for all $l \neq 1$.

Induction Step. Let $r \geq 1$ be an integer. In the $(r+1)^{th}$ turn, we choose vertex $A_i^{(r+1)}$ for the i^{th} cop by an algorithm of choosing vertices in the next turn for each cop in Figure 5.1

For more detail, we would like to give the example of how we use the algorithm of choosing vertices when $k = 3$ and $n = 8$.

Example 5.9. We choose vertices $\{1,2,3\}, \{1,2,5\}, \{3,4,5\}$ and $\{6,7,8\}$ as the four cops' beginning positions and the robber chooses $\{1,3,8\}$ as his beginning vertex.

We have $A_1^{(0)} = \{1,2,3\}, A_2^{(0)} = \{6,7,8\}, A_3^{(0)} = \{3,4,5\}, A_4^{(0)} = \{1,2,5\}$ and $B^{(0)} = \{1,3,8\}$.

In the first turn, each cop move from $A_1^{(0)}, A_2^{(0)}, A_3^{(0)}, A_4^{(0)}$ to $A_1^{(1)} = \{4,5,6\}$, $A_2^{(1)} = \{1,3,5\}, A_3^{(1)} = \{1,8,6\}, A_4^{(1)} = \{3,8,4\}$, respectively, see Remark 2. Then, we need to check that $\{A_i^{(1)}\}_{i=1}^{4}$ and $B^{(0)}$ satisfy Lemma 5.6 and found that they satisfy Lemma 5.6. Thus, the robber staying on vertex $B^{(0)}$ must be caught in the next turn of a group of cops.

Next, we give the example of how we use the algorithm of choosing vertices when $k = 4$ and $n = 10$.

Example 5.10. We choose vertices $\{1,2,3,4\}, \{1,2,4,5\}, \{1,2,5,6\}, \{3,4,5,6\}$ and $\{7,8,9,10\}$ as the five cops' beginning positions and the robber chooses $\{1,2,3,8\}$ as his beginning vertex.

Algorithm 1: How to choose vertices in the $(r+1)^{th}$ turn for $k+1$ cops

Result: $\{A_i^{(r)}\}_{i=1}^{k+1}$ and $B^{(r-1)}$ satisfy Lemma 5.6

$k \geq 3$, $2k+1 \leq n \leq k^2+k-1$, $2 \leq l \leq k+1$, $r \geq 1$, $\{A_i^{(r)}\}_{i=1}^{k+1} \subseteq [n]^{(k)}, B^{(r)} \in [n]^{(k)}$;

while $\{A_i^{(r)}\}_{i=1}^{k+1}$ *and* $B^{(r-1)}$ *does not satisfy Lemma 5.6* **do**

  compare $\sum_{l\geq 2}(k-|B^{(r)}-A_l^{(r)}|)$ and $\sum_{l\geq 2}(k-|B^{(r-1)}-A_l^{(r-1)}|)$;

  if $\sum_{l\geq 2}(k-|B^{(r)}-A_l^{(r)}|) < \sum_{l\geq 2}(k-|B^{(r-1)}-A_l^{(r-1)}|)$ **then**

    1. $A_l^{(r+1)} = (B^{(r)}-A_l^{(r)}) \cup C$ where $C \subseteq B^{(r-1)}-A_l^{(r)}$ and $|C| = k-|B^{(r)}-A_l^{(r)}|$;
    2. $A_1^{(r+1)} = B^{(r-1)}$;
    3. Check **MIN** and **GC**

  else

    if *there exist* l' *such that* $|B^{(r)}-A_{l'}^{(r)}| \neq k-1$ *and* $|B^{(r)}-A_{l'}^{(r)}| = |B^{(r-1)}-A_{l'}^{(r-1)}|$ **then**

      1. $A_{l'}^{(r+1)} = (B^{(r)}-A_{l'}^{(r)}) \cup C \cup \{y\}$ where $C \subseteq B^{(r-1)}-A_{l'}^{(r)}, |C| = k-|B^{(r)}-A_{l'}^{(r)}|-1$ and $y \in ([n]-B^{(r)})-(A_{l'}^{(r)} \cup B^{(r-1)})$;
      2. $A_l^{(r+1)} = (B^{(r)}-A_l^{(r)}) \cup C$ where $C \subseteq B^{(r-1)}-A_l^{(r)}, |C| = k-|B^{(r)}-A_l^{(r)}|$ and $l \neq l'$;
      3. $A_1^{(r+1)} = (B^{(r-1)}-\{x\}) \cup \{y\}$ where $x \in (B^{(r-1)}-A_{l'}^{(r+1)}) - \cup_{l\neq l'} A_l^{(r+1)}$;
      4. Check **MIN** and **GC**

    else

      if $|B^{(r)}-A_l^{(r)}| = k-1$ *for all* $2 \leq l \leq k+1$ *and there exists* $l_2 > l_1$ *such that* $B^{(r-1)}-A_{l_1}^{(r)} = B^{(r-1)}-A_{l_2}^{(r)}$ **then**

        1. $A_{l_2}^{(r+1)} = (B^{(r)}-A_{l_2}^{(r)}) \cup \{y\}$ where $y \in ([n]-B^{(r)})-(A_{l_2}^{(r)} \cup B^{(r-1)})$;
        2. $A_l^{(r+1)} = (B^{(r)}-A_l^{(r)}) \cup C$ where $C \subseteq B^{(r-1)}-A_l^{(r)}, |C| = k-|B^{(r)}-A_l^{(r)}|$ and $l \neq l_2$;
        3. $A_1^{(r+1)} = (B^{(r-1)}-\{x\}) \cup \{y\}$ where $x \in (B^{(r-1)}-A_{l_2}^{(r+1)}) - \cup_{l\neq l_2} A_l^{(r+1)}$;
        4. Check **MIN** and **GC**

      else

        1. $A_l^{(r+1)} = (B^{(r)}-A_l^{(r)}) \cup C$ where $C \subseteq B^{(r-1)}-A_l^{(r)}$ and $|C| = k-|B^{(r)}-A_l^{(r)}|$;
        2. $A_1^{(r+1)} = B^{(r-1)}$;
        3. Check **MIN** and **GC**

      end

    end

  end

end

Figure 5.1: An algorithm of choosing vertices in the next turn for each cop

The 1^{st} cop starts with a vertex $A_1^{(0)} = \{1,2,3,4\}$. We see that $\{1,2,3,4\}$ and $\{1,2,3,8\}$ satisfy Lemma 5.4. Next, the $2^{nd}, 3^{rd}, 4^{th}$ and 5^{th} cops start with vertices $A_2^{(0)} = \{7,8,9,10\}, A_3^{(0)} = \{3,4,5,6\}, A_4^{(0)} = \{1,2,5,6\}$ and $A_5^{(0)} = \{1,2,4,5\}$, respectively.

In the first turn, we begin with a choosing vertex for the 2^{nd} cop by following the basis step. We let $A_l^{(1)} = (B^{(0)} - A_l^{(0)}) \cup C$ where $C \subseteq [n] - (B^{(0)} \cup A_l^{(0)})$ and $|C| = k - |B^{(0)} - A_l^{(0)}|$.

Since $A_2^{(0)} = \{7,8,9,10\}$ and $B^{(0)} = \{1,2,3,8\}$, we have $B^{(0)} - A_2^{(0)} = \{1,2,3\}$. Next, we choose 5 from $[n] - (B^{(0)} \cup A_2^{(0)}) = \{4,5,6\}$. Therefore, the 2^{nd} cop moves to the vertex $\{1,2,3,5\}$.

Next, we consider how to choose vertices for $3^{rd}, 4^{th}$ and 5^{th} cops by the same idea as choosing for the 2^{nd} cop.

Since $A_3^{(0)} = \{3,4,5,6\}$ and $B^{(0)} = \{1,2,3,8\}$, we have $B^{(0)} - A_3^{(0)} = \{1,2,8\}$. Next, we choose 10 from $[n] - (B^{(0)} \cup A_3^{(0)}) = \{7,9,10\}$. Therefore, the 3^{rd} cop moves to the vertex $\{1,2,8,10\}$.

Since $A_4^{(0)} = \{1,2,5,6\}$ and $B^{(0)} = \{1,2,3,8\}$, we have $B^{(0)} - A_4^{(0)} = \{3,8\}$. Next, we choose 4 and 9 from $[n] - (B^{(0)} \cup A_4^{(0)}) = \{4,7,9,10\}$. Therefore, the 4^{th} cop moves to the vertex $\{3,8,4,9\}$.

Since $A_5^{(0)} = \{1,2,4,5\}$ and $B^{(0)} = \{1,2,3,8\}$, we have $B^{(0)} - A_4^{(0)} = \{3,8\}$. Next, we choose 6 and 7 from $[n] - (B^{(0)} \cup A_4^{(0)}) = \{6,7,9,10\}$. Therefore, the 4^{th} cop moves to the vertex $\{3,8,6,7\}$.

We see that $A_l^{(1)}$ satisfies $A_l^{(1)} \neq A_l^{(0)}$, $A_l^{(1)} \cap A_l^{(0)} = \varnothing$ and $(A_l^{(1)} - B^{(0)}) \cap A_1^{(0)} = \varnothing$ for all $2 \leq l \leq 5$ and $\{A_l^{(1)}\}_{l=2}^5$ satisfies **MIN**.

We obtain that the $2^{nd}, 3^{rd}, 4^{th}$ and 5^{th} cops move from vertices $A_2^{(0)} = \{7,8,9,10\}$, $A_3^{(0)} = \{3,4,5,6\}, A_4^{(0)} = \{1,2,5,6\}$ and $A_5^{(0)} = \{1,2,4,5\}$ to vertices $A_2^{(1)} = \{1,2,3,5\}, A_3^{(1)} = \{1,2,8,10\}, A_4^{(1)} = \{3,8,4,9\}$ and $A_5^{(1)} = \{3,8,6,7\}$, respectively. Next, we let the 1^{st} cop moves from $A_1^{(0)} = \{1,2,3,4\}$ to $A_1^{(1)} = \{5,6,9,10\}$.

We see that $A_1^{(1)} \neq A_1^{(0)}$, $A_1^{(1)} \cap A_1^{(0)} = \varnothing$, $A_1^{(1)} \cap B^{(0)} = \varnothing$ and $|A_1^{(1)} \cap A_l^{(1)}| \geq 1$ for all $l \neq 1$. Then, we need to check that $\{A_i^{(1)}\}_{i=1}^5$ and $B^{(0)}$ satisfy Lemma 5.6 and found that they does not satisfy Lemma 5.6. Thus, the robber can find free neigbors to stay in his next turn.

If the robber chooses $\{4,5,7,10\}$ as his next position, then we compare $\sum_{l\geq 2}(k-|B^{(1)}-A_l^{(1)}|)$ and $\sum_{l\geq 2}(k-|B^{(0)}-A_l^{(0)}|)$. Then, we obtain that $\sum_{l\geq 2}(k-|B^{(1)}-A_l^{(1)}|)=4$ and $\sum_{l\geq 2}(k-|B^{(0)}-A_l^{(0)}|)=6$. By our algorithm of choosing, we have $\sum_{l\geq 2}(k-|B^{(1)}-A_l^{(1)}|)<\sum_{l\geq 2}(k-|B^{(0)}-A_l^{(0)}|)$. Then, we let $A_l^{(r+1)}=(B^{(r)}-A_l^{(r)})\cup C$ where $C\subseteq[n]-(B^{(0)}\cup A_l^{(0)})$ and $|C|=k-|B^{(0)}-A_l^{(0)}|$ and $A_1^{(r+1)}=B^{(r-1)}$.

Therefore, $A_2^{(2)}=\{4,7,10\}\cup\{8\}=\{4,7,10,8\}, A_3^{(2)}=\{4,5,7\}\cup\{3\}=\{4,5,7,3\}, A_4^{(2)}=\{5,7,10\}\cup\{1\}=\{5,7,10,1\}, A_5^{(2)}=\{4,7,10\}\cup\{2\}=\{4,7,10,2\}$ and $A_1^{(3)}=B^{(0)}=\{1,2,3,8\}$. We see that $A_l^{(2)}$ for all $l\neq 1$ satisfies **MIN** and $A_1^{(2)}$ satisfies **GC**.

Thus, we need to check that $\{A_i^{(2)}\}_{i=1}^{5}$ and $B^{(1)}$ satisfy Lemma 5.6 and found that they satisfy Lemma 5.6. Hence, the robber staying on vertex $B^{(1)}$ must be caught in the next turn of a group of cops.

By Example 5.9 and 5.10, we can guarantee that the group of cops wins by these strategies. Then, we apply an algorithm of choosing vertices to the other cases depending on n,k and the positions of robber.

By following our algorithm of choosing vertices, we need to show that a game of cops and robbers is terminated at some point where $\{A_i\}_{i=1}^{k+1}$ and B satisfies Lemma 5.6 as follows.

Lemma 5.11. *By the algorithm of choosing vertices, there exists a turn f^{th} such that $\{A_i^{(f)}\}_{i=1}^{k+1}$ and $B^{(f-1)}$ satisfy Lemma 5.6.*

Proof. Assume that $\{A_i^{(r)}\}_{i=1}^{k+1}$ and $B^{(r-1)}$ does not satisfy Lemma 5.6 where $r\geq 1$. We consider each condition in our algorithm of choosing vertices.

Case I. If $\sum_{l\geq 2}(k-|B^{(r)}-A_l^{(r)}|)=\sum_{l\geq 2}(k-|B^{(r-1)}-A_l^{(r-1)}|)$, there exist l' such that $|B^{(r)}-A_{l'}^{(r)}|\neq k-1$ and $|B^{(r)}-A_{l'}^{(r)}|=|B^{(r-1)}-A_{l'}^{(r-1)}|$, then we let

- $A_{l'}^{(r+1)}=(B^{(r)}-A_{l'}^{(r)})\cup C\cup\{y\}$ where $C\subseteq[n]-(B^{(0)}\cup A_{l'}^{(0)}), |C|=k-|B^{(0)}-A_{l'}^{(0)}|-1$ and $y\in([n]-B^{(r)})-(A_{l'}^{(r)}\cup B^{(r-1)})$; and

- $A_1^{(r+1)} = (B^{(r-1)} - \{x\}) \cup \{y\}$ where $x \in (B^{(r-1)} - A_{l'}^{(r+1)}) - \cup_{l \neq l'} A_l^{(r+1)}$.

We claim that $\sum_{l\geq 2}(k - |B^{(r+1)} - A_l^{(r+1)}|) < \sum_{l\geq 2}(k - |B^{(r)} - A_l^{(r)}|)$.

Case 1. $y \notin B^{(r+1)}$. We have $|B^{(r+1)} - A_{l'}^{(r+1)}| > |B^{(r)} - A_{l'}^{(r)}|$, which implies that $k - |B^{(r+1)} - A_{l'}^{(r+1)}| > k - |B^{(r)} - A_{l'}^{(r)}|$. Then, $\sum_{l\geq 2}(k - |B^{(r+1)} - A_l^{(r+1)}|) < \sum_{l\geq 2}(k - |B^{(r)} - A_l^{(r)}|)$.

Case 2. $y \in B^{(r+1)}$. Since $|B^{(r)} - A_{l'}^{(r)}| \neq k-1$, we have $|B^{(r)} - A_{l'}^{(r)}| \leq k-2$, so $k - |B^{(r)} - A_{l'}^{(r)}| \geq 2$. Thus, there are at least one element other than y in $A_{l'}^{(r+1)} - B^{(r)}$, says z.

Case 2.1. $z \notin B^{(r+1)}$. We have $|B^{(r+1)} - A_{l'}^{(r+1)}| > |B^{(r)} - A_{l'}^{(r)}|$ which is same as Case 1.

Case 2.2. $z \in B^{(r+1)}$. Since $B^{(r+1)} \neq A_1^{(r+1)}$, we have $|A_1^{(r+1)} - B^{(r+1)}| \geq 1$. Let $w \in A_1^{(r+1)} - B^{(r+1)}$. Since $|A_1^{(r+1)} \cap A_j^{(r+1)}| \geq 1$ for all $2 \leq j \leq k+1$, there exists j' such that $w \in A_{j'}^{(r+1)}$. Then, $|B^{(r+1)} - A_{j'}^{(r+1)}| > |B^{(r)} - A_{j'}^{(r)}|$. Thus, $\sum_{l\geq 2}(k - |B^{(r+1)} - A_l^{(r+1)}|) < \sum_{l\geq 2}(k - |B^{(r)} - A_l^{(r)}|)$, which is the following case.

Case II. If $\sum_{l\geq 2}(k - |B^{(r)} - A_l^{(r)}|) < \sum_{l\geq 2}(k - |B^{(r-1)} - A_l^{(r-1)}|)$, then by our algorithm, for $2 \leq l \leq k+1$, $A_l^{(r+1)} - (B^{(r)} - A_l^{(r)}) \subseteq B^{(r-1)} - A_l^{(r)}$. Thus, $|A_l^{(r+1)} - (B^{(r)} - A_l^{(r)})| \leq |B^{(r-1)} - A_l^{(r)}|$. Since we know that $|A_l^{(r+1)} - B^{(r)}| < |A_l^{(r+1)} - (B^{(r)} - A_l^{(r)})|$, we have $|A_l^{(r+1)} - B^{(r)}| < |B^{(r-1)} - A_l^{(r)}|$. Thus, $|A_l^{(r+1)} \cap B^{(r)}| > |A_l^{(r)} \cap B^{(r-1)}|$ for $2 \leq l \leq k+1$.

Since each set is of size k, we obtain that, for $2 \leq l \leq k+1$, there exists a turn r_l^{th} such that $|A_l^{(r_l+1)} \cap B^{(r_l)}| = k-1$. By continue this process, we may have $|A_l^{(t)} \cap B^{(t-1)}| = k-1$ for all $2 \leq l \leq k+1$ at some turn t^{th}. When we consider Case I, we then obtain Case II in the next turn. Thus, we can conclude the same result. Next, we remain to show that there exists a turn f^{th} such that $A_{j_1}^{(f)} \cap A_1^{(f)} \neq A_{j_2}^{(f)} \cap A_1^{(f)}$ where $2 \leq j_1 \neq j_2 \leq k+1$.

Since $|B^{(t)} - A_l^{(t)}| = |A_l^{(t+1)} \cap B^{(t)}|$ and $|B^{(t+1)} - A_l^{(t+1)}| \geq |B^{(t)} - A_l^{(t)}|$, we have $|B^{(t+1)} - A_l^{(t+1)}| = k-1$ for all $2 \leq l \leq k+1$. It means that $\sum_{l\geq 2}(k - |B^{(t)} - A_l^{(t)}|) = k = \sum_{l\geq 2}(k - |B^{(t-1)} - A_l^{(t-1)}|)$. If $B^{(t)} - A_{j_1}^{(t+1)} \neq B^{(t)} - A_{j_2}^{(t+1)}$ for all $2 \leq j_1 \neq j_2 \leq k+1$, we let

- $A_1^{(t+2)} = B^{(t)}$ and
- $A_j^{(t+2)} = (B^{(t+1)} - A_j^{(t+1)}) \cup C$ where $C \subseteq B^{(t)} - A_j^{(t+1)}$ and $|C| = k - |B^{(t+1)} - A_j^{(t+1)}|$ for all $2 \leq j \leq k+1$.

Therefore, $A_{j_1}^{(t+2)} \cap A_1^{(t+2)} \neq A_{j_2}^{(t+2)} \cap A_1^{(t+2)}$ where $2 \leq j_1 \neq j_2 \leq k+1$.

If there exists $j_2 > j_1$ such that $B^{(t)} - A_{j_1}^{(t+1)} = B^{(t)} - A_{j_2}^{(t+1)}$, then we let

- $A_{j_2}^{(t+2)} = (B^{(t+1)} - A_{j_2}^{(t+1)}) \cup \{y\}$ where $y \in ([n] - B^{(t+1)}) - (A_{j_2}^{(t+1)} \cup B^{(t)})$;
- $A_j^{(t+2)} = (B^{(t+1)} - A_j^{(t+1)}) \cup C$ where $C \subseteq B^{(t)} - A_j^{(t+1)}, |C| = k - |B^{(t+1)} - A_j^{(t+1)}|$ and $j \neq j_2$; and
- $A_1^{(t+2)} = (B^{(t)} - \{x\}) \cup \{y\}$ where $x \in (B^{(t)} - A_{j_2}^{(t+2)}) - \cup_{j \neq j_2} A_j^{(t+2)}$

Therefore, $A_{j_1}^{(t+2)} \cap A_1^{(t+2)} \neq A_{j_2}^{(t+2)} \cap A_1^{(t+2)}$. Then, $A_{l_1}^{(t+2)} \cap A_1^{(t+2)} \neq A_{l_2}^{(t+2)} \cap A_1^{(t+2)}$ where $2 \leq l_1 \neq l_2 \leq k+1$.

From case I and case II, since $|A_j^{(t+2)} \cap A_1^{(t+2)}| = k - |B^{(t+2)} - A_j^{(t+2)}|$, we have $|A_j^{(t+2)} \cap A_1^{(t+2)}| = 1$ for all $2 \leq j \leq k+1$. In our consideration, each choosing step needs to satisfy **MIN** and **GC**. Therefore, $A_{l_1}^{(t+2)} \cap A_1^{(t+2)} \neq A_{l_2}^{(t+2)} \cap A_1^{(t+2)}$ where $2 \leq l_1 \neq l_2 \leq k+1$. Now, we obtain that $\{A_i^{(t+2)}\}_{i=1}^{k+1}$ and $B^{(t+1)}$ satisfy Lemma 5.6. Then, we will choose $f = t+2$. □

By the previous lemma, we can guarantee that our algorithm of choosing vertices leads us to obtain a final turn where the positions of the $k+1$ cops satisfy Lemma 5.6. It means that a group of $k+1$ cops can guard all robber's neighbors, which causes robber cannot run away. In our algorithm, we need to find a guarding cop $A_1^{(0)}$ which need to satisfies Lemma 5.4.

Note that, thereafter, we write $\{a, b, c\}$ as abc and write $\overline{p}$ to represent p where p is a positive integer and $p \geq 10$.

Lemma 5.12. *There exists a guarding cop on $KG(n,k)$ where $3 \leq k \leq 5$ and $2k+2 \leq n \leq k^2+k-1$.*

Proof. When $k = 3$, let $123, 125, 345$ and 678 be the starting vertices for four cops. We claim that there exists a guarding cop which satisfies Lemma 5.4. We consider all possible vertex for a robber. We separate into three generating sets; that is, $P = \{1, 2, 4\}, Q = \{3, 5\}$ and $R = \{6, 7, 8\}$. Let (p, q, r) denote the number of chosen elements of the position of the robber from P, Q and R, respectively.

Case 1. $(0, 2, 1)$. We choose 345 to be the position of the guarding cop.

Case 2. $(1, 1, 1)$ and 4 is not chosen. We choose one of $\{123, 125\}$ to be the position of the guarding cop depending on the chosen elements from set Q.

Case 3. $(2, 0, 1)$ and 4 is chosen. We choose 345 to be the position of the guarding cop.

When $k = 4$, let $1234, 1245, 1256, 3456$ and $789\overline{10}$ be the starting vertices for five cops. We claim that there exists a guarding cop which satisfies Lemma 5.4. We consider all possible vertex for a robber. We separate into three generating sets; that is, $P = \{1, 2\}, Q = \{3, 4, 5, 6\}$ and $R = \{7, 8, 9, \overline{10}\}$. Let (p, q, r) denote the number of chosen elements of the position of the robber from P, Q and R, respectively.

Case 1. $(0, 2, 2)$. We choose $789\overline{10}$ to be the position of the guarding cop.

Case 2. $(0, 3, 1)$. We choose 3456 to be the position of the guarding cop.

Case 3. $(1, 1, 2)$. We choose $789\overline{10}$ to be the position of the guarding cop.

Case 4. $(1, 2, 1)$. We choose 3456 to be the position of the guarding cop.

Case 5. $(2, 1, 1)$. We choose one of $\{1234, 1245, 1256\}$ to be the position of the guarding cop depending on the chosen elements from set Q.

When $k = 5$, let $12345, 12456, 12567, 12673, 34567$ and $89\overline{10}\,\overline{11}\,\overline{12}$ be the starting vertices for six cops. We claim that there exists a guarding cop which satisfies Lemma 5.4. We consider all possible vertex for a robber. We separate into three generating sets; that is, $P = \{1, 2\}, Q = \{3, 4, 5, 6, 7\}$ and $R = \{8, 9, \overline{10}, \overline{11}, \overline{12}\}$. Let (p, q, r) denote the number of chosen elements of the position of the robber from P, Q and R, respectively.

Case 1. $(0, 2, 3)$. We choose $89\overline{10}\,\overline{11}\,\overline{12}$ to be the position of the guarding cop.

Case 2. $(0, 3, 2)$. We choose 34567 to be the position of the guarding cop.

Case 3. $(0,4,1)$. We choose 34567 to be the position of the guarding cop.

Case 4. $(1,1,3)$. We choose $89\overline{10}\,\overline{11}\,\overline{12}$ to be the position of the guarding cop.

Case 5. $(1,2,2)$. We choose one of $\{12345, 12456, 12567, 12673\}$ to be the position of the guarding cop depending on the chosen elements from set Q.

Case 6. $(1,3,1)$. We choose 34567 to be the position of the guarding cop.

Case 7. $(2,1,2)$. We choose one of $\{12345, 12456, 12567, 12673\}$ to be the position of the guarding cop depending on the chosen elements from set Q.

Case 8. $(2,2,1)$. We choose one of $\{12345, 12456, 12567, 12673\}$ to be the position of the guarding cop depending on the chosen elements from set Q. □

Note that if there are more than one vertices to be the position of the guarding cop in each case, we can choose the other one instead.

By Lemma 5.12, we have the guarding cop to use with our algorithm of choosing vertices. By Lemma 5.11 and Lemma 5.6, the robber cannot escape and we obtain the theorem as follows

Theorem 5.13. *Let $3 \leq k \leq 5$ and $2k+2 \leq n \leq k^2+k-1$. Then, $k+1$ cops are enough to catch the robber.*

Next, it remains to show that k cops is not enough to catch the robber. A pitfall cover in Definition 5.1 leads us to show that the robber's neighbor in his next turn exists to evade.

We start with $k = 3$ and $8 \leq n \leq 11$. Then, we would like to investigate that there exists a 3-set which intersects at most three 3-sets, says $\{A_i\}$ and is disjoint from the other one, says B. Therefore, we obtain the following Lemma.

Lemma 5.14. *Let $k = 3$, $8 \leq n \leq 11$ and $1 \leq l \leq 3$. Let $\{A_i\}_{i=1}^{l} \subseteq [n]^{(3)}$ and $B \in [n]^{(3)}$ be distinct 3-sets such that $B \cap A_1 = \varnothing$. Then, there exists a 3-set $C \in [n]^{(3)}$ such that $C \neq A_i$ and $C \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$ and $C \cap B = \varnothing$.*

Proof. Let $A'_i = A_i - B$ for all $1 \leq i \leq l$. Note that $A'_1 = A_1$ and $A'_i \neq \varnothing$ for all $1 \leq i \leq l$. Let $X = \{x_1, x_2, x_3 \ldots, x_p\}$ be a pitfall cover for $\{A'_1, A'_2, A'_3, \ldots, A'_l\}$.

Since $A_1' = A_1$, $p \geq 3 \geq l$. We may assume that $x_i \in A_i'$ for all $1 \leq i \leq l$. We separate into three cases depending on l.

Case 1. $l = 1$. We have $|[n] - (A_1 \cup B)| \geq 8 - 6 = 2$. We may choose $y \in [n] - (A_1 \cup B)$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $x \in A_1$.

Case 2. $l = 2$.

Case 2.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A_2' \neq \varnothing$. Since $A_2' \subseteq A_1$ and $|B| = 3$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$. We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 2$ and $C \cap B = \varnothing$.

Case 2.2. $X \neq A_1$. We set $C = \{x_1, x_2', z\}$ where $x_1 \in A_1, x_2' \in A_2' - A_1$ and $z \in ([n] - \{x_2'\}) - (A_1 \cup B)$. We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 2$ and $C \cap B = \varnothing$.

Case 3. $l = 3$.

Case 3.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A_j' \neq \varnothing$ for all $2 \leq j \leq 3$. Since $A_2' \cup A_3' \subseteq A_1$ and $|B| = 3$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$ and see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 3$ and $C \cap B = \varnothing$.

Case 3.2. $X \neq A_1$. Let $x_1 \in A_1$. Assume that $x_i' \in A_i' - A_1$ for all $2 \leq i \leq 3$.

If $x_2' = x_3'$, then we set $C = \{x_1, x_2', z\}$ where $z \in ([n] - \{x_2'\}) - (A_1 \cup B)$.

If $x_2' \neq x_3'$, then we set $C = \{x_1, x_2', x_3'\}$.

We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 3$ and $C \cap B = \varnothing$. $\square$

For $k = 4$ and $10 \leq n \leq 19$, we use the same idea as the previous lemma when $k = 3$ to obtain the following lemma.

Lemma 5.15. *Let $k = 4$, $10 \leq n \leq 19$ and $1 \leq l \leq 4$. Let $\{A_i\}_{i=1}^{l} \subseteq [n]^{(4)}$ and $B \in [n]^{(4)}$ be distinct 4-sets such that $B \cap A_1 = \varnothing$. Then, there exists a 4-set $C \in [n]^{(4)}$ such that $C \neq A_i$ and $C \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$ and $C \cap B = \varnothing$.*

Proof. Let $A_i' = A_i - B$ for all $1 \leq i \leq l$. Note that $A_1' = A_1$ and $A_i' \neq \varnothing$ for all $1 \leq i \leq l$. Let $X = \{x_1, x_2, x_3, \ldots, x_p\}$ be a pitfall cover for $\{A_1', A_2', A_3' \ldots, A_l'\}$.

Since $A'_1 = A_1$, $p \geq 4 \geq l$. We may assume that $x_i \in A'_i$ for all $1 \leq i \leq l$. We separate into four cases depending on l.

Case 1. $l = 1$. We have $|[n] - (A_1 \cup B)| \geq 10 - 8 = 2$. We may choose $y \in [n] - (A_1 \cup B)$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $x \in A_1$.

Case 2. $l = 2$.

Case 2.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A'_2 \neq \varnothing$. Since $A'_2 \subseteq A_1$ and $|B| = 4$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$. We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 2$ and $C \cap B = \varnothing$.

Case 2.2. $X \neq A_1$. We set $C = \{x_1, x'_2, y_1, z\}$. where $x_1 \in A_1, x'_2 \in A'_2 - A_1, y_1 \in A_1 - \{x_1\}$ and $z \in ([n] - \{x'_2\}) - (A_1 \cup B)$. We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 2$ and $C \cap B = \varnothing$.

Case 3. $l = 3$.

Case 3.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A'_j \neq \varnothing$ for all $2 \leq j \leq 3$. Since $A'_2 \cup A'_3 \subseteq A_1$ and $|B| = 4$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$ and see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 3$ and $C \cap B = \varnothing$.

Case 3.2. $X \neq A_1$. Let $x_1 \in A_1$. Assume that $x'_i \in A'_i - A_1$ for all $2 \leq i \leq 3$.

If $x'_2 = x'_3$, then we set $C = \{x_1, x'_2, y_1, z\}$ where $y_1 \in A_1 - \{x_1\}$ and $z \in ([n] - \{x'_2\}) - (A_1 \cup B)$.

If $x'_2 \neq x'_3$, then we set $C = \{x_1, x'_2, x'_3, y_1\}$ where $y_1 \in A_1 - \{x_1\}$.

We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 3$ and $C \cap B = \varnothing$.

Case 4. $l = 4$.

Case 4.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A'_j \neq \varnothing$ for all $2 \leq j \leq 4$. Since $\cup_{j=2}^{4} A'_j \subseteq A_1$ and $|B| = 4$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$ and see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 4$ and $C \cap B = \varnothing$.

Case 4.2. $X \neq A_1$. Let $x_1 \in A_1$. Assume that $x'_i \in A'_i - A_1$ for all $2 \leq i \leq 4$.

Let M be a pitfall cover for $\{A'_2 - A_1, A'_3 - A_1, A'_4 - A_1\}$.

If $|M| = 1$, then $x'_{j_1} = x'_{j_2}$ where $2 \leq j_1 \neq j_2 \leq 4$. Thus, we set $C = \{x_1, x'_2, y_1, z\}$ where $y_1 \in A_1 - \{x_1\}$ and $z \in ([n] - \{x'_2\}) - (A_1 \cup B)$.

If $|M| = 2$, then without loss of generality, assume that $M = \{x'_2, x'_3\}$, we set $C = \{x_1, x'_2, x'_3, y_1\}$ where $y_1 \in A_1 - \{x_1\}$.

If $|M| = 3$, then we set $C = \{x_1, x'_2, x'_3, x'_4\}$.

We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 4$ and $C \cap B = \varnothing$. □

Besides $k = 3$ and $k = 4$, we also show the existence of a 5-set which intersects at most five 5-sets, but is disjoint from another 5-set.

Lemma 5.16. *Let $k = 5$, $12 \leq n \leq 29$ and $1 \leq l \leq 5$. Let $\{A_i\}_{i=1}^{l} \subseteq [n]^{(5)}$ and $B \in [n]^{(5)}$ be distinct 5-sets such that $B \cap A_1 = \varnothing$. Then, there exists a 5-set $C \in [n]^{(5)}$ such that $C \neq A_i$ and $C \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$ and $C \cap B = \varnothing$.*

Proof. Let $A'_i = A_i - B$ for all $1 \leq i \leq l$. Note that $A'_1 = A_1$ and $A'_i \neq \varnothing$ for all $1 \leq i \leq l$. Let $X = \{x_1, x_2, x_3, \ldots, x_p\}$ be a pitfall cover for $\{A'_1, A'_2, A'_3 \ldots, A'_l\}$. Since $A'_1 = A_1$, $p \geq 5 \geq l$. We may assume that $x_i \in A'_i$ for all $1 \leq i \leq l$. We separate into five cases depending on l.

Case 1. $l = 1$. We have $|[n] - (A_1 \cup B)| \geq 12 - 10 = 2$. We may choose $y \in [n] - (A_1 \cup B)$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $x \in A_1$.

Case 2. $l = 2$.

Case 2.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A'_2 \neq \varnothing$. Since $A'_2 \subseteq A_1$ and $|B| = 5$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$. We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 2$ and $C \cap B = \varnothing$.

Case 2.2. $X \neq A_1$. We set $C = \{x_1, x'_2, y_1, y_2, z\}$. where $x_1 \in A_1, x'_2 \in A'_2 - A_1, y_1, y_2 \in A_1 - \{x_1\}$ and $z \in ([n] - \{x'_2\}) - (A_1 \cup B)$. We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 2$ and $C \cap B = \varnothing$.

Case 3. $l = 3$.

Case 3.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A'_j \neq \varnothing$ for all $2 \leq j \leq 3$. Since $A'_2 \cup A'_3 \subseteq A_1$

and $|B| = 5$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$ and see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 3$ and $C \cap B = \varnothing$.

Case 3.2. $X \neq A_1$. Let $x_1 \in A_1$ Assume that $x'_i \in A'_i - A_1$ for all $2 \leq i \leq 3$.

If $x'_2 = x'_3$, then we set $C = \{x_1, x'_2, y_1, y_2, z\}$ where $y_1, y_2 \in A_1 - \{x_1\}$ and $z \in ([n] - \{x'_2\}) - (A_1 \cup B)$.

If $x'_2 \neq x'_3$, then we set $C = \{x_1, x'_2, x'_3, y_1, y_2\}$ where $y_1, y_2 \in A_1 - \{x_1\}$.

We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 3$ and $C \cap B = \varnothing$.

Case 4. $l = 4$.

Case 4.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A'_j \neq \varnothing$ for all $2 \leq j \leq 4$. Since $\cup_{j=2}^{4} A'_j \subseteq A_1$ and $|B| = 5$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$ and see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 4$ and $C \cap B = \varnothing$.

Case 4.2. $X \neq A_1$. Let $x_1 \in A_1$ Assume that $x'_i \in A'_i - A_1$ for all $2 \leq i \leq 4$. Let M be a pitfall cover for $\{A'_2 - A_1, A'_3 - A_1, A'_4 - A_1\}$.

If $|M| = 1$, then $x'_{j_1} = x'_{j_2}$ where $2 \leq j_1 \neq j_2 \leq 4$. Thus, we set $C = \{x_1, x'_2, y_1, y_2, z\}$ where $y_1, y_2 \in A_1 - \{x_1\}$ and $z \in ([n] - \{x'_2\}) - (A_1 \cup B)$.

If $|M| = 2$, then without loss of generality, assume that $M = \{x'_2, x'_3\}$, we set $C = \{x_1, x'_2, x'_3, y_1, y_2\}$ where $y_1, y_2 \in A_1 - \{x_1\}$.

If $|M| = 3$, then we set $C = \{x_1, x'_2, x'_3, x'_4, y_1\}$ where $y_1 \in A_1 - \{x_1\}$.

We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 4$ and $C \cap B = \varnothing$.

Case 5. $l = 5$.

Case 5.1. $X = A_1$. Since $|A_1| = k > l - 1$, there exists one element in A_1, says x, such that $(A_1 - \{x\}) \cap A'_j \neq \varnothing$ for all $2 \leq j \leq 5$. Since $\cup_{j=2}^{5} A'_j \subseteq A_1$ and $|B| = 5$, we have $|[n] - (A_1 \cup B)| \geq 2$. Then, we set $C = (A_1 - \{x\}) \cup \{y\}$ where $y \in [n] - (A_1 \cup B)$ and see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 5$ and $C \cap B = \varnothing$.

Case 5.2. $X \neq A_1$. Let $x_1 \in A_1$ Assume that $x'_i \in A'_i - A_1$ for all $2 \leq i \leq 5$. Let M be a pitfall cover for $\{A'_2 - A_1, A'_3 - A_1, A'_4 - A_1, A'_5 - A_1\}$.

If $|M| = 1$, then $x'_{j_1} = x'_{j_2}$ where $2 \leq j_1 \neq j_2 \leq 5$. Thus, we set $C = \{x_1, x'_2, y_1, y_2, z\}$ where $y_1, y_2 \in A_1 - \{x_1\}$ and $z \in ([n] - \{x'_2\}) - (A_1 \cup B)$.

If $2 \leq |M| = m \leq 3$, then without loss of generality, assume that $M = \{x'_2, x'_3, x'_4, \ldots, x'_{m+1}\}$, we set $C = \{x_1, x'_2, x'_3, \ldots, x'_{m+1}, y_1, \ldots, y_{k-(m+1)}\}$ where $y_1, y_2, y_3, \ldots, y_{k-(m+1)} \in A_1 - \{x_1\}$.

If $|M| = 4$, then we set $C = \{x_1, x'_2, x'_3, x'_4, x'_5\}$.

We see that $C \neq A_i$, $|C \cap A_i| \geq 1$ for all $1 \leq i \leq 5$ and $C \cap B = \varnothing$. $\square$

Therefore, we let the l k-sets $\{A_i\}_{i=1}^{l}$ be occupied by k cops and the other k-set B occupied by the robber where $1 \leq l \leq k$. Thus, we see that the k-set C can be the robber's position in his next turn. Then, we will use the Lemmas 5.14, 5.15 and 5.16 to show that no matter where the k cops stay, there exists a free neighbor for the robber. However, in our consideration, Kneser graphs $KG(n, k)$ need to have diameter 2. By Theorem 2.28, we have to consider the cases when $n \geq 3k - 1$.

Theorem 5.17. *$c(KG(n, 3)) = 4$ for all $8 \leq n \leq 11$.*

Proof. Let $8 \leq n \leq 11$. By Theorem 5.13, we have $c(KG(n, 3)) \leq 4$. By Theorem 2.28, we have $\text{diam}(KG(n, 3)) = 2$. We claim that the robber has a winning strategy when there are three cops. Since $KG(n, 3)$ has diameter 2, the robber must stay far from all cops by distance 2.

Suppose that 3 cops occupy the distinct vertices $A_1, A_2, A_3, \ldots, A_l$ for some $1 \leq l \leq 3$. Then, the robber chooses a vertex B as follows. Let $X \in V(KG(n, k)) - \{A_1, A_2, A_3, \ldots, A_l\}$. If $X \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$, then we choose $B = X$. Otherwise, say $X \cap A_1 = \varnothing$, by Lemma 5.14, there exists a vertex $C \in V(KG(n, 3)) - \{A_1, A_2, A_3, \ldots, A_l\}$ such that $C \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$, so we choose $B = C$. Then, $d(B, A_i) = 2$ for all $1 \leq i \leq l$.

Suppose that, at some stage, the robber is at distance two from every cop. Then, the cops make their move and see that they do not catch the robber. Assume that 3 cops occupy the distinct vertices $V_1, V_2, V_3, \ldots, V_r$ for some $1 \leq r \leq 3$, and the robber stay at vertex V. If $d(V, V_i) = 2$ for all $1 \leq i \leq r$, then the robber chooses not to move. Otherwise, say $d(V, V_1) = 1$. Thus, $V \cap V_1 = \varnothing$. By Lemma

5.14, there exists a vertex W such that $W \neq V_i$, $W \cap V_i$ for all $1 \leq i \leq r$, and $W \cap V = \varnothing$. Then, the robber decides to move from V to W; that is, $d(W, V_i) = 2$ for all $1 \leq i \leq r$. Hence, the robber always escape. □

Theorem 5.18. *$c(KG(n, 4)) = 5$ for all $11 \leq n \leq 19$.*

Proof. Let $11 \leq n \leq 19$. By Theorem 5.13, we have $c(KG(n, 4)) \leq 5$. By Theorem 2.28, we have $\operatorname{diam}(KG(n, 4)) = 2$. We claim that the robber has a winning strategy when there are four cops. Since $KG(n, 4)$ has diameter 2, the robber must stay far from all cops by distance 2.

Suppose that 4 cops occupy the distinct vertices $A_1, A_2, A_3, \ldots, A_l$ for some $1 \leq l \leq 4$. Then, the robber chooses a vertex B as follows. Let $X \in V(KG(n, k)) - \{A_1, A_2, A_3, \ldots, A_l\}$. If $X \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$, then we choose $B = X$. Otherwise, say $X \cap A_1 = \varnothing$, by Lemma 5.15, there exists a vertex $C \in V(KG(n, 4)) - \{A_1, A_2, A_3, \ldots, A_l\}$ such that $C \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$, so we choose $B = C$. Then, $d(B, A_i) = 2$ for all $1 \leq i \leq l$.

Suppose that, at some stage, the robber is at distance two from every cop. Then, the cops make their move and see that they do not catch the robber. Assume that 4 cops occupy the distinct vertices $V_1, V_2, V_3, \ldots, V_r$ for some $1 \leq r \leq 4$, and the robber stay at vertex V. If $d(V, V_i) = 2$ for all $1 \leq i \leq r$, then the robber chooses not to move. Otherwise, say $d(V, V_1) = 1$. Thus, $V \cap V_1 = \varnothing$. By Lemma 5.15, there exists a vertex W such that $W \neq V_i$, $W \cap V_i$ for all $1 \leq i \leq r$, and $W \cap V = \varnothing$. Then, the robber decides to move from V to W; that is, $d(W, V_i) = 2$ for all $1 \leq i \leq r$. Hence, the robber always escape. □

Theorem 5.19. *$c(KG(n, 5)) = 6$ for all $14 \leq n \leq 29$.*

Proof. Let $14 \leq n \leq 29$. By Theorem 5.13, we have $c(KG(n, 5)) \leq 6$. By Theorem 2.28, we have $\operatorname{diam}(KG(n, 5)) = 2$. We claim that the robber has a winning strategy when there are five cops. Since $KG(n, 5)$ has diameter 2, the robber must stay far from all cops by distance 2.

Suppose that 5 cops occupy the distinct vertices $A_1, A_2, A_3, \ldots, A_l$ for some $1 \leq l \leq 5$. Then, the robber chooses a vertex B as follows. Let $X \in V(KG(n, k)) -$

$\{A_1, A_2, A_3, \ldots, A_l\}$. If $X \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$, then we choose $B = X$. Otherwise, say $X \cap A_1 = \varnothing$, by Lemma 5.16, there exists a vertex $C \in V(KG(n,5)) - \{A_1, A_2, A_3, \ldots, A_l\}$ such that $C \cap A_i \neq \varnothing$ for all $1 \leq i \leq l$, so we choose $B = C$. Then, $d(B, A_i) = 2$ for all $1 \leq i \leq l$.

Suppose that, at some stage, the robber is at distance two from every cop. Then, the cops make their move and see that they do not catch the robber. Assume that 5 cops occupy the distinct vertices $V_1, V_2, V_3, \ldots, V_r$ for some $1 \leq r \leq 5$, and the robber stay at vertex V. If $d(V, V_i) = 2$ for all $1 \leq i \leq r$, then the robber chooses not to move. Otherwise, say $d(V, V_1) = 1$. Thus, $V \cap V_1 = \varnothing$. By Lemma 5.16, there exists a vertex W such that $W \neq V_i$, $W \cap V_i$ for all $1 \leq i \leq r$, and $W \cap V = \varnothing$. Then, the robber decides to move from V to W; that is, $d(W, V_i) = 2$ for all $1 \leq i \leq r$. Hence, the robber always escape. $\square$

By Theorems 5.17, 5.18 and 5.19, we can conclude that

Theorem 5.20. *Let* $3 \leq k \leq 5$ *and* $3k - 1 \leq n \leq k^2 + k - 1$. *Then, cop-number on* $KG(n,k)$ *is* $k+1$.

For $n = 2k+1$, we know that the lower bound of the cop-number may be obtained by considering the minimum degree of the graph under consideration.

Since $KG(n,k)$ is a regular graph, each vertex of $KG(n,k)$ has exactly $\binom{n-k}{k}$ neighbors. Thus, the degree of each vertex in $KG(n,k)$ is $\binom{n-k}{k}$. We know that $KG(2k+1,k)$ is a k-odd graph, its girth is 6 [17], which means it has no 3- or 4-cycles. Since its degree is $k+1$, by Theorem 2.36, $c(KG(2k+1,k)) \geq k+1$.

It is easy to see that the case $k = 3$ satisfies our choosing algorithm and the previous lemmas. First, we show the existence of a guarding cop.

Lemma 5.21. *There exists a guarding cop on* $KG(7,3)$.

Proof. When $k = 3$, let $123, 125, 345$ and 467 be the starting vertices for four cops. We claim that there exists a guarding cop which satisfies Lemma 5.4. We consider all possible vertices for a robber. We separate into three generate sets; that is, $P = \{1,2\}, Q = \{3,5\}$ and $R = \{4,6,7\}$. Let (p,q,r) denote the number of chosen elements of the position of the robber from P, Q and R, respectively.

Case 1. $(0, 2, 1)$ and 4 is not chosen. We choose 345 to be the position of the guarding cop.

Case 2. $(1, 0, 2)$ and 4 is chosen. We choose 467 to be the position of the guarding cop.

Case 3. $(1, 1, 1)$ and 4 is not chosen. We choose one of $\{123, 125\}$ to be the position of the guarding cop depending on the chosen elements from set Q.

Case 4. $(1, 1, 1)$ and 4 is chosen. We choose 345 to be the position of the guarding cop.

Case 5. $(2, 0, 1)$ and 4 is chosen. We choose one of $\{123, 125\}$ to be the position of the guarding cop. □

By Lemma 5.21, we have the guarding cop to use our choosing algorithm. By Lemma 5.11 and Lemma 5.6, the robber cannot escape. We obtain theorem as follows

Theorem 5.22. *In $KG(7, 3)$, four cops is enough to catch the robber; that is, $c(KG(7, 3)) \leq 4$.*

By Theorems 2.36 and 5.22, we can conclude that

Theorem 5.23. *Cop-number of $KG(7, 3)$ is 4.*

CHAPTER VI
CONCLUSION AND DISCUSSION

First, in Chapter III, we are interested in the results on cop-win hypergraphs. We characterize cop-win hypergraphs $\mathcal{H}$ by successively weak deletion corner (in any order) to reduce $\mathcal{H}$ to be a trivial hypergraph and provide some results on each product of hypergraphs in Table 6.1.

	Cop-win hypergraphs	**Robber-win hypergraphs**
Cartesian product	Robber-win hypergraph	Robber-win hypergraph
Minimal rank preserving direct product	Robber-win hypergraph	Robber-win hypergraph
Maximal rank preserving direct product **(uniformity of each hypergraph is needed)**	Robber-win hypergraph	Robber-win hypergraph
Normal (Standard) strong product	Cop-win hypergraph	still open

Table 6.1: The results on product of hypergraphs

Next, in Chapter IV, we give a better chance to cop to allow more than one cop to play this game and investigate the minimum number of cops to guarantee that they have a winning strategy, called a cop-number. We focus on two structures of hypergraphs, namely, the complete k-partite hypergraphs and n-prisms over a hypergraph. Thus, we conclude that the cop-number of each structure of hypergraphs in Table 6.2.

Finally, by the concept on cop-number in cops and robbers on graphs, it leads us to study on the Kneser graphs $KG(n, k)$ in Chapter V. We observe that that $c(KG(n, k))$ can be concluded in Table 6.3.

As for the future research, we suggest one to investigate the cop-number of n-Prisms over a robber-win hypergraph $\mathcal{H}_0^{(2)}$ and Kneser graphs in the remaining cases.

	Cop-number
Complete k-uniform k-partite hypergraph	2 with $\|\mathcal{V}_i\| \geq 2$ for all $1 \leq i \leq k$
Complete σ-uniform k-partie hypergraph	1 where $\sigma \geq 3$
	2 where $\sigma = 2, k \geq 3$ and $\|\mathcal{V}_i\| \geq 2$ for all $1 \leq i \leq k$
n-Prisms over a cop-win hypergraph $\mathcal{H}_0^{(k)}$	1 where $k \geq 3$
	2 where $k = 2$
n-Prisms over a robber-win hypergraph $\mathcal{H}_0^{(k)}$	$c(\mathcal{H}_0^{(k)})$ where $k \geq 3$

Table 6.2: Cop-number of certain hypergraphs

	Cop-number $c(KG(n,k))$			
	$n = 2k+1$	$2k+2 \leq n \leq 3k-2$	$3k-1 \leq n \leq k^2+k-1$	$n \geq k^2+k$
$k = 2$	3	N/A	3	3
$k = 3$	4	N/A	4	4
$k = 4$	Still open	Still open	5	5
$k = 5$	Still open	Still open	6	6
$k \geq 6$	Still open	Still open	Still open	$k+1$

Table 6.3: Cop-number of Kneser graphs

REFERENCES

[1] Aigner, M., Fromme, M.: A Game of cops and robbers, *Discrete Appl Math.* **8**, 1–12 (1984).

[2] Baird, W.D.: Cops and Robbers on Graphs and Hypergraphs, Ph.D. thesis, Ryerson University, 2011, Theses and dissertations. Paper 821.

[3] Baird, W., Beveridge, A., Bonato, A., Codenotti, P., Maurer, A., McCauley, J., Valeva, S.: On the minimum order of k-cop-win graphs, *Contrib. Discrete Math.* **9(1)**, 70–84 (2014).

[4] Bollobás, B.: *Extremal Graph Theory*, Academic Press (1978).

[5] Bollobás, B.: *Modern Graph Theory*, Springer-Verlag (1998).

[6] Boonklurb, R., Singhun, S., Termtanasombat S.: Hamiltonian decompositions of prisms over complete 3-uniform hypergraphs *Discrete Math. Algorithms Appl.* **7(3) 1550031**, 11 pages (2015).

[7] Diestel, R.: *Graph Theory*, 5th Edition, Springer-Verlag (2018).

[8] Hellmuth, M., Ostermeier, L., Stadler, P.F.: A Survey on Hypergraph Products, *Math. Comput. Sci.* **6**, 1–32 (2012).

[9] Jirimutu, Wang, J.: Hamiltonian decomposition of complete bipartite r-hypergraphs *Acta Math. Appl. Sin.* **17**, 563–566 (2001).

[10] Kuhl, J., Schroeder, M.W.: Hamilton cycle decompositions of k-uniform k-partite hypergraphs *Australas. J. Combin.* **56**, 23–37 (2013).

[11] Nowakowski, R., Winkler, P.: Vertex-to-vertex pursuit in a graph, *Discrete Math.* **43**, 235–239 (1983).

[12] Ostergard, P.R.J., Shao, Z., Xu, X.: Bounds on the domination number of Kneser graphs, *Ars Math. Contemp.* **9**, 187–195 (2015).

[13] Quilliot, A.: Jeux et pointes fixes sur les graphes. Ph.D thesis, Université de Paris VI, 1978.

[14] Tošić, R.: Search number of cartesian product, *Univ. Novom Sabu Zb. Rad. Prirod. Mat Fak. Ser. Mat.* **17**, 239–243 (1987).

[15] Valencia-Pabon, M., Vera, J-C.: On the diameter of Kneser graphs, *Discrete Math.* **305**, 383–385 (2005).

[16] Voloshin, V.I.: *Introduction to Graph and Hypergraph Theory*, Nova Science Publishers, Inc. (2009).

[17] West, D. B.: *Introduction to Graph Theory (Exercise 1.1.28, p.17)*, Pearson Education (2000).

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