ทางเดินของม้าหมากรุกบนกระดานวงแหวน กระดานหมากรุก $4 \times n$ แบบพร่อง และกระดานแอลบางแบบ

นายวสุพล ศรีโชติ

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KNIGHT'S TOURS ON RINGBOARDS, DEFICIENT $4 \times n$ CHESSBOARDS AND SOME L-BOARDS

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การเดินของม้าหมากรุก คือ ผลลัพธ์ของการเคลื่อนที่ของม้าหมากรุกจากช่องหนึ่งไปอีกช่องหนึ่งโดยเดินสองช่องในแนวตั้งหรือแนวนอนบนกระดานและเดินในแนวตั้งฉากอีกหนึ่งช่อง ส่วนการเดินของม้าหมากรุกแบบปิด คือ การเคลื่อนที่ของม้าหมากรุกที่เดินผ่านทุกช่องบนกระดานที่กำหนดเพียงหนึ่งครั้งและกลับมาที่จุดเริ่มต้นเดิน การเดินของม้าหมากรุกแบบปิดในรูปแบบต่าง ๆ มีการศึกษาอย่างกว้างขวางบนกระดานรูปสี่เหลี่ยมผืนผ้าหรือบนกระดานสามมิติ สำหรับ $m, n > 2r$ กระดานวงแหวน (m, n, r) หรือ $\mathrm{RB}(m, n, r)$ เป็นกระดานขนาด $m \times n$ หรือ $\mathrm{CB}(m \times n)$ ที่มีส่วนตรงกลางหายไปและขอบของกระดานมี r แถว และ r หลัก ต่อมาถ้าให้ A เป็นเซตของสองช่องบน $\mathrm{CB}(m \times n)$ แล้ว $\mathrm{CB}(m \times n) - A$ คือ บอร์ดที่เหลือจากการลบสองช่องนั้นออกไป ในวิทยานิพนธ์ฉบับนี้ เราศึกษาการมีอยู่ของการเดินม้าแบบปิดบนกระดาน $\mathrm{RB}(m, n, r)$ และ $\mathrm{CB}(m \times n) - A$

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WASUPOL SRICHOTE : KNIGHT'S TOURS ON RINGBOARDS, DEFICIENT $4 \times n$ CHESSBOARDS AND SOME L-BOARDS.
ADVISOR : ASSOC. PROF. RATINAN BOONKLURB, Ph.D.
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A (legal) knight's move is the result of moving the knight two squares horizontally or vertically on the board and then turning and moving one square in a perpendicular direction. A closed knight's tour is a sequence of knight's moves that visits every square on a given chessboard exactly once and returns to its start square. A closed knight's tour and its variations are studied widely over the rectangular chessboard or a three-dimensional rectangular box. For $m, n > 2r$, an (m, n, r)-ringboard or $\mathrm{RB}(m, n, r)$ is defined to be an $m \times n$ chessboard, denoted by $\mathrm{CB}(m \times n)$, with the middle part missing and the rim contains r rows and r columns. Next, if A is a set of two squares of $\mathrm{CB}(4 \times n)$, then $\mathrm{CB}(4 \times n) - A$ is the deficient board after deleting these two squares. In this dissertation, we study the existence of closed knight's tours on $\mathrm{RB}(m, n, r)$ and $\mathrm{CB}(4 \times n) - A$.

Department : Mathematics and Computer Science Student's Signature ....

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CHAPTER I
INTRODUCTION AND PRELIMINARIES

In the first part of this chapter, we give some definitions, notations and the basic knowledge of this dissertation.

1.1 Basic Knowledge

Since this dissertation is about the Hamiltonian cycle of a graph, we first give a definition of graphs that we use here.

Definition 1.1. [4] A ***graph*** G consists of a finite nonempty set V of objects called vertices and a set E of 2-element subsets of $V(G)$ called edges. The sets V and E are the ***vertex set*** and the ***edge set*** of G.

Definition 1.2. [4] A ***subgraph*** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that G ***contains*** H.

Definition 1.3. [4] A $u - v$ ***path*** in a graph G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent and no vertices are repeated. In the case that $u = v$, it is called a ***cycle***. A graph G is ***connected*** if it has a $u-v$ path whenever $u, v \in V(G)$. The ***components*** of a graph G are its maximal connected subgraphs of G.

Definition 1.4. [4] A path in a graph G that contains every vertex of G is called a ***Hamiltonian path***. Also, a cycle in G that contains every vertex of G is called a ***Hamiltonian cycle***.

The fact about the existence of a Hamiltonian cycle and a Hamiltonian path in a graph G that we use in this dissertation is the following theorem.

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Theorem 1.5. [4, 5] Let $G = (V, E)$ be a graph, S be a proper subset of V and $\omega(G - S)$ is the number of components of $G - S$.

(a) If $\omega(G - S) > |S|$, then G does not contain any Hamiltonian cycle.

(b) If $\omega(G - S) > |S| + 1$, then G does not contain any Hamiltonian path.

In the second part of this chapter, we give our motivation, literature reviews and some known results which is useful for this dissertation.

1.2 Motivation of This Dissertation

The $m \times n$ chessboard, denoted by, CB$(m \times n)$ is the generalization of the regular CB(8×8). It consists of m rows and n arrays of squares. Suppose the squares of the CB$(m \times n)$ are labeled by (i, j) in the matrix fashion. A *legal knight's move* is the result of a moving the knight two squares horizontally or vertically on the CB$(m \times n)$ and then turning and moving one square in a perpendicular direction. That is, if we start at (i, j), then the knight can move to one of eight squares: $(i \pm 2, j \pm 1)$ or $(i \pm 1, j \pm 2)$ (if exists) as shown in Figure 1.1. Here, we let the square (i, j) be colored by black if $i + j$ is even. Otherwise, it is colored by white.

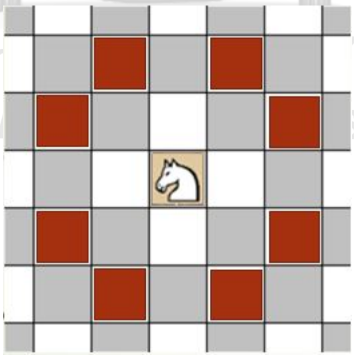

Figure 1.1: legal knight's move

A *closed knight's tour* (CKT) is a sequence of legal knight's moves that visit every squares on a given chessboard exactly once and return to its start square. While, an *open knight's tour* (OKT) is a sequence of legal knight's moves that visit every squares on a given chessboard exactly once and the starting and terminating squares are different. Both CKT and OKT problems on a 2-dimensional or 3-dimensional chessboards are one of the interesting mathematical problems as you can see some of them listed in [3], [6], [8], [10], [13] and [14]. Not only the legal knight's move, but some researchers also extended it to be an (a, b)-knight's move which is the result of a moving the knight a squares horizontally or vertically on the CB$(m \times n)$ and then turning and moving b squares in a perpendicular direction. Several mathematical problems along this direction were considered, see for examples [1], [5], [12] and references therein for details.

In 1991, Schwenk [11] obtained necessary and sufficient conditions for the existence of a CKT on the CB$(m \times n)$ as follows.

Theorem 1.6. [11] A CB$(m \times n)$ with $m \leq n$ admits a CKT unless one or more of the following conditions holds: (i) mn is odd or (ii) $m \in \{1, 2, 4\}$ or (iii) $m = 3$ and $n \in \{4, 6, 8\}$. Furthermore, this CKT contains a knight's move from square $(1, n-1)$ to square $(3, n)$ and square $(m, 2)$ to square $(m-1, 4)$.

In 2005, Chia and Ong [5] obtained necessary and sufficient conditions for the existence of an OKT on the CB$(m \times n)$ as follows.

Theorem 1.7. [5] A CB$(m \times n)$ with $m \leq n$ admits an OKT unless one or more of the following conditions holds: (i) $m \in \{1, 2\}$ or (ii) $m = 3$ and $n \in \{3, 5, 6\}$ or (iii) $m = 4$ and $n = 4$.

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In this dissertation, we first consider one of the variations of the CKT problem by considering the chessboard that the middle part is missing which is called $(m, n, r)-$*ringboard* or $(m, n, r)-$*annulus* board and we denote it by RB(m, n, r).

Definition 1.8. Let m, n and r be integers such that $m, n > 2r$. An RB(m, n, r) is defined to be an CB$(m \times n)$ with the middle part missing and the rim contains exactly r rows and r columns.

In 1996, Wiitala [15] showed that the RB$(m, m, 2)$ contains no CKT. However, the characterization of the general RB(m, n, r) has not been given.

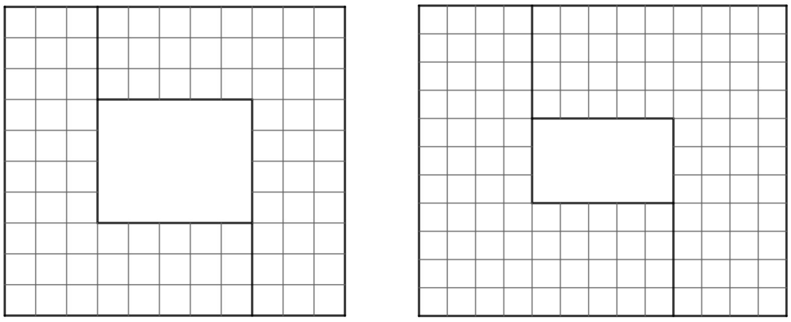
Theorem 1.9. [15] The $\mathrm{RB}(m,m,2)$ contains no CKT for all $m \geq 5$.

Thus, we try to establish the characterization like the one given by Schwenk [11]. Actually, the CKT problem on the $\mathrm{RB}(m,n,r)$ can be converted to a certain graph problem. If we regard each square of the $\mathrm{RB}(m,n,r)$ as a vertex, then the knight graph $G(m,n,r)$ represented all legal knight's moves on $\mathrm{RB}(m,n,r)$ is a graph with $2r(m+n-2r)$ vertices and two vertices (a,b) and (c,d) are joined by an edge whenever the knight can be moved from one square to another by a legal knight's move and this edge is denoted by $(a,b)-(c,d)$. Then, a CKT (respectively, OKT) on the $\mathrm{RB}(m,n,r)$ is a Hamiltonian cycle (respectively, Hamiltonian path) in $G(m,n,r)$.

The first goal of this dissertation is to prove that for $m,n > 2r$, the $\mathrm{RB}(m,n,r)$ admits a closed knight's tour if and only if (a) $m=n=3$ and $r=1$ or (b) $r \geq 3$. In order to reach our goal, we need to divide our $\mathrm{RB}(m,n,r)$ into small pieces depending on r. If $r \geq 5$ is even, then $\mathrm{RB}(m,n,r)$ is divided into four smaller rectangular chessboard and we can use Theorem 1.6 to construct the CKT for $\mathrm{RB}(m,n,r)$ which will be elaborated in the Case 3.1 of Theorem 3.4 in Section 3.2 of Chapter III. However, if $r \geq 5$ is odd and $\mathrm{RB}(m,n,r)$ is divided into four smaller rectangular chessboard, then there is a case that Theorem 1.6 cannot be used (Case 3.2 of Theroem 3.4). Thus, we need to construct our own CKT base on the existence of an OKT on some rectangular chessboards which will be constructed in Theorem 2.3 in Section 2.2 of Chapter II. For small r, namely $r \in \{3,4\}$, we need to divide $\mathrm{RB}(m,n,r)$ into two parts, namely L-board and 7-board of width 3 or 4 which we denote them by $\mathrm{LB}(r,c,3)$, $\mathrm{LB}(r,c,4)$, $7\mathrm{B}(r,c,3)$ and $7\mathrm{B}(r,c,4)$ depends on the numbers of rows r and columns c (see Cases 1 and 2 of Theorem 3.4). For example, Figure 1.2 illustrates that $\mathrm{RB}(10,11,3)$ is divided into $\mathrm{LB}(10,8,3)$ and $7\mathrm{B}(10,8,3)$ and $\mathrm{RB}(11,13,4)$ is divided into $\mathrm{LB}(11,9,4)$ and $7\mathrm{B}(11,9,4)$.

Note that the definitions of L-board and 7-board are given as follows

Definition 1.10. Let m, n and r be integers such that $m,n > r$. An $\mathrm{LB}(m,n,r)$ is defined to be an $\mathrm{CB}(m \times n)$ with the upper right (lower left) part missing, the



(a) RB(10,11,3) (b) RB(11,13,4)

Figure 1.2: (a) LB$(10, 8, 3)$ and 7B$(10, 8, 3)$ and (b) LB$(11, 9, 4)$ and 7B$(11, 9, 4)$

lower leg contains r rows (columns) and the upper leg contains r columns (rows). A 7B(m, n, r) is defined to be an CB$(m \times n)$ with the lower left part missing, the lower leg contains r columns and the upper leg contains r rows.

Therefore, to construct the CKT on the ringboard for this case, we prove the existence of some special OKTs on LB$(m, n, 3)$ and 7B$(m, n, 3)$ and the existence of a CKT on LB$(m, n, 4)$ and 7B$(m, n, 4)$ are given in Theorems 2.1 and 3.1 in section 2.1 of Chapter II and Section 3.1 of Chapter III, respectively. For $r = 2$, we prove the extension of Wiitala's result in [15] which is the non-existence of the CKT on the RB$(m, n, 2)$ in Theorem 3.3 in Section 3.2 of Chapter III.

However, there is another interesting research that we have covered in our dissertation. In Theorem 1.6, for those CB$(m \times n)$ which do not admit a CKT, one can notice that if we ignore some squares of them, a CKT can be constructed on those deficient CB$(m \times n)$. In 2009, DeMaio and Hippchen [7] found $T(m, n)$, the minimum number of squares removal from CB$(m \times n)$, so that a CKT on the deficient CB$(m \times n)$ exists but they did not determine the exact position of each removal square. In particular, it is stated that (i) for $m, n \geq 3$ are odd and $(m, n) \neq (3, 5)$, $T(m, n) = 1$ and (ii) for $n \geq 3$, $T(4, n) = 2$.

Consequently, in 2013, Miller and Farnsworth [9] determined the exact position of the one square to be removed from CB$(3 \times n)$ where $n \neq 5$ so that a CKT exists. While, in 2015, Bi et al. [2] determined the exact position of the one square to be removed from CB$(m \times n)$ where $m, n \geq 3$ are odd and $(m, n) \neq (3, 5)$ so that a

CKT exists. In [2], they also tried to consider the exact positions of two squares be removed from $\mathrm{CB}(4 \times n)$, where $n \geq 3$. One useful proposition is stated here for ease of reference. The first part was proved by [2] and the second part is from the fact that a knight's move always moves from black to white or white to black square.

Proposition 1.11. [2] If two squares in $\mathrm{CB}(4 \times n)$ are deleted and a CKT exists for the remaining board, then (i) neither square could come from the middle two rows and (ii) these two squares have different color.

They also gave the following conjecture.

Conjecture 1 [2] Consider $\mathrm{CB}(4 \times n)$ with $n \geq 7$. For any pair of squares, with one of each parity of color and neither coming from the middle two rows, there is a CKT on $\mathrm{CB}(4 \times n)$ that avoids only these two squares.

Therefore, the second goal of this dissertation is to prove the Conjecture 1 in Section 3.3.2 of Chapter III and also determine the exact pair of squares removal from $\mathrm{CB}(4 \times n)$ for $3 \leq n \leq 6$ in Section 3.3.1 of Chapter III. If A is a set of two squares of $\mathrm{CB}(4 \times n)$, then $\mathrm{CB}(4 \times n) - A$ is the deficient board after deleting these two squares. In addition, we use $G(m \times n) - A$ to represent the knight graph after deleting these two vertices. Therefore, the existence of a CKT on $\mathrm{CB}(4 \times n) - A$ is simply the existence of a Hamiltonian cycle on $G(4 \times n) - A$.

To proof the Conjecture 1 and to determine all possible set A, some special open knight's tours (OKTs) on $\mathrm{CB}(4 \times n) - \{(i, j)\}$ for $n \geq 5$ are required. Actually the OKT is the Hamiltonian path on $G(4 \times n) - \{(i, j)\}$. These OKTs are constructed in Section 2.3 of Chapter II. Finally, conclusion and discussion are given in Chapter IV.

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CHAPTER II
OPEN KNIGHT'S TOURS ON SOME BOARDS

In this chapter, we give the existence of some special OKTs on some boards which is divided into three sections as follows.

2.1 Existence of Some Special OKTs on LBs and 7Bs

First, let us construct two OKTs on LB$(m, n, 3)$ and two OKTs on 7B$(m, n, 3)$ for $m, n \geq 4$.

Theorem 2.1. *Let $m, n \geq 4$.*

(a) *LB$(m, n, 3)$ contains an OKT from $(1, 2)$ to $(1, 3)$ if and only if $m+n$ is odd and $(m, n) \neq (4, 5)$.*

(b) *LB$(m, n, 3)$ contains an OKT from $(1, 3)$ to $(2, 2)$ if and only if $m+n$ is even and $(m, n) \notin \{(4, 4), (4, 6), (5, 5)\}$.*

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Proof. Let $m, n \geq 4$.

(a) We assume that LB$(m, n, 3)$ contains an OKT from $(1, 2)$ to $(1, 3)$ and let $m+n$ be even or $(m, n) = (4, 5)$.

If $m+n$ is even, then the numbers of white square and black square are not the same. Thus, the two end-points of this OKT must have the same color. However, $(1, 2)$ and $(1, 3)$ are next to each other and have different color, contradiction.

For $(m, n) = (4, 5)$, let G be the knight graph of LB$(4, 5, 3)$. Consider $G' = G - \{(1, 2), (1, 3)\}$. Since LB$(4, 5, 3)$ contains an OKT from $(1, 2)$ to $(1, 3)$, G' has a Hamiltonian path. Let $S = \{(2, 3), (3, 2), (3, 3), (3, 4), (4, 3)\}$. Then, $\omega(G' - S) = 7 > 6 = |S| + 1$ as shown in Figure 2.1. By Theorem 1.5(b), we obtain a contradiction.

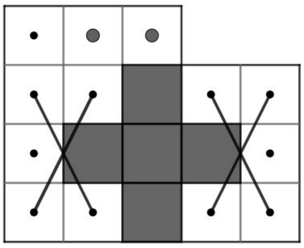


Figure 2.1: Components of $G' - S$

On the other hand, let us assume that $m+n$ is odd and $(m,n) \neq (4,5)$.

If $m+n$ is odd and $m+n < 11$, then $(m,n) = (5,4)$. The required OKT on the LB$(5,4,3)$ from $(1,2)$ to $(1,3)$ that contains an edge $(3,3)-(5,4)$ presented in Figure 2.2.

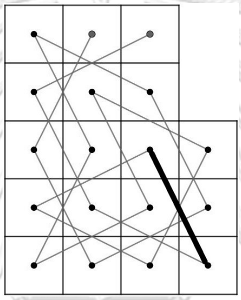
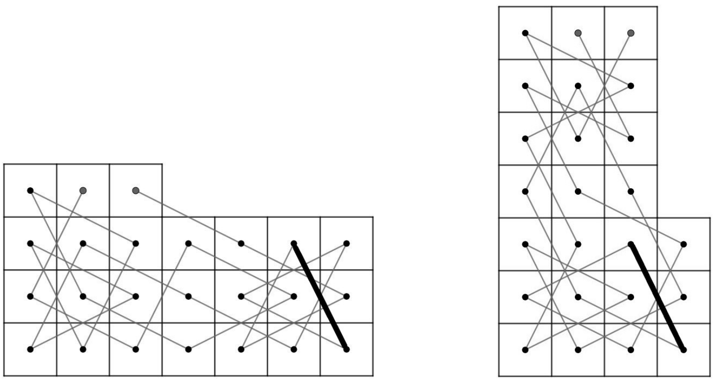


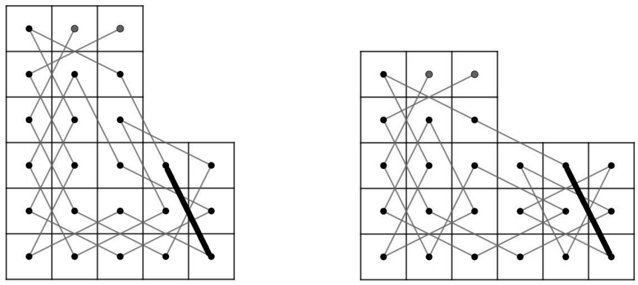
Figure 2.2: Required OKT on the LB$(5,4,3)$

If $m+n$ is odd and $m+n \geq 11$, we construct OKTs from $(1,2)$ to $(1,3)$ containing the edge $(m-2,n-1)-(m,n)$ on some small LB$(m,n,3)$ according to the remainders of m and n after divided by 4 as the following Figures 2.3 - 2.6.



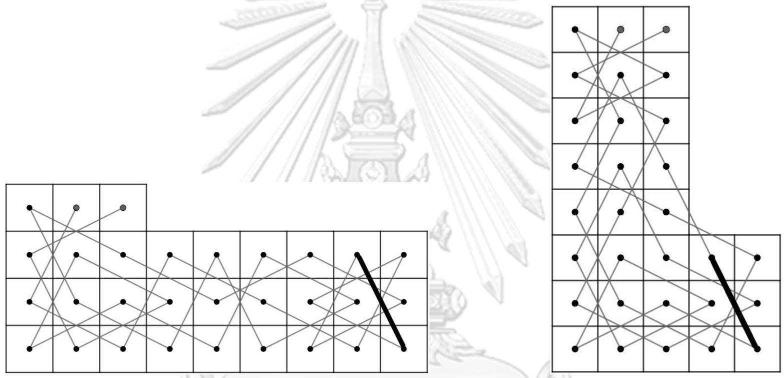
(a) (b)

Figure 2.3: OKTs on the LB$(4,7,3)$ and LB$(7,4,3)$



(a) (b)

Figure 2.4: OKTs on the LB$(6, 5, 3)$ and LB$(5, 6, 3)$



(a) (b)

Figure 2.5: OKTs on the LB$(4, 9, 3)$ and LB$(8, 5, 3)$

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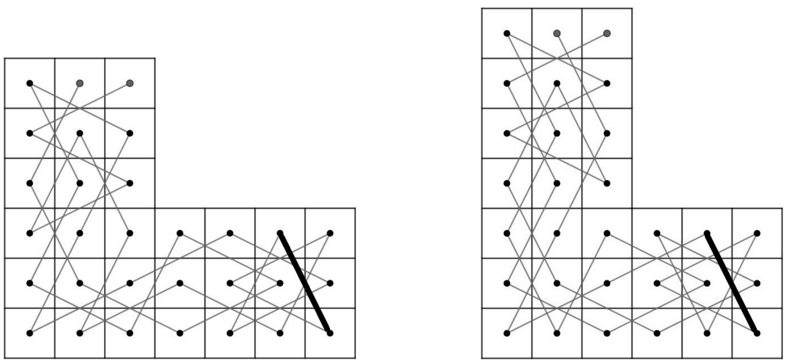

(a) (b)

Figure 2.6: OKTs on the LB$(6, 7, 3)$ and LB$(7, 6, 3)$

Next, for the larger L-boards, denoted by LB, we start by constructing an OKT on CB(3×4) from $(1,1)$ to $(2,1)$ that contains an edge $(1,3)-(3,4)$ as shown in Figure 2.7.

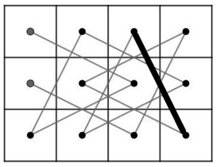

Figure 2.7: An OKT on CB(3×4)

Then, we construct an OKT on the CB($3 \times 4t$), where $t \geq 2$. Let us connect t CB(3×4)'s in Figure 2.7 to the right of each other and do the following.

(i) For $1 \leq i \leq t-1$, delete $(1,3)-(3,4)$ from the OKT of the ith CB(3×4);

(ii) For $1 \leq i \leq t-1$, join $(1,3)$ and $(3,4)$ of the ith CB(3×4) to $(2,1)$ and $(1,1)$ of the $(i+1)$th CB(3×4), respectively.

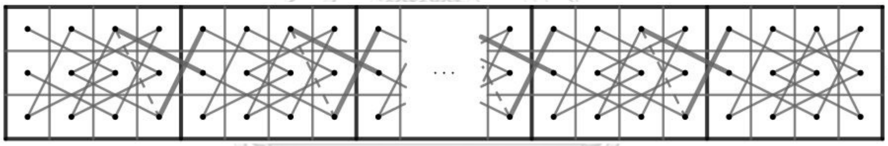

Figure 2.8: An OKT on CB($3 \times 4t$)

By rotating Figure 2.8 clockwise for 90 degrees, we also obtain an OKT on CB($4s \times 3$) from $(1,2)$ to $(1,3)$ as shown in Figure 2.9.

Now, we are ready to construct an OKT on a larger LB by placing the CB($3 \times 4t$) to the right and the CB($4s \times 3$) above each smaller LB in Figures 2.2 - 2.6, respectively.

Since $m+n$ is odd, $m, n \geq 4$ and $(m,n) \neq (4,5)$, there exist nonnegative integers s, t such that $m = a+4s$ and $n = b+4t$ where $(a,b) \in \{(5,4),(4,7),(7,4),$ $(6.5),(5,6),(4,9),(8,5),(6,7),(7,6)\}$. We divided the LB$(m,n,3)$ into subboards, CB($4s \times 3$) (Figure 2.9) and LB$(a,b,3)$ (Figures 2.2 - 2.6) if $s > 0$ and $t = 0$ and LB$(a,b,3)$ (Figures 2.2 - 2.6) and CB($3 \times 4t$) (Figure 2.8) if $s = 0$ and $t > 0$. Otherwise, we divide into three subboards, CB($4s \times 3$) (Figure 2.9), LB$(a,b,3)$

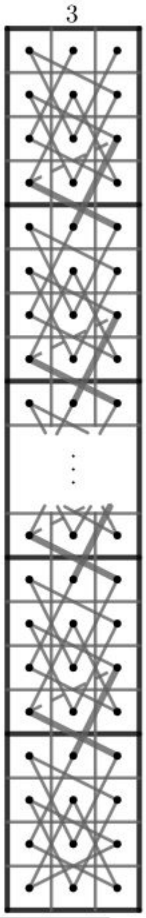

Figure 2.9: An OKT on CB($4s \times 3$)

(Figures 2.2 - 2.6) and CB($3 \times 4t$) (Figure 2.8). Then, we construct the required OKT by the following two steps.

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(i) If $s > 0$ and $t = 0$, then delete $(4s, 1) - (4s - 1, 3)$ of the OKT on the CB($4s \times 3$) in Figure 2.9. If $s = 0$ and $t > 0$, then delete $(a-2, b-1) - (a, b)$ of the OKT on the LB($a, b, 3$) in Figures 2.2 - 2.6. Otherwise, delete both edges.

(ii) If $s > 0$ and $t = 0$, then join $(4s, 1)$ and $(4s - 1, 3)$ of the CB($4s \times 3$) to $(1, 3)$ and $(1, 2)$ of the LB($a, b, 3$), respectively. If $s = 0$ and $t > 0$, then join $(a-2, b-1)$ and (a, b) of the LB($a, b, 3$) to $(2, 1)$ and $(1, 1)$ of the CB($3 \times 4t$) chessboard, respectively. Otherwise, join four pairs of vertices together.

This completes the proof of (a).

(b) We assume that the LB$(m, n, 3)$ contains an OKT from $(1, 3)$ to $(2, 2)$ and let $m + n$ be odd or $(m, n) \in \{(4, 4), (4, 6), (5, 5)\}$.

If $m+n$ is odd, then the numbers of white square and black square are the same. Thus, the two end-points of this OKT must have the different color. However, $(1, 3)$ and $(2, 2)$ have the same color, contradiction. Next, we consider the cases that $(m, n) = (4, 4)$; or $(m, n) = (4, 6)$; or $(m, n) = (5, 5)$.

For $(m, n) = (4, 4)$, let G_1 be the knight graph of the LB$(4, 4, 3)$. Consider $G_1' = G_1 - \{(1, 3), (2, 2)\}$. Since the LB$(4, 4, 3)$ contains an OKT from $(1, 3)$ to $(2, 2)$, G_1' has a Hamiltonian path. Let $S = \{(2, 3), (3, 2), (3, 3)\}$. Then, $\omega(G_1' - S) = 5 > 4 = |S| + 1$ as shown in Figure 2.10. By Theorem 1.5(b), we obtain a contradiction.

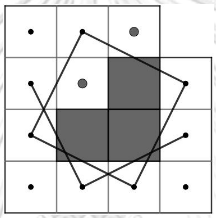

Figure 2.10: Components of $G_1' - S$

For $(m, n) = (4, 6)$, let G_2 be the knight graph of the LB$(4, 6, 3)$. Consider $G_2' = G_2 - \{(1, 3), (2, 2)\}$. Since the LB$(4, 6, 3)$ contains an OKT from $(1, 3)$ to $(2, 2)$, G_2' has an Hamiltonian path. Let $S = \{(2, 3), (2, 4), (3, 3), (3, 4), (4, 3), (4, 4)\}$. Then, $\omega(G_2' - S) = 8 > 7 = |S| + 1$ as shown in Figure 2.11. By Theorem 1.5(b), we obtain a contradiction.

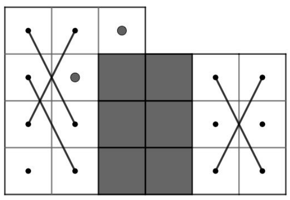

Figure 2.11: Components of $G_2' - S$

For $(m, n) = (5, 5)$, let G_3 be the knight graph of the LB$(5, 5, 3)$. Consider $G_3' = G_3 - \{(1, 3), (2, 2)\}$. Since the LB$(5, 5, 3)$ contains an OKT from $(1, 3)$ to $(2, 2)$, G_3' has an Hamiltonian path. Let $S = \{(2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (5, 3)\}$. Then,

$\omega(G'_3 - S) = 8 > 7 = |S| + 1$ as shown in Figure 2.12. By Theorem 1.5(b), we obtain a contradiction.

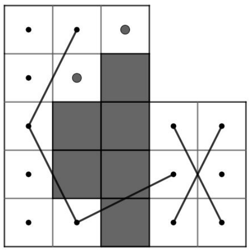

Figure 2.12: Components of $G'_3 - S$

On the other hand, let us assume that $m + n$ is even and $(m, n) \notin \{(4, 4), (4, 6), (5, 5)\}$.

If $m + n$ is even and $m + n < 12$, then $(m, n) = (6, 4)$. Then the required OKT on the LB$(6, 4, 3)$ from $(1, 3)$ to $(2, 2)$ that contains an edge $(4, 3) - (6, 4)$ is presented in Figure 2.13.

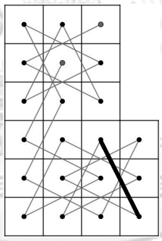

Figure 2.13: Required OKT on the LB$(6, 4, 3)$

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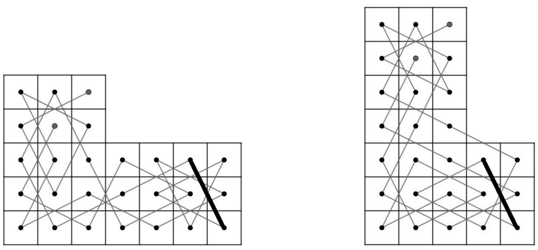
If $m + n$ is even and $m + n \geq 12$, we construct OKTs from $(1, 3)$ to $(2, 2)$ containing the edge $(m-2, n-1) - (m, n)$ on some small LB$(m, n, 3)$ according to the remainders of m and n after divided by 4 as the following Figures 2.14 - 2.19.



(a) (b)

Figure 2.14: OKTs on the LB$(5, 7, 3)$ and LB$(7, 5, 3)$

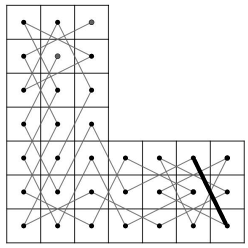
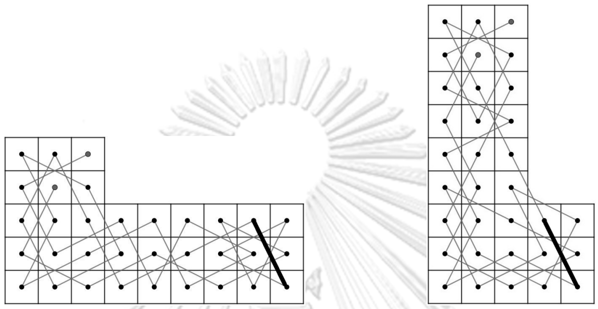


Figure 2.15: An OKT on the LB$(7,7,3)$



(a) (b)

Figure 2.16: OKTs on the LB$(5,9,3)$ and LB$(9,5,3)$

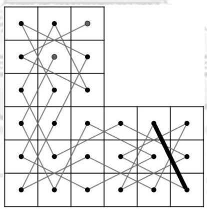
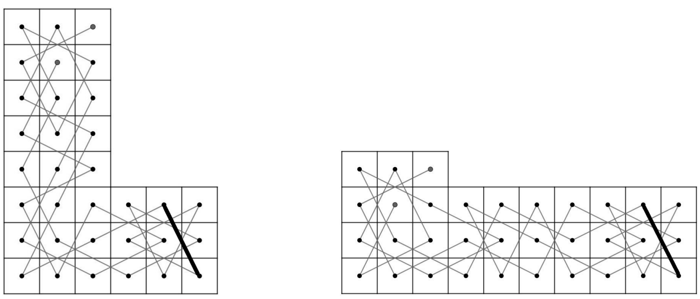
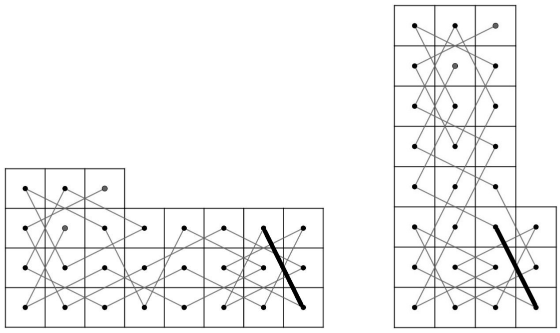


Figure 2.17: An OKT on the LB$(6,6,3)$



(a) (b)

Figure 2.18: OKTs on the LB$(8,6,3)$ and LB$(4,10,3)$



(a) (b)

Figure 2.19: OKTs on the LB$(4, 8, 3)$ and LB$(8, 4, 3)$

Next, for the larger LBs, we start by constructing an OKT on CB(4×3) from $(1, 3)$ to $(4, 1)$ and contains an edge $(2, 2) - (4, 3)$ as shown in Figure 2.20.

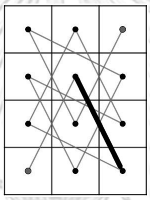


Figure 2.20: An OKT on CB(4×3)

Then, we construct two paths on CB$(4s \times 3)$, where $s \geq 2$. Let us connect s CB(4×3)'s in Figure 2.20 on the top of each other and do the following.

(i) For $1 \leq i \leq s$, delete $(2, 2) - (4, 3)$ from the OKT of the ith CB(4×3);

(ii) For $1 \leq i \leq s - 1$, join $(4, 1)$ and $(4, 3)$ of the ith CB(4×3) to $(1, 3)$ and $(2, 2)$ of the $(i + 1)$th CB(4×3), respectively.

We can see from Figure 2.21 that either s is odd or s is even, there is one path that has $(1, 3)$ as its end-point and another path that has $(2, 2)$ as its end-point.

Now, we are ready to construct an OKT on a larger LB by placing the CB$(3\times 4t)$ in Figure 2.8 to right and the CB$(4s \times 3)$ above each smaller LB in Figures 2.13 - 2.19, respectively.

Since $m + n$ is even, $m, n \geq 4$ and $(m, n) \notin \{(4, 4), (4, 6), (5, 5)\}$, there exist nonnegative integers s, t such that $m = a + 4s$ and $n = b + 4t$ where $(a, b) \in$

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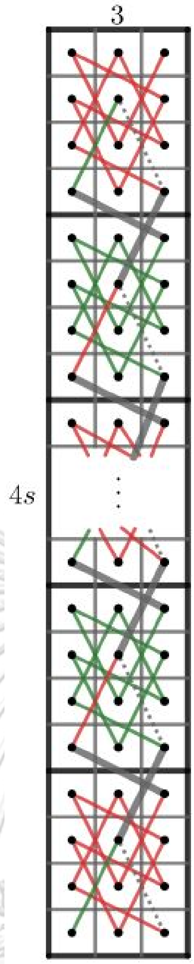

Figure 2.21: Two paths on $4s \times 3$ chessboard

$\{(6,4),(5,7),(7,5),(7.7),(5,9),(9,5),(6,6),(8,6),(4,10),(4,8),(8,4)\}$. We divide the LB$(m,n,3)$ into subboards, CB$(4s \times 3)$ (Figure 2.21) and LB$(a,b,3)$ (Figures 2.13 - 2.19) if $s > 0$ and $t = 0$ and LB$(a,b,3)$ (Figures 2.13 - 2.19) and CB$(3 \times 4t)$ (Figure 2.8) if $s = 0$ and $t > 0$. Otherwise, we divide into three subboards, CB$(4s \times 3)$ (Figure 2.21), LB$(a,b,3)$ (Figures 2.13 - 2.19) and CB$(3 \times 4t)$ (Figure 2.8). Then, we construct the required OKT by the followings.

(i) If $s \geq 0$ and $t > 0$, then delete $(a-2,b-1)-(a,b)$ of the OKT on the LB$(a,b,3)$ in Figures 2.13 - 2.19.

(ii) If $s > 0$ and $t = 0$, then join $(4s,1)$ and $(4s,3)$ of the CB$(4s \times 3)$ in Figure 2.21 to $(1,3)$ and $(2,2)$ of the LB$(a,b,3)$, respectively. If $s = 0$ and $t > 0$, then join $(a-2,b-1)$ and (a,b) of the LB$(a,b,3)$ to $(2,1)$ and $(1,1)$ of the

CB($3 \times 4t$) in Figure 2.8, respectively. Otherwise, join four pairs of vertices together.

This completes the proof. $\square$

Next, we get the following Corollary by flipping vertically and rotating 90 degrees clockwise the LB in the above Theorem.

Corollary 2.2. *Let $m, n \geq 4$.*

(a) *The 7B$(m, n, 3)$ contains an OKT from $(2, 1)$ to $(3, 1)$ if and only if $m + n$ is odd and $(m, n) \neq (5, 4)$.*

(b) *The 7B$(m, n, 3)$ contains an OKT from $(3, 1)$ to $(2, 2)$ if and only if $m + n$ is even and $(m, n) \notin \{(4, 4), (5, 5), (6, 4)\}$.*

We note that Theorem 2.1(b) and Corollary 2.2(b) will be used in Case 1.1 of Theorem 3.4 in Section 3.2 of Chapter III. While, Theorem 2.1(a) and Corollary 2.2(a) will be used in Case 1.2 of Theorem 3.4 in Section 3.2 of Chapter III.

2.2 Existence of Some Special OKTs on CB($m \times n$)

The following theorem gives necessary and sufficient conditions on the existence of some special OKTs on CB($m \times n$) from $(m, 1)$ to $(2, n-1)$. This OKT will be used to prove the existence of a CKT on RB(m, n, r) for $r \geq 5$ when r is odd (Case 3.2 of Theorem 3.4 in Section 3.2 of Chapter III).

Theorem 2.3.

(a) *Let $m \leq 4$ and $n \geq m$. Then, a CB($m \times n$) contains an OKT from $(m, 1)$ to $(2, n-1)$ if and only if $m = 3$ and $n \geq 7$.*

(b) *Let $n \geq m \geq 5$. Then, a CB($m \times n$) contains an OKT from $(m, 1)$ to $(2, n-1)$ if and only if m and n are not both even.*

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Proof. (a) Let $m \leq 4$. We assume that a CB$(m \times n)$ contains an OKT from $(m, 1)$ to $(2, n-1)$ and let $m \neq 3$; or $n \leq 6$. Then, we consider four cases as follows.

Case 1: $m = 1$ and $n \geq 1$; or $m = 2$ and $n \geq 2$; or $m = 3$ and $n \in \{3, 5, 6\}$. A CB$(m \times n)$ contains no OKT by using Theorem 1.7, contradiction.

Case 2: $m = 3$ and $n = 4$. Let G_1 be the knight graph of the CB(3×4). We assume that G_1 contains a Hamiltonian path from $(3, 1)$ to $(2, 3)$. Consider $G_1' = G_1 - \{(2, 3)\}$. By assumption, G_1' has a Hamiltonian path. Let $S = \{(1, 2), (3, 2)\}$. Then, $\omega(G_1' - S) = 4 > 3 = |S| + 1$ as shown in Figure 2.22. By Theorem 1.5(b), we obtain a contradiction.

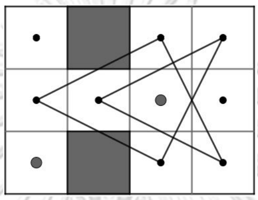


Figure 2.22: Components of $G_1' - S$

Case 3: $m = 4$ and n is odd such that $n \geq 5$. Let G_2 be the knight graph of the CB$(4 \times n)$. We assume that G_2 contains a Hamiltonian path from $(m, 1)$ to $(2, n-1)$. Consider $G_2' = G_2 - \{(2, n-1)\}$. Let $S = \{(2, j), (3, l) \mid j \text{ is even, } 2 \leq j \leq n-3, l \text{ is odd and } 1 \leq l \leq n\}$. Then, $\omega(G_2' - S) = n + 1 > n = |S| + 1$ as shown in Figure 2.23. By Theorem 1.5(b), we have a contradiction.

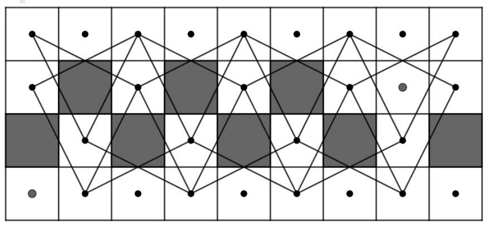


Figure 2.23: Components of $G_2' - S$, where $n = 9$

Case 4: $m = 4$ and n is even such that $n \geq 4$. Assume that CB$(4 \times n)$ contains an OKT from $(4, 1)$ to $(2, n-1)$. Since CB$(4 \times n)$ contains the same numbers of black and white squares, this OKT must have end-points at two squares with different colors. However, $4 + 1 = 5$ and $2 + (n-1) = n + 1$ are odd. Thus, $(m, 1)$ and $(2, n-1)$ are two squares of the same color, contradiction.

On the other hand, let us assume that $m = 3$ and $n \geq 7$.

Let us construct OKTs from $(3, 1)$ to $(2, n-1)$ on some small size $\mathrm{CB}(3 \times n)$ where $n \in \{7, 8, 9, 10\}$ as shown in Figure 2.24.

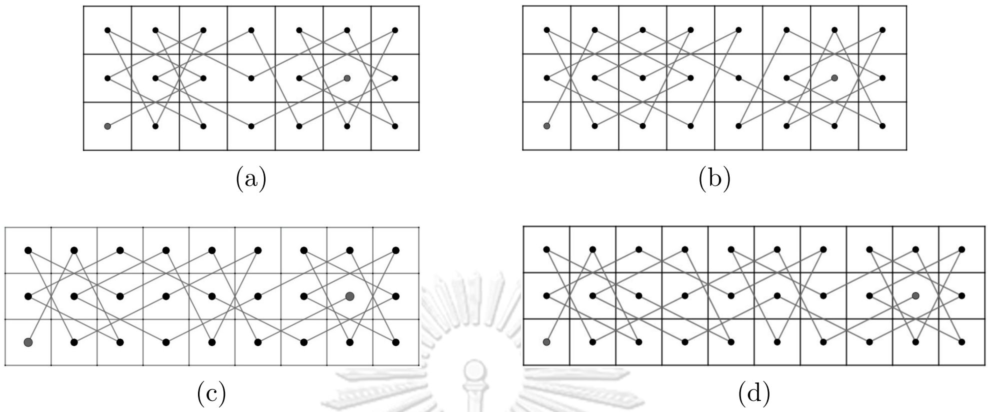


Figure 2.24: OKTs from $(3, 1)$ to $(2, n-1)$ on the $\mathrm{CB}(3 \times n)$ where $n \in \{7, 8, 9, 10\}$

Before we continue further, let us rotate the $\mathrm{CB}(4 \times 3)$ shown in Figure 2.20 for 90 degrees clockwise and flip it to obtain an OKT from $(3, 1)$ to $(1, 4)$ on $\mathrm{CB}(3 \times 4)$. We can place t of these $\mathrm{CB}(3 \times 4)$ to the right of each other to extend this OKT into an OKT on $\mathrm{CB}(3 \times 4t)$ by connecting $(1, 4)$ on the ith $\mathrm{CB}(3 \times 4)$ to $(3, 1)$ on the $(i + 1)$th $\mathrm{CB}(3 \times 4)$ for all $1 \leq i \leq t - 1$ as shown in Figure 2.25. Note that this extended OKT starts from $(3, 1)$ to $(1, 4t)$.

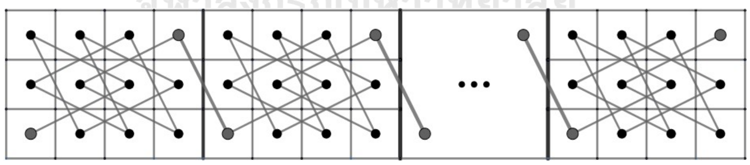


Figure 2.25: An OKT from $(3, 1)$ to $(1, 4t)$ on $\mathrm{CB}(3 \times 4t)$

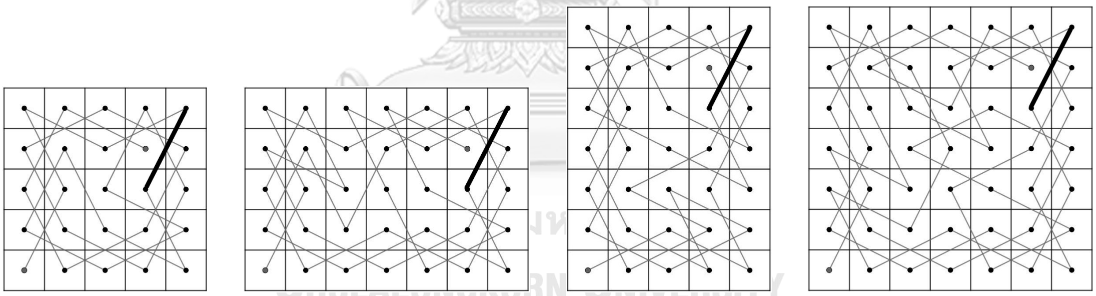
Next, let n be a positive integer such that $n \geq 11$.

If $n \equiv 3 \pmod 4$ (respectively, $n \equiv 0 \pmod 4$, $n \equiv 1 \pmod 4$, $n \equiv 2 \pmod 4$), then there is a positive integer t such that $n = 7 + 4t$ (respectively, $n = 8 + 4t, n = 9 + 4t, n = 10 + 4t$). We divide the $\mathrm{CB}(3 \times n)$ into subboards, $\mathrm{CB}(3 \times 4t)$ (Figure 2.25) and $\mathrm{CB}(3 \times 7)$ (Figure 2.24(a)) (respectively, $\mathrm{CB}(3 \times 8)$ (Figure 2.24(b)), $\mathrm{CB}(3 \times 9)$ (Figure 2.24(c)), $\mathrm{CB}(3 \times 10)$ (Figure 2.24(d))). Then, we construct the required OKT by connecting $(1,4t)$ of the OKT on the CB$(3\times 4t)$ in Figure 2.25 to $(3,1)$ of the OKT on the CB(3×7) in Figure 2.24(a) (respectively, CB(3×8) in Figure 2.24(b), CB(3×9) in Figure 2.24(c), CB(3×10) in Figure 2.24(d)).

(b) Let $n \geq m \geq 5$. We assume that a CB$(m\times n)$ contains an OKT from $(m,1)$ to $(2,n-1)$ and let m and n are both even. Since CB$(m\times n)$ contains the same numbers of black and white squares, this OKT must have end-points at two squares with different colors. However, $m+1$ and $2+(n-1)=n+1$ are odd. Thus, $(m,1)$ and $(2,n-1)$ are two squares of the same color, contradiction.

On the other hand, let us assume that m and n are not both even such that $n \geq m \geq 5$. Then, we consider three cases as follows.

Case 1: m and n are both odd such that $m,n \geq 5$. Let us construct OKTs from $(m,1)$ to $(2,n-1)$ containing the edge $(1,n)-(3,n-1)$ on some small size CB$(m\times n)$ where $m,n \in \{5,7\}$ as shown in Figure 2.26.

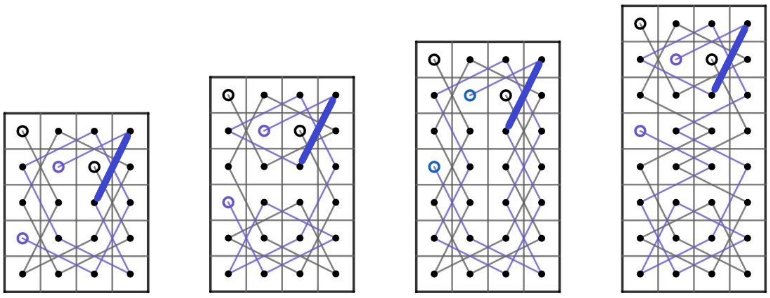


(a) (b) (c) (d)

Figure 2.26: OKTs from $(m,1)$ to $(2,n-1)$ on the CB$(m\times n)$ where $m,n \in \{5,7\}$

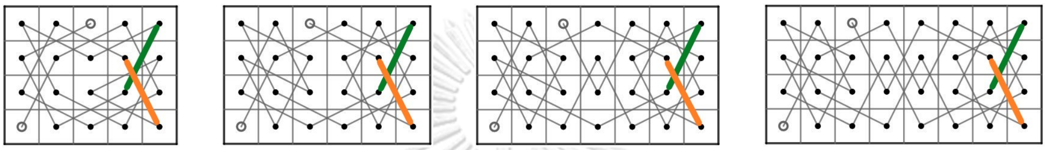
For the larger CB$(m\times n)$, we start by constructing two paths on CB$(m\times 4)$. The first path starts from $(1,1)$ to $(2,3)$ and the second path starts from $(2,2)$ to $(4,1)$ containing the edge $(1,4)-(3,3)$ where $n\in\{5,6,7,8\}$ as shown in Figure 2.27.

Next, we construct an OKT from $(1,3)$ to $(4,1)$ containing the edges $(1,n)-(3,n-1)$ and $(2,n-1)-(4,n)$ on the CB$(4\times n)$ where $n\in\{5,6,7,8\}$ as shown in Figure 2.28.



(a) (b) (c) (d)

Figure 2.27: Two paths on CB($m \times 4$) where $m \in \{5, 6, 7, 8\}$



(a) (b) (c) (d)

Figure 2.28: OKTs on CB($4 \times n$) where $n \in \{5, 6, 7, 8\}$

Let m, n be odd integers such that $m \geq 5$ and $n \geq 9$. We consider four cases according to the remainders after dividing m and n by 4.

Case 1.1: $m, n \equiv 1 \pmod 4$. There are integers s and t such that $0 \leq s \leq t$, $t \neq 0$, $m = 5 + 4s$ and $n = 5 + 4t$. If $s = 0$, then we divide the CB($5 \times n$) into subboards, CB(5×5) (Figure 2.26(a)) and t CB(5×4)'s (Figure 2.27(a)). Then, we construct the required OKT by the followings.

(i) We delete $(1, 5) - (3, 4)$ of the OKT on the CB(5×5) and connect $(2, 4), (1, 5)$ and $(3, 4)$ of the CB(5×5) to $(1, 1), (2, 2)$ and $(4, 1)$ of the 1st CB(5×4), respectively.

(ii) We delete $(1, 4) - (3, 3)$ of the second path of the ith CB(5×4) for all $1 \leq i \leq t - 1$. Then, we connect $(2, 3), (1, 4)$ and $(3, 3)$ of the ith CB(5×4) to $(1, 1), (2, 2)$ and $(4, 1)$ of the $(i + 1)$th CB(5×4).

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($5 \times n$) (Figure 2.29) and s CB($4 \times n$)'s from the top to the bottom. We start by constructing two paths P_1 (dash line) and P_2 (solid line) on the CB(4×4) as shown in Figures 2.30.

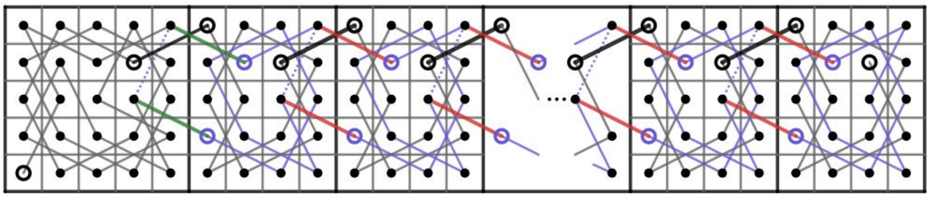


Figure 2.29: An OKT on CB($5 \times n$) in Case 1.1

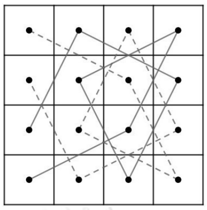


Figure 2.30: Two paths P_1 and P_2 on the CB(4×4)

Then, we construct two paths P_1' and P_2' on the CB($4 \times 4t$) where $t \geq 2$. Let us connect t CB(4×4)'s in Figure 2.30 to the right of each other and do the followings.

(a) For $1 \leq i \leq t-1$, delete $(2,3)-(4,4)$ from P_1 and $(1,4)-(3,3)$ from P_2 of the ith CB(4×4).

(b) For $1 \leq i \leq t-1$, join $(2,3)$ and $(4,4)$ of the ith CB(4×4) to $(1,1)$ and $(2,1)$ of the $(i+1)$th CB(4×4), respectively

(c) For $1 \leq i \leq t-1$, join $(1,4)$ and $(3,3)$ of the ith CB(4×4) to $(3,1)$ and $(4,1)$ of the $(i+1)$th CB(4×4), respectively.

Thus, we have two paths P_1' and P_2' on the CB($4 \times 4t$) as shown in Figure 2.31, where $t \geq 2$.

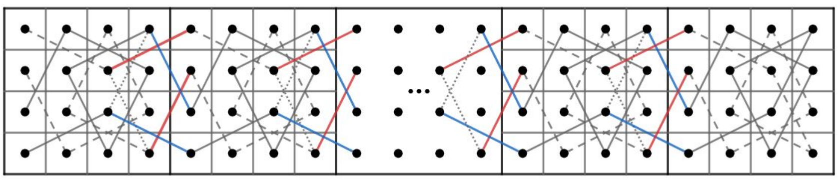


Figure 2.31: Two paths P_1' and P_2' on the CB($4 \times 4t$)

Next, we construct the required OKT on CB($m \times n$) as follows.

(i′) For each $1 \leq i \leq s$, we divide the ith CB($4 \times n$) into subboards, CB(4×5) (Figure 2.28(a)) and CB($4 \times 4t$) (Figure 2.31). Delete $(1,5)-(3,4)$ and

$(2,4)-(4,5)$ of the OKT on CB(4×5). Then, join $(2,4),(4,5),(1,5)$ and $(3,4)$ of the CB(4×5) to $(1,1),(2,1),(3,1)$ and $(4,1)$ of the CB$(4\times 4t)$, respectively, to obtain an OKT on the CB$(4\times n)$ as shown in Figure 2.32.

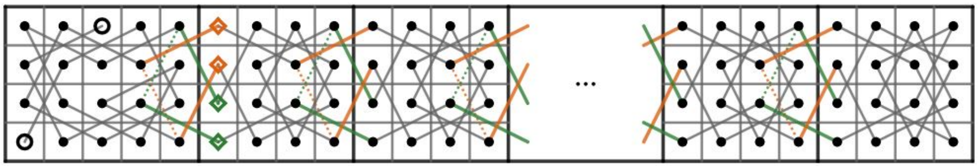


Figure 2.32: An OKT on CB$(4\times n)$ in Case 1.1

(ii′) Join $(5,1)$ of the OKT on CB$(5\times n)$ (Figure 2.29) to $(1,3)$ of the 1st CB$(4\times n)$ (Figure 2.32).

(iii′) For each $1\le i\le s-1$, we join $(4,1)$ of the OKT on the ith CB$(4\times n)$ (Figure 2.32) to $(1,3)$ of the OKT on the $(i+1)$th CB$(4\times n)$ (Figure 2.32).

Case 1.2: $m\equiv 1 \pmod 4$ and $n\equiv 3 \pmod 4$. There are integers s and t such that $0\le s\le t$, $t\ne 0$, $m=5+4s$ and $n=7+4t$. If $s=0$, then we divide the CB$(5\times n)$ into subboards, CB(5×7) (Figure 2.26(b)) and t CB(5×4)'s (Figure 2.27(a)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) by CB(5×7) (Figure 2.26(b)) and $(1,5)$, $(3,4)$ and $(2,4)$ by $(1,7)$, $(3,6)$ and $(2,6)$, respectively.

If $s>0$, then we divide the CB$(m\times n)$ into subboards, CB$(5\times n)$ (Figure 2.33) and s CB$(4\times n)$'s (Figure 2.34) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (i′) replace CB(4×5) by CB(4×7) (Figure 2.28(c)) and $(1,5)$, $(3,4)$, $(2,4)$ and $(4,5)$ by $(1,7)$, $(3,6)$, $(2,6)$ and $(4,7)$, respectively.

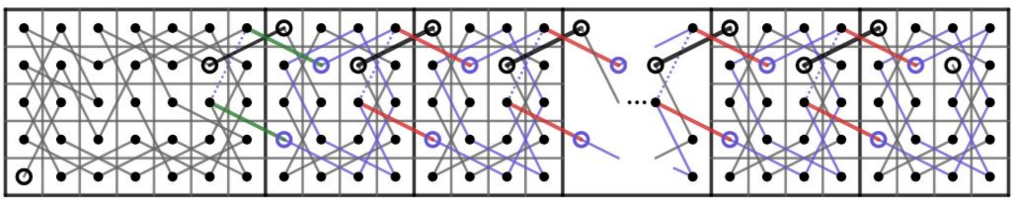


Figure 2.33: An OKT on CB$(5\times n)$ in Case 1.2

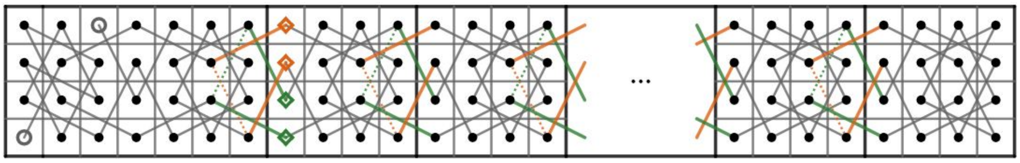


Figure 2.34: An OKT on CB($4 \times n$) in Case 1.2

Case 1.3: $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$. There are integers s and t such that $0 \leq s < t$, $m = 7 + 4s$ and $n = 5 + 4t$. If $s = 0$, then we divide the CB($7 \times n$) into subboards, CB(7×5) (Figure 2.26(c)) and t CB(7×4)'s (Figure 2.27(c)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) and CB(5×4) by CB(7×5) (Figure 2.26(c)) and CB(7×4) (Figure 2.27(c)), respectively.

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($7 \times n$) (Figure 2.35) and s CB($4 \times n$) (Figure 2.32) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (ii′) replace CB($5 \times n$) by CB($7 \times n$) (Figure 2.35) and $(5, 1)$ by $(7, 1)$.

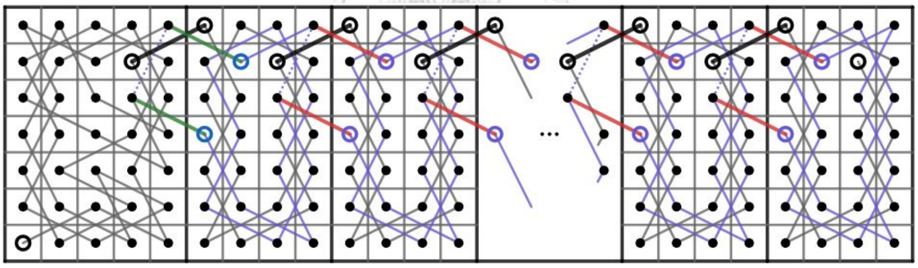


Figure 2.35: An OKT on CB($7 \times n$) in Case 1.3

Case 1.4: $m \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$. There are integers s and t such that $0 \leq s \leq t$, $t \neq 0$, $m = 7 + 4s$ and $n = 7 + 4t$. If $s = 0$, then we divide the CB($7 \times n$) into subboards, CB(7×7) (Figure 2.26(d)) and t CB(7×4)'s (Figure 2.27(c)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) and CB(5×4) by CB(7×7) (Figure 2.26(d)) and CB(7×4) (Figure 2.27(c)) and $(1, 5)$, $(3, 4)$ and $(2, 4)$ by $(1, 7)$, $(3, 6)$ and $(2, 6)$, respectively.

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($7 \times n$) (Figure 2.36) and s CB($4 \times n$)'s (Figure 2.34) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (i′) and (ii′) replace CB(4×5) and CB($5 \times n$) by CB(4×7) (Figure 2.28(c)) and CB($7 \times n$) (Figure 2.36) and replace $(1,5)$, $(3,4)$, $(2,4)$, $(4,5)$ and $(5,1)$ by $(1,7)$, $(3,6)$, $(2,6)$, $(4,7)$ and $(7,1)$, respectively.

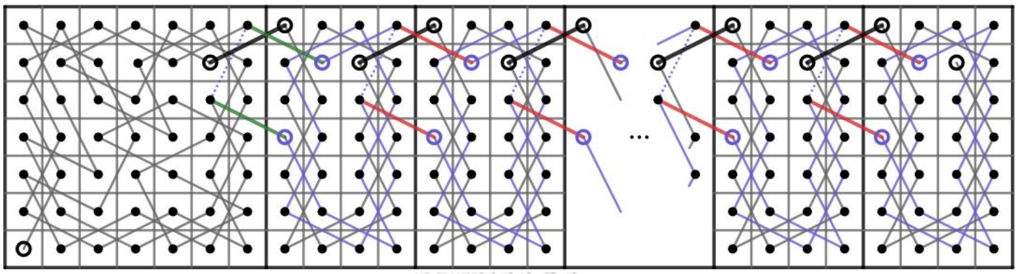
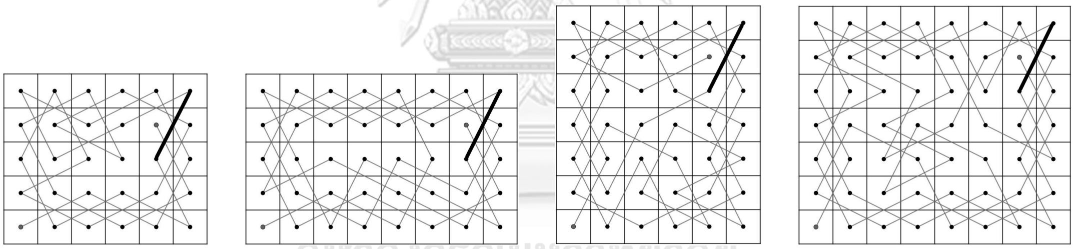


Figure 2.36: An OKT on CB($7 \times n$) in Case 1.4

Case 2: m is odd such that $m \geq 5$ and n is even such that $n \geq 6$. Let us construct OKTs from $(m,1)$ to $(2,n-1)$ containing the edge $(1,n)-(3,n-1)$ on some small size CB($m \times n$) where $m \in \{5,7\}$ and $n \in \{6,8\}$ as shown in Figure 2.37.



(a) (b) (c) (d)

Figure 2.37: OKTs from $(m,1)$ to $(2,n-1)$ on the CB($m \times n$) where $m \in \{5,7\}$ and $n \in \{6,8\}$

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Let m be an odd integer such that $m \geq 5$ and let n be an even integer such that $n \geq 10$. We consider four cases according to the remainders after dividing m and n by 4.

Case 2.1: $m \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 4$. There are integers s and t such that $0 \leq s \leq t$, $t \neq 0$, $m = 5 + 4s$ and $n = 8 + 4t$. If $s = 0$, then we divide the CB($5 \times n$) into subboards, CB(5×8) (Figure 2.37(b)) and t CB(5×4)'s (Figure 2.27(a)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) by CB(5×8) (Figure 2.37(b)) and $(1,5)$, $(3,4)$ and $(2,4)$ by $(1,8)$, $(3,7)$ and $(2,7)$, respectively.

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($5 \times n$) (Figure 2.38) and s CB($4 \times n$)'s (Figure 2.39) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (i′) replace CB(4×5) by CB(4×8) (Figure 2.28(d)) and $(1,5)$, $(3,4)$, $(2,4)$ and $(4,5)$ by $(1,8)$, $(3,7)$, $(2,7)$ and $(4,8)$, respectively.

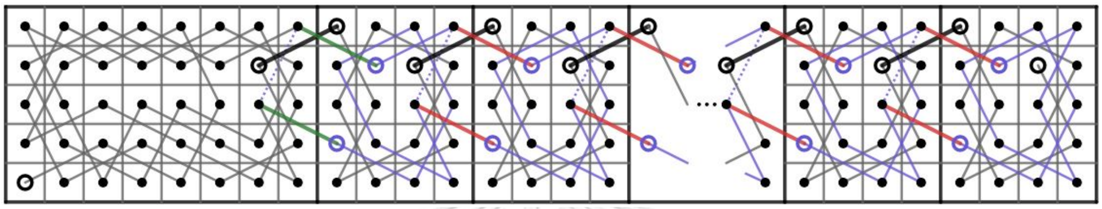

Figure 2.38: An OKT on CB($5 \times n$) in Case 2.1

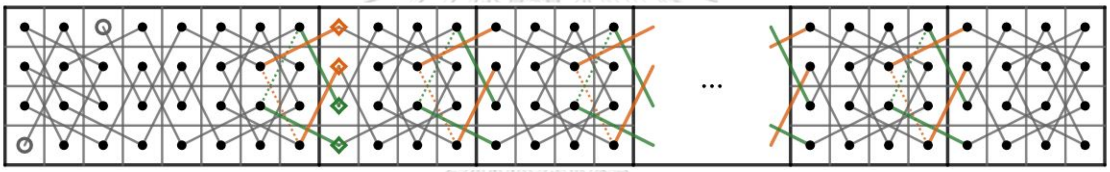

Figure 2.39: An OKT on CB($4 \times n$) in Case 2.1

Case 2.2: $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$. There are integers s and t such that $0 \leq s \leq t$, $t \neq 0$, $m = 5 + 4s$ and $n = 6 + 4t$. If $s = 0$, then we divide the CB($5 \times n$) into subboards, CB(5×6) (Figure 2.37(a)) and t CB(5×4)'s (Figure 2.27(a)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) by CB(5×6) (Figure 2.37(a)) and $(1,5)$, $(3,4)$ and $(2,4)$ by $(1,6)$, $(3,5)$ and $(2,5)$, respectively.

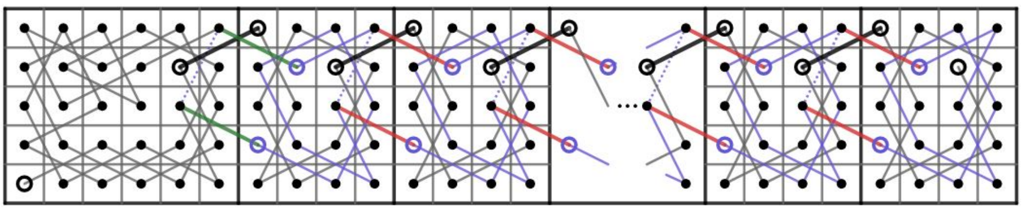

Figure 2.40: An OKT on CB($5 \times n$) in Case 2.2

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($5 \times n$) (Figure 2.40) and s CB($4 \times n$)'s (Figure 2.41) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (i′) replace CB(4×5) by CB(4×6) (Figure 2.28(b)) and $(1, 5)$, $(3, 4)$, $(2, 4)$ and $(4, 5)$ by $(1, 6)$, $(3, 5)$, $(2, 5)$ and $(4, 6)$, respectively.

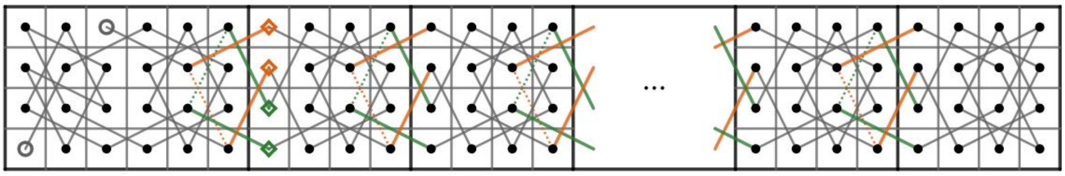

Figure 2.41: An OKT on CB($4 \times n$) in Case 2.2

Case 2.3 $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$. There are integers s and t such that $0 \le s \le t$, $t \ne 0$, $m = 7 + 4s$ and $n = 8 + 4t$. If $s = 0$, then we divide CB($7 \times n$) into subboards, CB(7×8) (Figure 2.37(d)) and t CB(7×4)'s (Figure 2.27(c)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) and CB(5×4) by CB(7×8) (Figure 2.37(d)) and CB(7×4) (Figure 2.27(c)) and $(1, 5)$, $(3, 4)$ and $(2, 4)$ by $(1, 8)$, $(3, 7)$ and $(2, 7)$, respectively.

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($7 \times n$) (Figure 2.42) and s CB($4 \times n$)'s (Figure 2.39) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (i′) and (ii′) replace CB(4×5) and CB($5 \times n$) by CB(4×8) (Figure 2.28(d)) and CB($7 \times n$) (Figure 2.42) and $(1, 5)$, $(3, 4)$, $(2, 4)$ and $(4, 5)$ by $(1, 8)$, $(3, 7)$, $(2, 7)$ and $(4, 8)$, respectively.

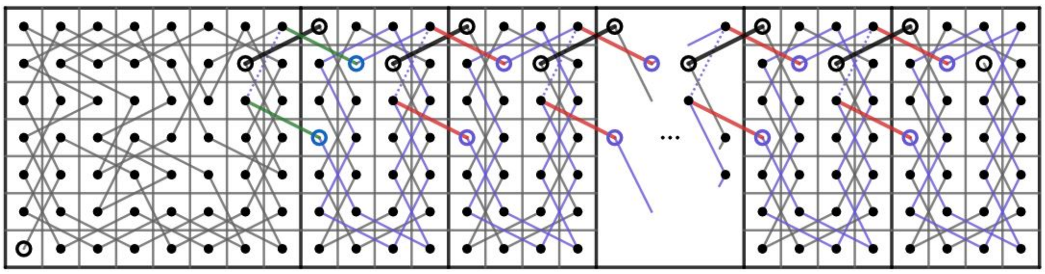

Figure 2.42: An OKT on CB($7 \times n$) in Case 2.3

Case 2.4: $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$. There are integers s and t

such that $0 \leq s < t$, $m = 7 + 4s$ and $n = 6 + 4t$. If $s = 0$, then we divide the CB$(7 \times n)$ into subboards, CB(7×6) (Figure 2.37(c)) and t CB(7×4)'s (Figure 2.27(c)). Then, we construct the required OKT by (i) and (ii) in Case 1.1. but replace CB(5×5) and CB(5×4) by CB(7×6) (Figure 2.37(c)) and CB(7×4) (Figure 2.27(c)) and $(1, 5)$, $(3, 4)$ and $(2, 4)$ by $(1, 6)$, $(3, 5)$ and $(2, 5)$, respectively.

If $s > 0$, then we divide the CB$(m \times n)$ into subboards, CB$(7 \times n)$ (Figure 2.43) and s CB$(4 \times n)$'s (Figure 2.41) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1. but in (i′) and (ii′) replace CB(4×5) and CB$(5 \times n)$ by CB(4×6) (Figure 2.28(b)) and CB$(7 \times n)$ (Figure 2.43) and $(1, 5)$, $(3, 4)$, $(2, 4)$, $(4, 5)$ and $(5, 1)$ by $(1, 6)$, $(3, 5)$, $(2, 5)$, $(4, 6)$ and $(7, 1)$, respectively.

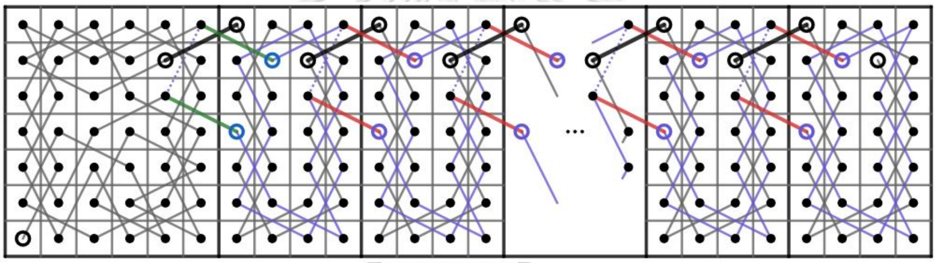

Figure 2.43: An OKTs on CB$(7 \times n)$ in Case 2.4

Case 3: m is even such that $m \geq 6$ and n is odd such that $n \geq 5$. Let us construct OKTs from $(m, 1)$ to $(2, n-1)$ containing the edge $(1, n) - (3, n-1)$ on some small size CB$(m \times n)$ where $m \in \{6, 8\}$ and $n \in \{5, 7\}$ as shown in Figure 2.44.

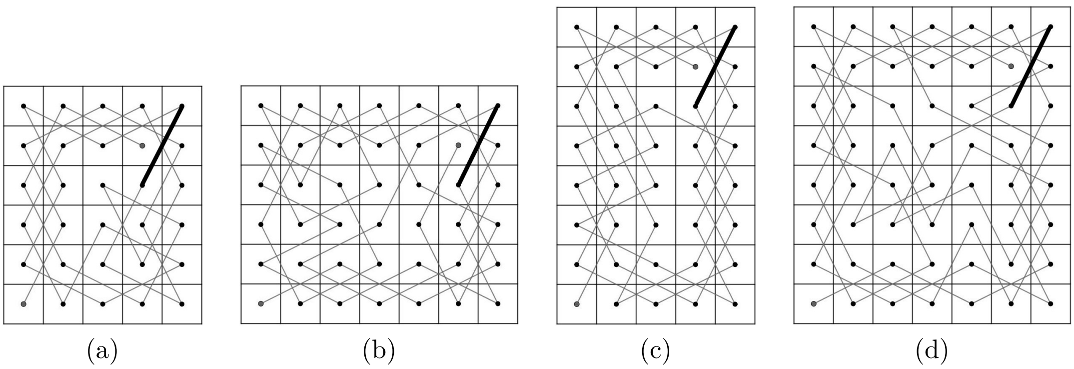

Figure 2.44: OKTs from $(m, 1)$ to $(2, n-1)$ on the CB$(m \times n)$ where $m \in \{6, 8\}$ and $n \in \{5, 7\}$

Let m be an even integer such that $m \geq 6$ and let n be an odd integer such that $n \geq 9$. We consider four cases according to the remainders after dividing m and n by 4.

Case 3.1: $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$. There are integers s and t such that $0 \leq s < t$, $m = 8 + 4s$ and $n = 5 + 4t$. If $s = 0$, then we divide the CB($8 \times n$) into subboards, CB(8×5) (Figure 2.44(c)) and t CB(8×4) (Figure 2.27(d)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 replace CB(5×5) and CB(5×4) by CB(8×5) (Figure 2.44(c)) and CB(8×4) (Figure 2.27(d)), respectively.

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($8 \times n$) (Figure 2.45) and s CB($4 \times n$)'s (Figure 2.32) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (ii′) replace CB($5 \times n$) by CB($8 \times n$) (Figure 2.45) and $(5, 1)$ by $(8, 1)$.

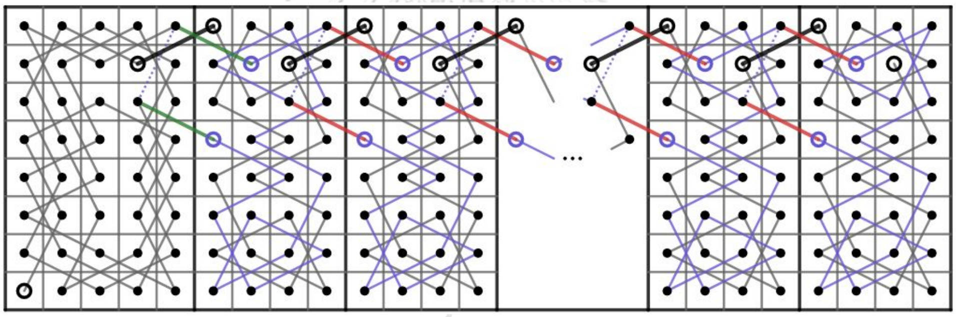

Figure 2.45: An OKTs on CB($8 \times n$) in Case 3.1

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Case 3.2: $m \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$. There are integers s and t such that $0 \leq s < t$, $t \neq 0$, $m = 8 + 4s$ and $n = 7 + 4t$. If $s = 0$, then we divide the CB($8 \times n$) into subboards, CB(8×7) (Figure 2.44(d)) and t CB(8×4)'s (Figure 2.27(d)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) and CB(5×4) by CB(8×7) (Figure 2.44(d)) and CB(8×4) (Figure 2.27(d)) and $(1, 5)$, $(3, 4)$ and $(2, 4)$ by $(1, 7)$, $(3, 6)$ and $(2, 6)$, respectively.

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($8 \times n$) (Figure 2.46) and s CB($4 \times n$)'s (Figure 2.34) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (i′) and (ii′) replace

CB(4×5) and CB$(5 \times n)$ by CB(4×7) (Figure 2.28(c)) and CB$(8 \times n)$ (Figure 2.46) and $(1,5)$, $(3,4)$, $(2,4)$, $(4,5)$ and $(5,1)$ by $(1,7)$, $(3,6)$, $(2,6)$, $(4,7)$ and $(8,1)$, respectively.

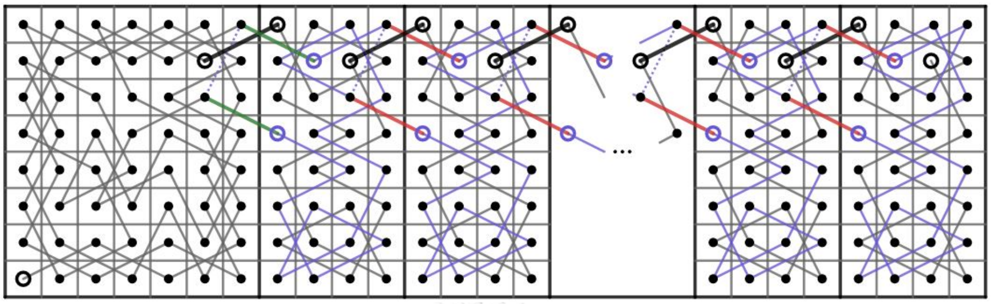

Figure 2.46: An OKT on CB$(8 \times n)$ in Case 3.2

Case 3.3: $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$. There are integers s and t such that $0 \leq s < t$, $t \neq 0$, $m = 6 + 4s$ and $n = 5 + 4t$. If $s = 0$, then we divide the CB$(6 \times n)$ into subboards, CB(6×5) (Figure 2.44(a)) and t CB(6×4) (Figure 2.27(b)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) and CB(5×4) by CB(6×5) (Figure 2.44(a)) and CB(6×4) (Figure 2.27(b)), respectively.

If $s > 0$, then we divide the CB$(m \times n)$ into subboards, CB$(6 \times n)$ (Figure 2.47) and s CB$(4 \times n)$'s (Figure 2.32) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (ii′) replace CB$(5 \times n)$ by CB$(6 \times n)$ (Figure 2.47) and $(5,1)$ by $(6,1)$.

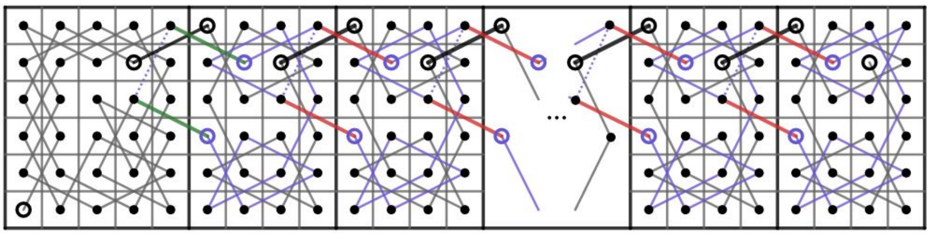

Figure 2.47: An OKT on CB$(6 \times n)$ in Case 3.3

Case 3.4: $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$. There are integers s and t such that $0 \leq s \leq t$, $t \neq 0$, $m = 6 + 4s$ and $n = 7 + 4t$. If $s = 0$, then we divide the CB$(6 \times n)$ into subboards, CB(6×7) (Figure 2.44(b)) and t CB(6×4) (Figure

2.27(b)). Then, we construct the required OKT by (i) and (ii) in Case 1.1 but replace CB(5×5) and CB(5×4) by CB(6×7) (Figure 2.44(b)) and CB(6×4) (Figure 2.27(b)) and $(1, 5)$, $(3, 4)$ and $(2, 4)$ by $(1, 7)$, $(3, 6)$ and $(2, 6)$, respectively.

If $s > 0$, then we divide the CB($m \times n$) into subboards, CB($6 \times n$) (Figure 2.48) and s CB($4 \times n$)'s (Figure 2.34) from the top to the bottom. Then, we construct the required OKT by (i′), (ii′) and (iii′) in Case 1.1 but in (i′) and (ii′) replace CB(4×5) and CB($5 \times n$) by CB(4×7) (Figure 2.28(c)) and CB($6 \times n$) (Figure 2.48) and $(1, 5)$, $(3, 4)$, $(2, 4)$, $(4, 5)$ and $(5, 1)$ by $(1, 7)$, $(3, 6)$, $(2, 6)$, $(4, 7)$ and $(6, 1)$, respectively.

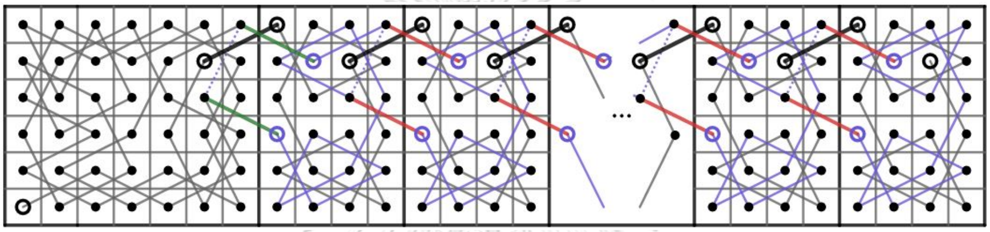

Figure 2.48: An OKT on CB($6 \times n$) in Case 3.4

This completes the proof. □

2.3 Existence of Some Special OKTs on CB($m \times n$) $- \{(i, j)\}$

The following lemmas give necessary and sufficient conditions on the existence of some special OKTs on CB($4 \times n$) $- \{(i, j)\}$ where $n \geq 5$ and (i, j) is a square on CB($4 \times n$). These OKTs will be used to prove the existence of a CKT on CB($m \times n$) $- \{(i, j)\}$ in Section 3.3 of Chapter III. First, we consider the case where $n \geq 5$ and n is odd.

Lemma 2.4. *Let $n \geq 5$ and n is odd. Then,*

(a) *CB($4 \times n$) $- \{(i, j)\}$ contains an OKT from $(2, n)$ to $(4, n)$ if and only if ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even).*

(b) *$CB(4 \times n) - \{(i,j)\}$ contains an OKT from $(1,n)$ to $(3,n)$ if and only if ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd).*

(c) *$CB(4 \times n) - \{(i,j)\}$ contains an OKT from $(1,1)$ to $(3,1)$ if and only if ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd).*

(d) *$CB(4 \times n) - \{(i,j)\}$ contains an OKT from $(2,1)$ to $(4,1)$ if and only if ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even).*

Proof. Let $n \geq 5$ and n is odd. We consider $\text{CB}(4 \times n) - \{(i,j)\}$ where (i,j) is a square on $\text{CB}(4 \times n)$.

(a) Assume that ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even). Let $n = a + 4k$ where $k \in \mathbb{N} \cup \{0\}$ and $a \in \{5, 7\}$. We prove by the mathematical induction on k.

First, for $k = 0$, we construct the required OKTs from $(2,a)$ to $(4,a)$ on $\text{CB}(4 \times a) - \{(i,j)\}$ as shown in Figures 2.49 and 2.50. Note that four $\text{CB}(4 \times 5) - \{(i,j)\}$ in Figure 2.49 contain edges $(1,1) - (3,2)$ and $(2,2) - (4,1)$. In addition, four $\text{CB}(4 \times 7) - \{(i,j)\}$ in Figure 2.50 contain edges $(1,1) - (3,2)$ and $(2,2) - (4,1)$.

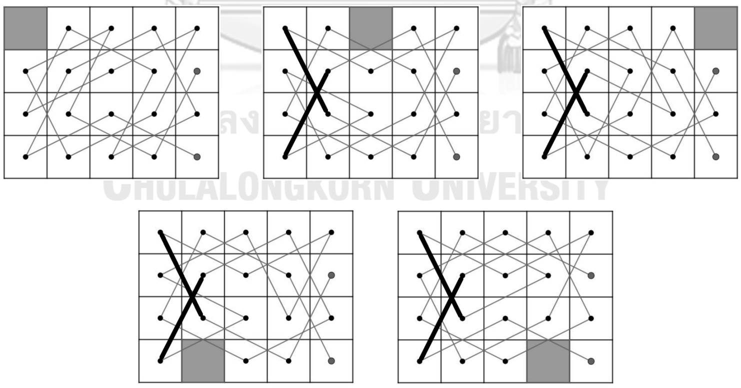

Figure 2.49: The required OKTs on $\text{CB}(4 \times 5) - \{(i,j)\}$ where ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even)

Next, let $k \geq 0$ be an integer. Assume that $\text{CB}(4 \times (a + 4k)) - \{(i,j)\}$ contains an OKT from $(2, a+4k)$ to $(4, a+4k)$. Consider two cases of $\text{CB}(4 \times (a + 4(k + 1))) - \{(i,j)\}$.

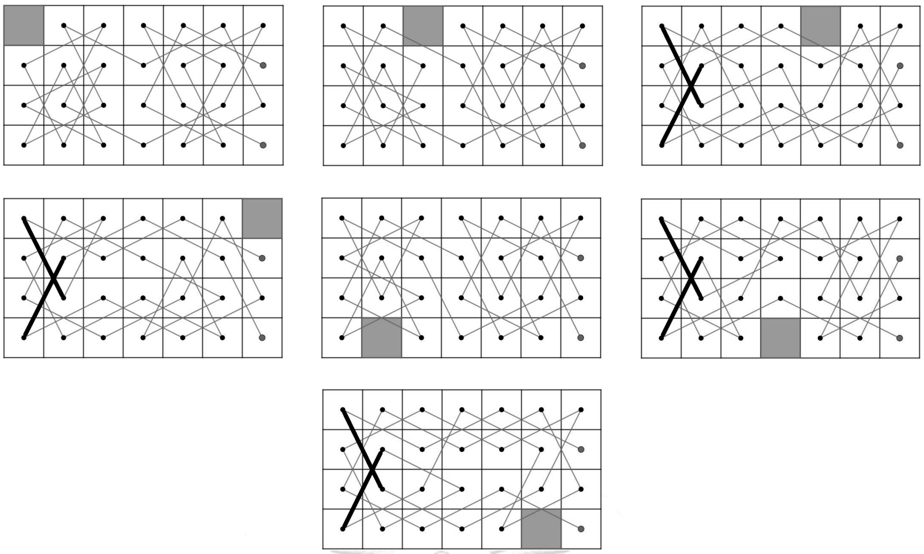

Figure 2.50: The required OKTs on $\mathrm{CB}(4\times 7)-\{(i,j)\}$ where ($i=1$ and $i+j$ is even) or ($i=4$ and $i+j$ is even)

Case 1: $1\le j\le a+4k$. We separate $\mathrm{CB}(4\times(a+4(k+1)))-\{(i,j)\}$ into two sub-boards, $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$ and $\mathrm{CB}(4\times 4)$ as shown in Figure 2.51 with $(i,j)=(1,1)$.

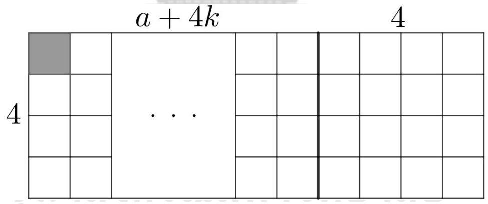

Figure 2.51: $\mathrm{CB}(4\times(a+4(k+1)))-\{(1,1)\}$ with two sub-boards

By the induction hypothesis, the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$ contains an OKT from $(2,a+4k)$ to $(4,a+4k)$. For the sub-board $\mathrm{CB}(4\times 4)$, we construct two paths P_1 from $(2,1)$ to $(4,4)$ and P_2 from $(4,1)$ to $(2,4)$ as shown in Figure 2.52.

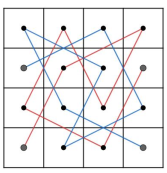

Figure 2.52: Two paths P_1 and P_2 on $\mathrm{CB}(4\times 4)$

Then, we construct the required OKT on $CB(4 \times (a+4(k+1))) - \{(i,j)\}$ by joining $(2, a+4k)$ and $(4, a+4k)$ of the OKT on the sub-board $CB(4 \times (a+4k)) - \{(i,j)\}$ to $(4,1)$ of P_2 and $(2,1)$ of P_1 on the sub-board $CB(4 \times 4)$, respectively, as shown in Figure 2.53 with $(i,j) = (1,1)$.

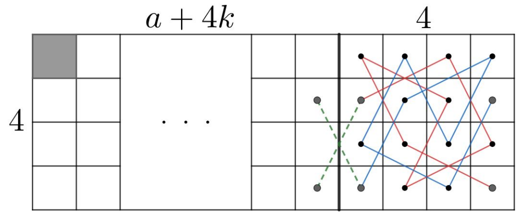

Figure 2.53: The required OKT from $(2, a+4(k+1))$ to $(4, a+4(k+1))$ on $CB(4 \times (a+4(k+1))) - \{(1,1)\}$

Case 2: $a+4k+1 \le j \le a+4(k+1)$. We separate $CB(4 \times (a+4(k+1))) - \{(i,j)\}$ into two sub-boards, $CB(4 \times 4)$ and $CB(4 \times (a+4k)) - \{(i,j)\}$ as shown in Figure 2.54 with $(i,j) = (1, a+4(k+1))$.

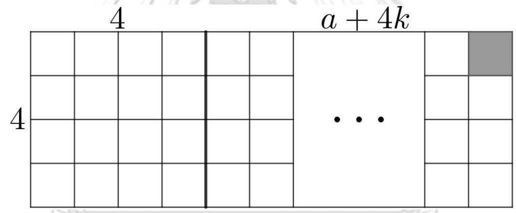

Figure 2.54: $CB(4 \times (a+4(k+1))) - \{(1, a+4(k+1))\}$ with two sub-boards

For the sub-board $CB(4 \times 4)$, we construct two paths P_1' from $(1,4)$ to $(2,4)$ and P_2' from $(3,4)$ to $(4,4)$ as shown in Figure 2.55.

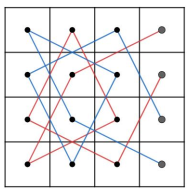

Figure 2.55: Two paths P_1' and P_2' on $CB(4 \times 4)$

By the induction hypothesis, the sub-board $CB(4 \times (a+4k)) - \{(i,j)\}$ contains an OKT from $(2, a+4k)$ to $(4, a+4k)$. Since $(1,1)$ and $(4,1)$ have degree 2 in $G(4 \times (a+4k)) - \{(i,j)\}$, $(1,1) - (3,2)$ and $(2,2) - (4,1)$ are two edges of the OKT.

Then, we construct the required OKT by the following two steps:

(i) delete $(1,1)-(3,2)$ and $(2,2)-(4,1)$ of the OKT on the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$;

(ii) join $(1,4)$ and $(2,4)$ of P_1' to $(2,2)$ and $(4,1)$ of the OKT on the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$, respectively and join $(3,4)$ and $(4,4)$ of P_2' to $(1,1)$ and $(3,2)$ of the OKT on the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$, respectively.

The required OKT is shown in Figure 2.56 with $(i,j)=(1,a+4(k+1))$.

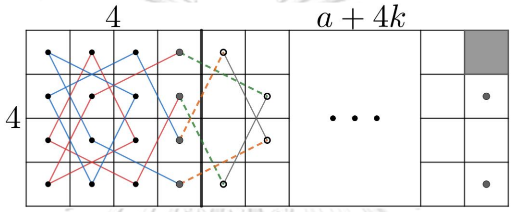

Figure 2.56: The required OKT on $\mathrm{CB}(4\times(a+4(k+1)))-\{(1,4(k+1))\}$

Hence, by the mathematical induction, if ($i=1$ and $i+j$ is even) or ($i=4$ and $i+j$ is even), then there exist an OKT on $\mathrm{CB}(4\times n)-\{(i,j)\}$ from $(2,n)$ to $(4,n)$.

Conversely, assume that ($i\neq 1$ or $i+j$ is odd) and ($i\neq 4$ or $i+j$ is odd) and $\mathrm{CB}(4\times n)-\{(i,j)\}$ contains an OKT from $(2,n)$ to $(4,n)$. Note that (i,j) is the black square when $i+j$ is even and the white square when $i+j$ is odd.

If $(i,j)=(2,n)$ or $(i,j)=(4,n)$, then it contradicts with our assumption about the existence of the OKT from $(2,n)$ to $(4,n)$.

If $i+j$ is odd and $(i,j)\notin\{(2,n),(4,n)\}$, then the number of black squares is greater than the number of white squares on $\mathrm{CB}(4\times n)-\{(i,j)\}$. Since $\mathrm{CB}(4\times n)-\{(i,j)\}$ contains an OKT from $(2,n)$ to $(4,n)$, $(2,n)$ and $(4,n)$ must be black. However, $2+n$ and $4+n$ are odd, then $(2,n)$ and $(4,n)$ are white, a contradiction.

For ($i=2$ and $i+j$ is even) or ($i=3$ and $i+j$ is even), let $G_1=G(4\times n)-\{(i,j)\}$. Consider $G_1'=G_1-\{(2,n)\}$. Let $S=\{(2,s),(3,t)\mid s$ is even, $2\leq s\leq n-1, t$ is odd and $1\leq t\leq n\}-\{(i,j)\}$. Then, $\omega(G_1'-S)=n+1>n=|S|+1$,

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see Figure 2.57 for the case $(i, j) = (3, 1)$ and $n = 9$. By Theorem 1.5(b), we have a contradiction.

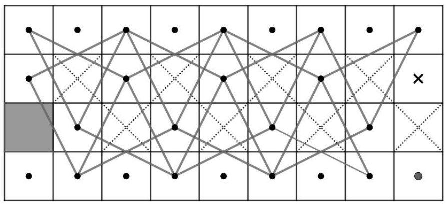

Figure 2.57: Components of $G'_1 - S$ where $(i, j) = (3, 1)$ and $n = 9$

(b) The required OKT can be obtained by horizontally flipping the OKT of $\mathrm{CB}(4 \times n) - \{(i, j)\}$ in (a).

(c) The required OKT can be obtained by rotating 180 degrees of the OKT of $\mathrm{CB}(4 \times n) - \{(i, j)\}$ in (a).

(d) The required OKT can be obtained by rotating 180 degrees of the OKT of $\mathrm{CB}(4 \times n) - \{(i, j)\}$ in (b). $\square$

Next, we give the existence of some special OKTs on $\mathrm{CB}(4 \times n) - \{(i, j)\}$ for $n \geq 6$ and n is even.

Lemma 2.5. *Let $n \geq 6$ and n is even. Then,*

(a) CB$(4 \times n) - \{(i, j)\}$ contains an OKT from $(2, n)$ to $(4, n)$ if and only if ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd).

(b) CB$(4 \times n) - \{(i, j)\}$ contains an OKT from $(1, n)$ to $(3, n)$ if and only if ($i = 1$ $i + j$ is even) or ($i = 4$ and $i + j$ is even).

Proof. Let $n \geq 6$ and n is even. We consider $\mathrm{CB}(4 \times n) - \{(i, j)\}$ where (i, j) is a square on $\mathrm{CB}(4 \times n)$.

(a) Assume that ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd). Let $n = a + 4k$ where $k \in \mathbb{N} \cup \{0\}$ and $a \in \{6, 8\}$. We prove by the mathematical induction on k.

For $k = 0$, we construct the required OKTs from $(2, a)$ to $(4, a)$ on $\mathrm{CB}(4 \times a) - \{(i, j)\}$ as shown in Figures 2.58 and 2.59. Note that four $\mathrm{CB}(4 \times 6) - \{(i, j)\}$

in Figure 2.58 contain edges $(1,1)-(3,2)$ and $(2,2)-(4,1)$. In addition, four $\text{CB}(4\times 8)-\{(i,j)\}$ in Figure 2.59 contain edges $(1,1)-(3,2)$ and $(2,2)-(4,1)$.

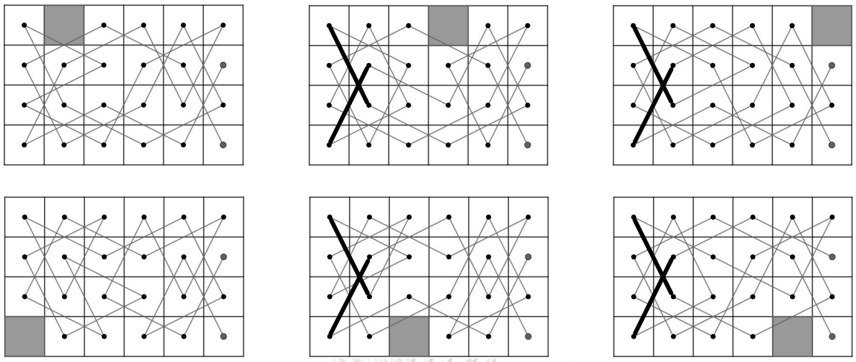

Figure 2.58: The required OKTs on $\text{CB}(4\times 6)-\{(i,j)\}$ where ($i=1$ and $i+j$ is odd) or ($i=4$ and $i+j$ is odd)

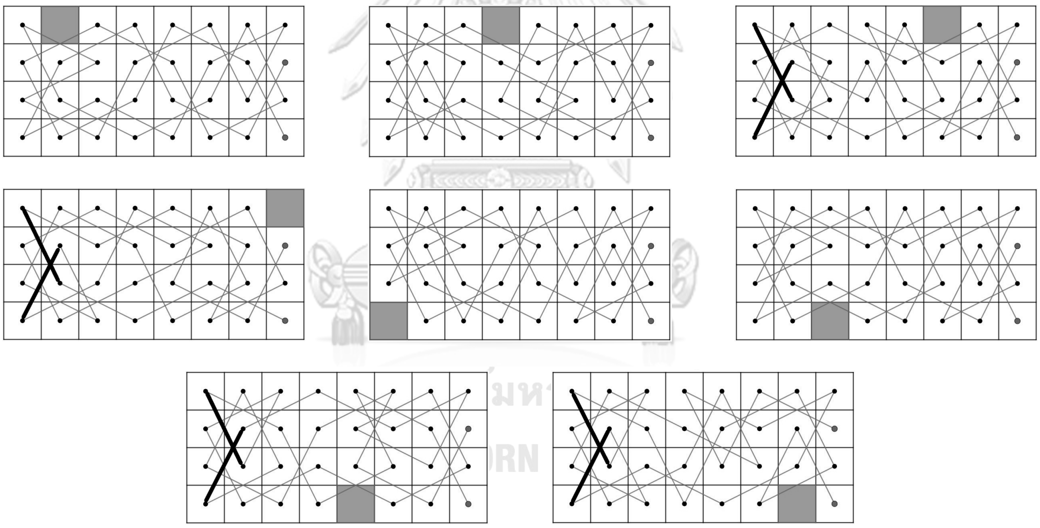

Figure 2.59: The required OKTs on $\text{CB}(4\times 8)-\{(i,j)\}$ where ($i=1$ and $i+j$ is odd) or ($i=4$ and $i+j$ is odd)

Next, let $k\geq 0$ be an integer. Assume that $\text{CB}(4\times(a+4k))-\{(i,j)\}$ contains an OKT from $(2,a+4k)$ to $(4,a+4k)$. Consider two cases of $\text{CB}(4\times(a+4(k+1)))-\{(i,j)\}$.

Case 1: $1\leq j\leq a+4k$. We separate $\text{CB}(4\times(a+4(k+1)))-\{(i,j)\}$ into two sub-boards, $\text{CB}(4\times(a+4k))-\{(i,j)\}$ and $\text{CB}(4\times 4)$ as shown in Figure 2.60 with $(i,j)=(1,2)$.

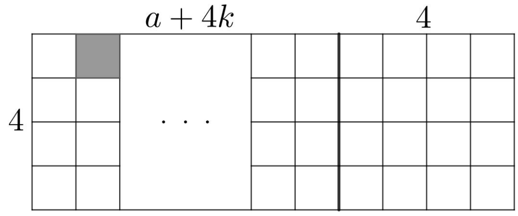

Figure 2.60: CB$(4 \times (a + 4(k + 1))) - \{(1, 2)\}$ with two sub-boards

By the induction hypothesis, the sub-board CB$(4 \times (a+4k)) - \{(i, j)\}$ contains an OKT from $(2, a + 4k)$ to $(4, a + 4k)$. Then, as shown in Figure 2.61 for $(i, j) = (1, 2)$, we construct the required OKT by joining $(2, a + 4k)$ and $(4, a + 4k)$ of the OKT on the sub-board CB$(4 \times (a + 4k)) - \{(i, j)\}$ to $(4, 1)$ of P_2 and $(2, 1)$ of P_1 on the sub-board CB(4×4) (Figure 2.52), respectively.

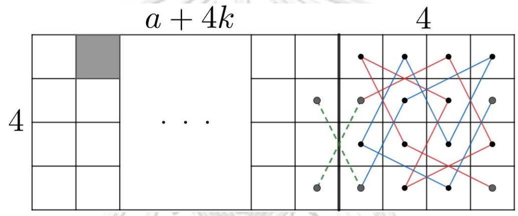

Figure 2.61: The required OKT from $(2, a + 4(k + 1))$ to $(4, a + 4(k + 1))$ on CB$(4 \times (a + 4(k + 1))) - \{(1, 2)\}$

Case 2: $a+4k+1 \leq j \leq a+4(k+1)$. We separate CB$(4\times(a+4(k+1)))-\{(i, j)\}$ into two sub-boards, CB(4×4) and CB$(4 \times (a + 4k)) - \{(i, j)\}$ as shown in Figure 2.62 with $(i, j) = (1, a + 4(k + 1))$.

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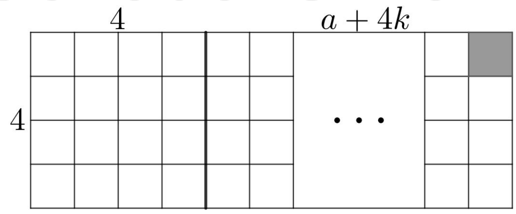

Figure 2.62: CB$(4 \times (a + 4(k + 1))) - \{(1, a + 4(k + 1))\}$ with two sub-boards

By the induction hypothesis, the sub-board CB$(4 \times (a+4k)) - \{(i, j)\}$ contains an OKT from $(2, a + 4k)$ to $(4, a + 4k)$. Since $(1, 1)$ and $(4, 1)$ have degree 2 in $G(4 \times (a + 4k)) - \{(i, j)\}$, $(1, 1) - (3, 2)$ and $(2, 2) - (4, 1)$ are two edges of the OKT.

Then, we construct the required OKT by the following two steps:

(i) delete $(1,1)-(3,2)$ and $(2,2)-(4,1)$ of the OKT on the sub-board $\mathrm{CB}(4\times(a+4k))-(i,j)$;

(ii) by using P_1' and P_2' on $\mathrm{CB}(4\times 4)$ shown in Figure 2.55, we join $(1,4)$ and $(2,4)$ of P_1' on the sub-board $\mathrm{CB}(4\times 4)$ to $(2,2)$ and $(4,1)$ of the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$, respectively and join $(3,4)$ and $(4,4)$ of P_2' on the sub-board $\mathrm{CB}(4\times 4)$ to $(1,1)$ and $(3,2)$ of the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$, respectively.

The required OKT is shown in Figure 2.63.

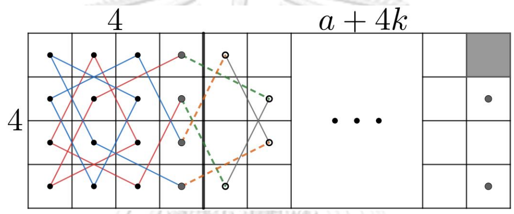

Figure 2.63: The required OKT on $\mathrm{CB}(4\times(a+4(k+1)))-\{(1,4(k+1))\}$

Hence, by the mathematical induction, if ($i=1$ and $i+j$ is odd) or ($i=4$ and $i+j$ is odd), then there exist an OKT on $\mathrm{CB}(4\times n)-\{(i,j)\}$ from $(2,n)$ to $(4,n)$.

Conversely, assume that ($i\neq 1$ or $i+j$ is even) and ($i\neq 4$ or $i+j$ is even) and $\mathrm{CB}(4\times n)-\{(i,j)\}$ contains an OKT from $(2,n)$ to $(4,n)$. Note that (i,j) is the black square when $i+j$ is even and the white square when $i+j$ is odd.

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If $(i,j)=(2,n)$ or $(i,j)=(4,n)$, then it contradicts with our assumption about the existence of the OKT from $(2,n)$ to $(4,n)$.

If $i+j$ is even and $(i,j)\notin\{(2,n),(4,n)\}$, then the number of black squares is less than the number of white squares on $\mathrm{CB}(4\times n)-\{(i,j)\}$. Since $\mathrm{CB}(4\times n)-\{(i,j)\}$ contains an OKT from $(2,n)$ to $(4,n)$, $(2,n)$ and $(4,n)$ must be white. Since $2+n$ and $4+n$ are even, $(2,n)$ and $(4,n)$ are black, a contradiction.

For ($i=2$ and $i+j$ is odd) or ($i=3$ and $i+j$ is odd), let $G_1=G(4\times n)-\{(i,j)\}$. Consider $G_1'=G_1-\{(2,n)\}$. Let $S=\{(2,s),(3,t)\mid s \text{ is odd}, 1\leq s\leq n-1, t \text{ is even and } 2\leq t\leq n\}-\{(i,j)\}$. Then, $\omega(G_1'-S)=n+1>n=|S|+1$, see

Figure 2.64 for the case $(i, j) = (2, 1)$ and $n = 10$. By Theorem 1.5(b), we have a contradiction.

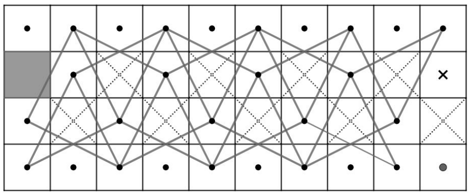

Figure 2.64: Components of $G_1' - S$ where $(i, j) = (2, 1)$ and $n = 10$

(b) The required OKT can be obtained by horizontally flipping the OKT of $\mathrm{CB}(4 \times n) - \{(i, j)\}$ in (a). $\square$

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CHAPTER III

CLOSED KNIGHT'S TOURS ON SOME BOARDS

In this chapter, we give the existence of CKTs on some boards which is divided into 3 sections. Sections 3.2 and 3.3 are two main results of our dissertation.

3.1 CKTs on some LB(m, n, r)

First, let us construct the CKT on LB$(m, n, 4)$, where $m, n \geq 5$.

Theorem 3.1. *An LB*$(m, n, 4)$ *has a CKT containing an edge* $(1, 4) - (3, 3)$ *for all* $m, n \geq 5$.

Proof. First, let us construct CKTs on some small size LBs of the same and different parity of m and n as in the following Figures 3.1 - 3.3. Note that Figure 3.1(a) is a result of this Theorem, while Figure 3.1(b) is constructed for extension to larger LBs.

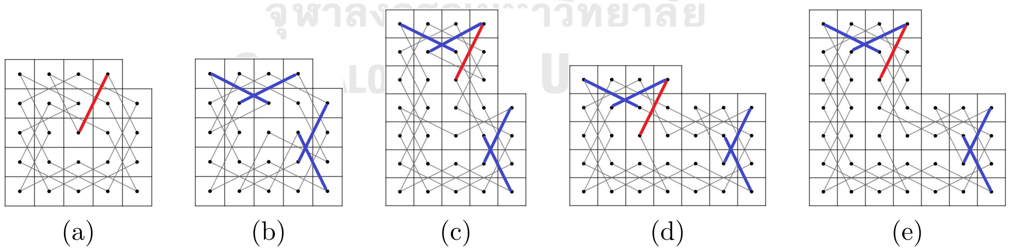

Figure 3.1: CKTs for LB$(5, 5, 4)$, LB$(7, 5, 4)$, LB$(5, 7, 4)$ and LB$(7, 7, 4)$

In addition, from Figures 3.1(b) - 3.1(e), 3.2 and 3.3, each CKT of the LB$(a, b, 4)$ contains edges $(1, 1) - (2, 3)$, $(1, 4) - (2, 2)$, $(a - 3, b) - (a - 1, b - 1)$ and $(a - 2, b - 1) - (a, b)$. Furthermore, from Figures 3.1(a), 3.1(c) - 3.1(e), 3.2 and 3.3, each CKT of the LB$(a, b, 4)$ contains the edge $(1, 4) - (3, 3)$.

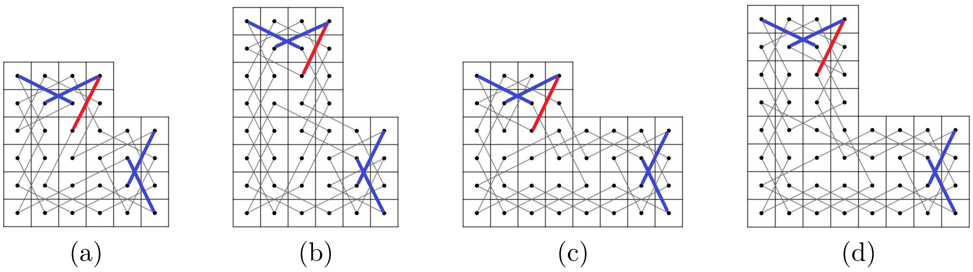

Figure 3.2: CKTs for LB$(6, 6, 4)$, LB$(8, 6, 4)$, LB$(6, 8, 4)$ and LB$(8, 8, 4)$

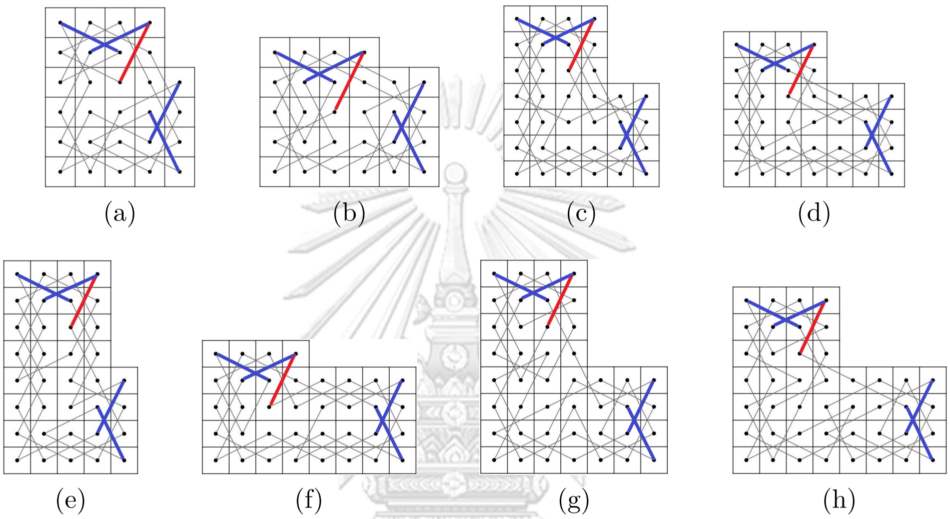

Figure 3.3: CKTs for LB$(6, 5, 4)$, LB$(5, 6, 4)$, LB$(7, 6, 4)$, LB$(6, 7, 4)$, LB$(8, 5, 4)$, LB$(5, 8, 4)$, LB$(8, 7, 4)$ and LB$(7, 8, 4)$

Next, for the larger LBs, we already have two paths P_1' and P_2' on the CB$(4 \times 4t)$ as shown in Figure 2.31, where $t \geq 1$.

Notice that $(1, 1)$, $(2, 1)$, $(3, 1)$ and $(4, 1)$ are four end-points of two paths of the CB$(4 \times 4t)$ for $t \geq 1$. By rotating Figure 2.31 anti-clockwise for 90 degrees, we also obtain two paths P_1'' and P_2'' on the CB$(4s \times 4)$ where $s \geq 1$ as shown in Figure 3.4. Also, notice that $(4s, 1)$, $(4s, 2)$, $(4s, 3)$ and $(4s, 4)$ are four end-points of two paths and the edge $(1, 4) - (3, 3)$ contained in one path of the CB$(4s \times 4)$ for $s \geq 1$.

Now, we are ready to construct a CKT on a larger LB by placing the CB$(4 \times 4t)$ (Figure 2.31) to the right and the CB$(4s \times 4)$ (Figure 3.4) above each smaller LB in Figures 3.1 - 3.3, respectively. WLOG, let $m \geq n \geq 5$.

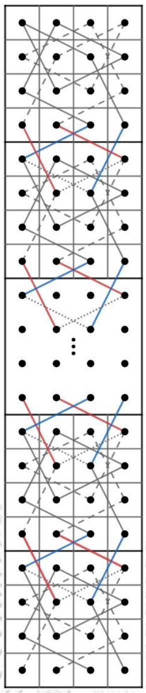

Figure 3.4: Two paths P_1'' and P_2'' on the CB($4s \times 4$)

- If m and n are odd integers, then $m \equiv 1$ or $3 \pmod 4$ and $n \equiv 1$ or 3 (mod 4).
- If m and n are even integers, then $m \equiv 0$ or $2 \pmod 4$ and $n \equiv 0$ or 2 (mod 4).
- If m and n are different parity, then $m \equiv 1$ or $3 \pmod 4$ and $n \equiv 0$ or 2 (mod 4); and $m \equiv 0$ or $2 \pmod 4$ and $n \equiv 1$ or $3 \pmod 4$.

Recall that the LB$(a, b, 4)$ has a CKT for all $a, b \in \{5, 6, 7, 8\}$. Thus, it is enough to show that the LB$(a + 4s, b + 4t, 4)$ has a CKT for any nonnegative s, t and $a, b \in \{5, 6, 7, 8\}$ such that $s \geq t$ and $s \neq 0$. First, if $t = 0$, then let us divide the LB$(a + 4s, b, 4)$ into two subboards, CB$(4s \times 4)$ and LB$(a, b, 4)$. Otherwise, we divide into three subboards, CB$(4s \times 4)$, LB$(a, b, 4)$ and CB$(4 \times 4t)$. Then, we construct the required CKT by the followings.

(i) if $t = 0$, then delete $(1, 1) - (2, 3)$ and $(1, 4) - (2, 2)$ from the CKT of the LB$(a, b, 4)$. If $t > 0$, then further delete $(a - 3, b) - (a - 1, b - 1)$ and $(a - 2, b - 1) - (a, b)$ from the CKT of the LB$(a, b, 4)$.

(ii) If $t = 0$, then join $(4s, 1)$, $(4s, 2)$, $(4s, 3)$ and $(4s, 4)$ which are four end-points of two paths of the CB$(4s \times 4)$ to $(2, 2)$, $(1, 4)$, $(1, 1)$ and $(2, 3)$ of the LB$(a, b, 4)$, respectively. If $t > 0$, then further join $(1, 1)$, $(2, 1)$, $(3, 1)$ and $(4, 1)$ which are four end-points of two paths of the CB$(4 \times 4t)$ to $(a-2, b-1)$, (a, b), $(a-3, b)$ and $(a-1, b-1)$ of the LB$(a, b, 4)$, respectively.

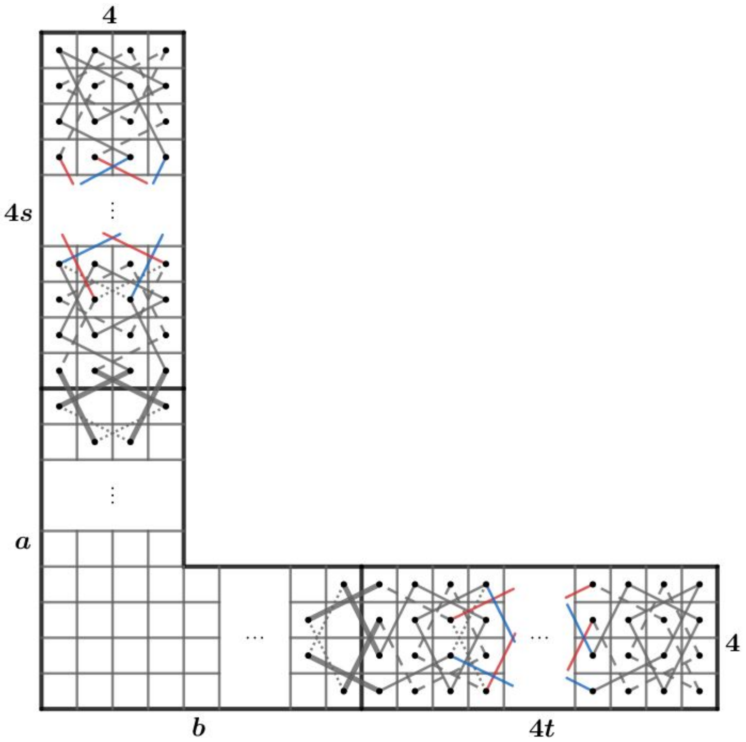

Figure 3.5: A CKT on the LB$(a + 4s, b + 4t, 4)$

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Figure 3.5 illustrates the constructed CKT on the LB$(a + 4s, b + 4t, 4)$. This completes the proof. □

By properly rotating counter-clockwise 90 degrees and then flipping vertically the LB$(m, n, 4)$, where $m, n \geq 5$, we obtain the following result immediately.

Corollary 3.2. *A 7B$(m, n, 4)$ has a CKT containing an edge $(4, 1) - (2, 2)$ for all $m, n \geq 5$.*

We note that Theorem 3.1 and Corollary 3.2 will be used in Case 2 of Theorem 3.4 in Section 3.2 of Chapter III.

3.2 CKTs on RB(m, n, r)

To characterize the RB(m, n, r) according to the existence of its CKT, let us first consider the case when $r = 2$. It is known from Theorem 1.9 that RB$(m, m, 2)$ admits no CKTs. The following theorem can be regarded as an extended result of Theorem 1.9. Recall that $G(m, n, r)$ is the knight graph of the RB(m, n, r).

Theorem 3.3. *There are no CKT on RB$(m, n, 2)$ for all $n \geq m \geq 5$.*

Proof. Let m and n be integers such that $n \geq m \geq 5$. In the case that $m = n$, the result is obtained by Theorem 1.9. Assume that $n > m$. Then, there exist positive integers k and l and $r, q \in \{1, 2, 3, 4\}$ such that $m = 4k + r$ and $n = 4l + q$. Assume that there exists a CKT H on RB$(m, n, 2)$ which is a Hamiltonian cycle on $G(m, n, 2)$. We separate our consideration into 16 cases according to the values of r and q.

Case 1: $k < l$ and $r = q = 1$. Since all vertices in $\{(1, 4i+1), (m, 4i+1) | 0 \leq i \leq l\}$ and $\{(4i + 1, 1), (4i + 1, n) | 0 \leq i \leq k\}$ have only 2 incident edges which must be in H and we collect all incident edges from these two sets, it happens to form a cycle $(1, 1)$, $(2, 3)$, $(1, 5)$, $(2, 7)$,..., $(2, n-2)$, $(1, n)$, $(3, n-1)$, $(5, n)$, $(7, n-1)$,..., $(m-2, n-1)$, (m, n), $(m-1, n-2)$, $(m, n-4)$, $(m-1, n-6)$,..., $(m-1, 3)$, $(m, 1)$, $(m-2, 2)$, $(m-4, 1)$, $(m-6, 2)$,..., $(3, 2)$, $(1, 1)$, see Figure 3.6 for a cycle on $G(13, 17, 2)$. This is a contradiction since this cycle does not contain all vertices of $G(m, n, 2)$.

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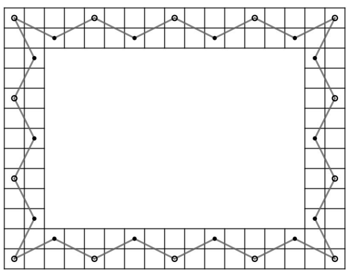


Figure 3.6: A cycle on $G(13, 17, 2)$

Case 2: $k < l$ and $r = q = 2$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+1), (m-1, 4i+1) | 0 \leq i \leq l\}$ and $\{(4i+1, 1), (4i+1, n-1) | 0 \leq$

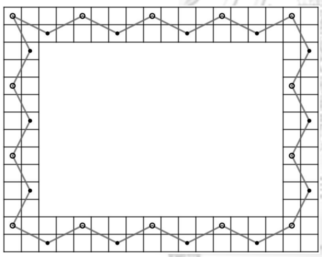
$i \leq k\}$ instead, see Figure 3.7(a) for a cycle on $G(14, 18, 2)$.

Case 3: $k < l$ and $r = q = 3$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+4), (m, 4i+4) | 0 \leq i \leq l-1\}$ and $\{(4i+4, 1), (4i+4, n) | 0 \leq i \leq k-1\}$ instead, see Figure 3.7(b) for a cycle on $G(15, 19, 2)$.

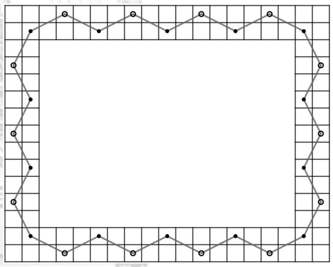
Case 4: $k < l$ and $r = q = 4$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+1), (m, 4i+4) | 0 \leq i \leq l\}$ and $\{(4i+1, 1), (4i+4, n) | 0 \leq i \leq k\}$ instead, see Figure 3.7(c) for a cycle on $G(16, 20, 2)$.

Case 5: $k \leq l$, $r = 1$ and $q = 2$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+1), (m, 4i+1) | 0 \leq i \leq l\}$ and $\{(4i+1, 1), (4i+1, n-1) | 0 \leq i \leq k\}$ instead, see Figure 3.7(d) for a cycle on $G(13, 14, 2)$.

Case 6: $k \leq l$, $r = 1$ and $q = 3$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+2), (m, 4i+2) | 0 \leq i \leq l\}$ and $\{(4i+1, 2), (4i+1, n-1) | 0 \leq i \leq k\}$ instead, see Figure 3.7(e) for a cycle on $G(13, 15, 2)$.



(a) A cycle on $G(14, 18, 2)$



(b) A cycle on $G(15, 19, 2)$

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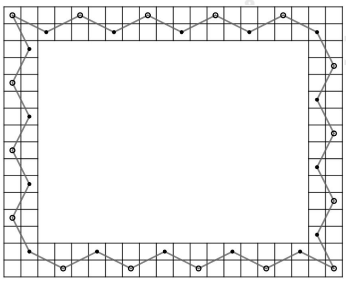
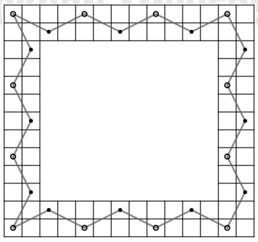
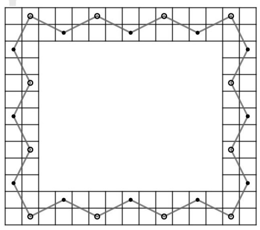


(c) A cycle on $G(16, 20, 2)$



(d) A cycle on $G(13, 14, 2)$



(e) A cycle on $G(13, 15, 2)$

Figure 3.7: Cycles on $G(14, 18, 2)$, $G(15, 19, 2)$, $G(16, 20, 2)$, $G(13, 14, 2)$ and $G(13, 15, 2)$, respectively.

Case 7: $k \leq l$, $r = 1$ and $q = 4$.

If $k = 1$, then there are some vertices (i.e., $(2, n-4)$ and $(4, n-4)$ which are indicated by "+" in Figure 3.8) that have degree more than the degree of the same vertices in the case when $k \geq 2$.

Case 7.1: $k = 1$. Since $(1, n)$ and $(5, n)$ have only 2 incident edges on the $G(5, n, 2)$, $(1, n) - (3, n-1)$ and $(3, n-1) - (5, n)$ must be in H and it forces that $(1, n-2) - (3, n-1)$ and $(3, n-1) - (5, n-2)$ must not be in H. Then, it also forces that $(2, n-4) - (1, n-2)$, $(1, n-2) - (2, n)$, $(4, n-4) - (5, n-2)$ and $(5, n-2) - (4, n)$ must be in H. Next, since all vertice in $\{(1, 4i+1), (5, 4i+1) | 0 \leq i \leq l\}$, $\{(1, 4i+2), (5, 4i+2) | 0 \leq i \leq l-1\}$ and $\{(2, n), (4, n)\}$ have only 2 incident edges. Collect $(2, n-4) - (1, n-2)$, $(1, n-2) - (2, n)$, $(4, n-4) - (5, n-2)$ and $(5, n-2) - (4, n)$ which must be in H together with all incident edges from these three sets, it happen to form a cycle $(1, 1)$, $(2, 3)$, $(1, 5)$, $(2, 7)$,..., $(1, n-3)$, $(2, n-1)$, $(4, n)$, $(5, n-2)$, $(4, n-4)$, $(5, n-6)$,..., $(4, 4)$, $(5, 2)$, $(3, 1)$, $(1, 2)$, $(2, 4)$, $(1, 6)$,..., $(1, n-6)$, $(2, n-4)$, $(1, n-2)$, $(2, n)$, $(4, n-1)$, $(5, n-3)$,..., $(5, 5)$, $(4, 3)$, $(5, 1)$, $(3, 2)$, $(1, 1)$, see Figure 3.8 for a cycle on $G(5, 12, 2)$. This is a contradiction since this cycle does not contain all vertices of $G(5, n, 2)$.

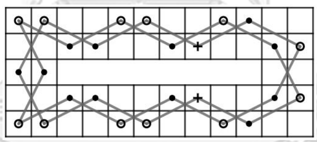


Figure 3.8: A cycle on $G(5, 12, 2)$

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Case 7.2: $k \geq 2$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+1), (1, 4i+2), (m, 4i+1), (m, 4i+2) | 0 \leq i \leq l-1\}$, $\{(1, n-3), (2, n-4), (m-1, n-4), (m, n-3)\}$ and $\{(4i+1, 1), (4i+1, 2), (4i+2, n), (4i+4, n) | 0 \leq i \leq k-1\}$ instead, see Figure 3.9 for a cycle on $G(13, 16, 2)$.

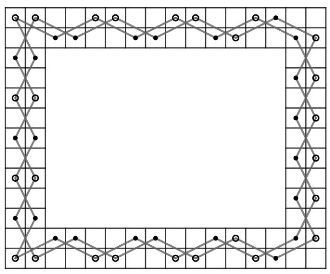
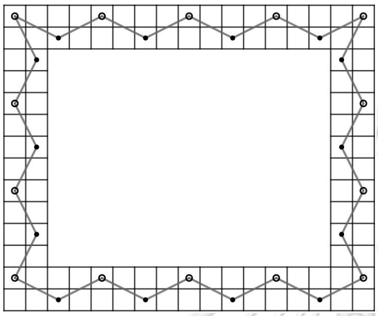


Figure 3.9: A cycle on $G(13, 16, 2)$

Case 8: $k < l$, $r = 2$ and $q = 1$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+1), (m-1, 4i+1) | 0 \leq i \leq l\}$ and $\{(4i+1, 1), (4i+1, n) | 0 \leq i \leq k\}$ instead, see Figure 3.10(a) for a cycle on $G(14, 17, 2)$.

Case 9: $k \leq l$, $r = 2$ and $q = 3$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+2), (m-1, 4i+2) | 0 \leq i \leq l\}$ and $\{(4i+1, 2), (4i+1, n-1) | 0 \leq i \leq k\}$ instead, see Figure 3.10(b) for a cycle on $G(14, 15, 2)$.


(a) A cycle on $G(14, 17, 2)$

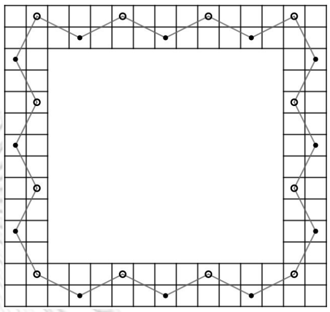

(b) A cycle on $G(14, 15, 2)$

Figure 3.10: Cycles on $G(14, 17, 2)$ and $G(14, 15, 2)$, respectively.

Case 10: $k \leq l$, $r = 2$ and $q = 4$.

If $k = 1$ and $l = 1$, then it is similar to Case 7.1. We can see that $(m-1, 5)$ (indicated by "+" in Figure 3.11) has degree 3 which is more than the degree of the same vertex in the case when $k \geq 2$ and $l \geq 2$.

Case 10.1: $k = 1$ and $l = 1$. Since $(2, 8)$ and $(6, 8)$ have only 2 incident edges on the $G(6, 8, 2)$, $(2, 8) - (4, 7)$ and $(4, 7) - (6, 8)$ must be in H and it forces that $(5, 5) - (4, 7)$ must not be in H. Then, it also forces that $(6, 3) - (5, 5)$ and $(5, 5) - (6, 7)$ must be in H. Next, since all vertice in $\{(1, 1), (1, 5), (2, 7), (6, 7), (5, 1)\}$ have only 2 incident edges. Collect $(6, 3) - (5, 5)$ and $(5, 5) - (6, 7)$ which must be in H and together with all incident edges from the set $\{(1, 1), (1, 5), (2, 7), (6, 7), (5, 1)\}$, it happens to form a cycle $(1, 1)$, $(2, 3)$, $(1, 5)$, $(2, 7)$, $(4, 8)$, $(6, 7)$, $(5, 5)$, $(6, 3)$, $(5, 1)$, $(3, 2)$, $(1, 1)$, see Figure 3.11. This is a contradiction since this cycle does not contain all vertices of $G(6, 8, 2)$.

Case 10.2: $k \geq 1$ and $l \geq 2$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+1) | 0 \leq i \leq l\}$, $\{(4i+2, n-1), (4i+1, 1) | 0 \leq i \leq k\}$,

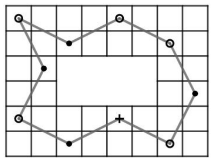


Figure 3.11: A cycle on $G(6,8,2)$

$\{(m-1,5)\}$ and $\{(m,4i+3)|1 \leq i \leq l\}$ instead, see Figure 3.12 for a cycle on $G(14,16,2)$.

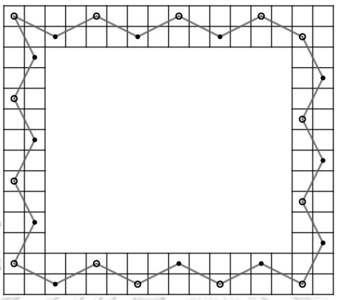


Figure 3.12: A cycle on $G(14,16,2)$

Case 11: $k < l$, $r = 3$ and $q = 1$. We obtain a contradiction similar to Case 1 by considering $\{(2,4i+1),(m-1,4i+1)|0 \leq i \leq l\}$ and $\{(4i+2,1),(4i+2,n)|0 \leq i \leq k\}$ instead, see Figure 3.13(a) for a cycle on $G(15,17,2)$.

Case 12: $k < l$, $r = 3$ and $q = 2$. We obtain a contradiction similar to Case 1 by considering $\{(2,4i+1),(m-1,4i+1)|0 \leq i \leq l\}$ and $\{(4i+2,1),(4i+2,n-1)|0 \leq i \leq k\}$ instead, see Figure 3.13(b) for a cycle on $G(15,18,2)$.

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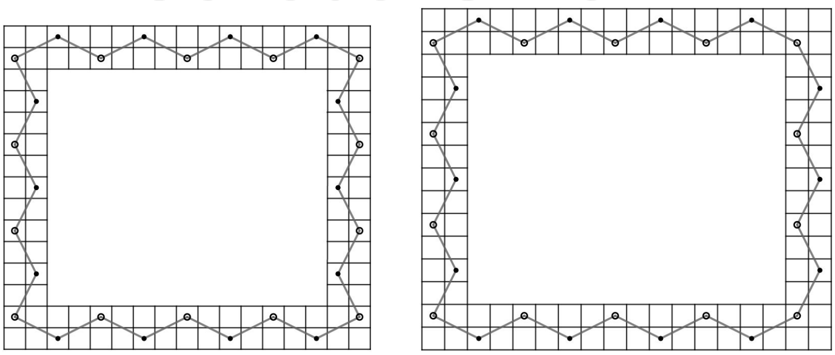

(a) A cycle on $G(15,17,2)$ (b) A cycle on $G(15,18,2)$

Figure 3.13: Cycles on $G(15,17,2)$ and $G(15,18,2)$, respectively.

Case 13: $k \leq l$, $r = 3$ and $q = 4$.

If $k = 1$ and $l = 1$ or $k = 1$ and $l \geq 2$, then it is similar to Case 7.1. For $k = 1$ and $l = 1$, there are some vertices (i.e., $(2,4)$, $(3,1)$, $(5,1)$ and $(6,4)$ which are indicated by "+" in Figure 3.14) that have degree more than the degree of the same vertices in the case when $k \geq 2$. For $k = 1$ and $l \geq 2$, there are some vertices (i.e., $(3,1)$ and $(5,1)$ which are indicated by "+" in Figure 3.15) that have degree more than the degree of the same vertices in the case when $k \geq 2$.

Case 13.1: $k = 1$ and $l = 1$. Since $(1,1)$, $(1,5)$, $(7,1)$ and $(7,5)$ have only 2 incident edges on $G(7,8,2)$, $(1,1)-(2,3)$, $(2,3)-(1,5)$, $(7,1)-(6,3)$ and $(6,3)-(7,5)$ must be in H and it forces that $(3,1)-(2,3)$ and $(5,1)-(6,3)$ must not be in H. Then, it also forces that $(1,2)-(3,1)$, $(3,1)-(5,2)$, $(3,2)-(5,1)$ and $(5,1)-(7,2)$ must be in H. Thus, $(3,2)$ and $(5,2)$ already have 2 incident edges on H and it forces again that $(3,2)-(2,4)$ and $(5,2)-(6,4)$ must not be in H. Next, since all vertice in $\{(1,1),(1,2),(1,5),(2,8),(4,8),(6,8),(7,5),(7,2),(7,1)\}$ have only 2 incident edges. Collect $(2,4)-(1,6)$, $(3,1)-(5,2)$, $(3,2)-(5,1)$ and $(6,4)-(7,6)$ which must be in H together with all incident edges from $\{(1,1),(1,2),(1,5),(2,8),(4,8),(6,8),(7,5),(7,2),(7,1)\}$, it happens to form a cycle $(1,1)$, $(2,3)$, $(1,5)$, $(2,7)$, $(4,8)$, $(6,7)$, $(7,5)$, $(6,3)$, $(7,1)$, $(5,2)$, $(3,1)$, $(1,2)$, $(2,4)$, $(1,6)$, $(2,8)$, $(4,7)$, $(6,8)$, $(7,6)$, $(6,4)$, $(7,2)$, $(5,1)$, $(3,2)$, $(1,1)$, see Figure 3.14. This is a contradiction since this cycle does not contain all vertices of $G(7,8,2)$.

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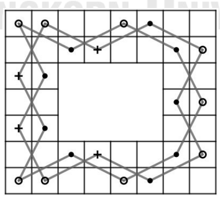

Figure 3.14: A cycle on $G(7,8,2)$

Case 13.2: $k = 1$ and $l \geq 2$. Since $(1,1)$, $(1,5)$, $(7,1)$ and $(7,5)$ have only 2 incident edges on $G(7,n,2)$, $(1,1)-(2,3)$, $(2,3)-(1,5)$, $(7,1)-(6,3)$ and $(6,3)-(7,5)$ must be in H and it forces that $(3,1)-(2,3)$ and $(5,1)-(6,3)$ must not be in H. Then, it also forces that $(1,2)-(3,1)$, $(3,1)-(5,2)$, $(3,2)-(5,1)$ and $(5,1)-(7,2)$ must be in H. Next, since all vertice in $\{(1,4i+1),(7,4i+1)|0 \leq i \leq$

$l\}$, $\{(1,4i+2),(7,4i+2)|0 \le i \le l-1\}$ and $\{(2,n-4),(6,n-4),(2,n),(4,n),(6,n)\}$ have only 2 incident edges. Collect $(3,1)-(5,2)$ and $(3,2)-(5,1)$ which must be in H together with all incident edges from these two sets, it happen to form a cycle $(1,1)$, $(2,3)$, $(1,5)$..., $(2,n-5)$, $(1,n-3)$, $(2,n-1)$, $(4,n)$, $(6,n-1)$, $(7,n-3)$,..., $(7,5)$, $(6,3)$, $(7,1)$, $(5,2)$, $(3,1)$, $(1,2)$, $(2,4)$,..., $(1,n-6)$, $(2,n-4)$, $(1,n-2)$, $(2,n)$, $(4,n-1)$, $(6,n)$, $(7,n-2)$, $(6,n-4)$, $(7,n-6)$,..., $(6,4)$, $(7,2)$, $(5,1)$, $(3,2)$, $(1,1)$, see Figure 3.15 for a cycle on $G(7,16,2)$. This is a contradiction since this cycle does not contain all vertices of $G(7,4l+4,2)$.

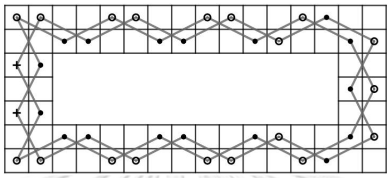

Figure 3.15: A cycle on $G(7,16,2)$

Case 13.3: $k \ge 2$. We obtain a contradiction similar to Case 1 by considering $\{(1,4i+1),(m,4i+1)|0 \le i \le l\}$, $\{(1,4i+2),(m,4i+2)|0 \le i \le l-1\}$, $\{(2,n-4),(m-1,n-4)\}$, $\{(4i+2,n)|0 \le i \le k\}$, $\{(4i+4,n)|0 \le i \le k-1\}$, $\{(4i+1,1),(4i+1,2)|0 \le i \le k-2\}$ and $\{(4i+3,1),(4i+3,2)|1 \le i \le k\}$ instead, see Figure 3.16 for a cycle on $G(15,16,2)$.

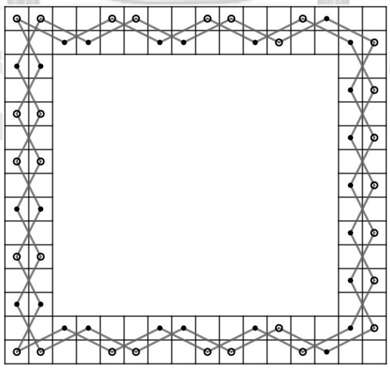

Figure 3.16: A cycle on $G(15,16,2)$

Case 14: $k < l$, $r = 4$ and $q = 1$.

If $k = 1$, then it is similar to Case 7.1. There are some vertices (i.e., $(4,2)$ and $(4,n-1)$ which are indicated by "+" in Figure 3.17) that have degree more than the degree of the same vertices in the case when $k \ge 2$.

Case 14.1: $k = 1$. Since $(1,1)$, $(1,5)$, $(1,n-4)$ and $(1,n)$ have only 2 incident edges on $G(8,4l+1,2)$, $(1,1)-(2,3)$, $(2,3)-(1,5)$, $(1,n-4)-(2,n-2)$ and $(2,n-2)-(1,n)$ must be in H and it forces that $(4,2)-(2,3)$ and $(4,n-1)-(2,n-2)$ must not be in H. Then, it also forces that $(4,2)-(6,1)$ and $(4,n-1)-(6,n)$ must be in H. Next, since all vertice in $\{(1,4i+1),(2,4i+1)|0 \le i \le l\}$, $\{(5,1),(5,n)\}$ and $\{(8,4i+2),(8,4i+4)|0 \le i \le l-1\}$ have only 2 incident edges. Collect $(4,2)-(6,1)$ and $(4,n-1)-(6,n)$ which must be in H together with all incident edges from these three sets, it happens to form a cycle $(1,1)$, $(2,3)$, $(1,5)$,..., $(1,n-4)$, $(2,n-2)$, $(1,n)$, $(3,n-1)$, $(5,n)$, $(7,n-1)$, $(8,n-3)$,..., $(7,4)$, $(8,2)$, $(6,1)$, $(4,2)$, $(2,1)$, $(1,3)$, $(2,5)$,..., $(2,n-4)$, $(1,n-2)$, $(2,n)$, $(4,n-1)$, $(6,n)$, $(8,n-1)$, $(7,n-3)$,..., $(8,4)$, $(7,2)$, $(5,1)$, $(3,2)$, $(1,1)$, see Figure 3.17 for a cycle on $G(8,13,2)$. This is a contradiction since this cycle does not contain all vertices of $G(8,4l+1,2)$.

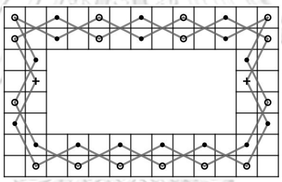

Figure 3.17: A cycle on $G(8,13,2)$

Case 14.2: $k \ge 2$. We obtain a contradiction similar to Case 1 by considering $\{(1,4i+1),(2,4i+1)|0 \le i \le l\}$, $\{(4i+1,1),(4i+1,n)|0 \le i \le k\}$, $\{(4i+2,1),(4i+2,n)|0 \le i \le k-1\}$, $\{(m-4,2),(m-4,n-1)\}$ and $\{(m,4i+2),(m,4i+4)|0 \le i \le l-1\}$ instead, see Figure 3.18 for a cycle on $G(16,17,2)$.

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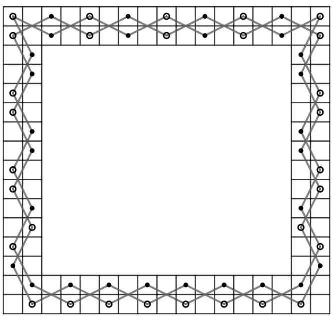

Figure 3.18: A cycle on $G(16,17,2)$

Case 15: $k < l$, $r = 4$ and $q = 2$.

If $k = 1$ and $l \geq 2$, then it is similar to Case 7.1. The vertex $(3, n)$ (indicated by "+" in Figure 3.19) has degree 3 which is more than the degree of the same vertex in the case when $k \geq 2$ and $l \geq 2$.

Case 15.1: $k = 1$ and $l \geq 2$. Since $(1, n-4)$ and $(1, n)$ have only 2 incident edges on $G(8, 4l+2, 2)$, $(1, n-4) - (2, n-2)$ and $(2, n-2) - (1, n)$ must be in H and it forces that $(2, n-2) - (3, n)$ must not be in H. Then, it also forces that $(1, n-1) - (3, n)$ and $(3, n) - (5, n-1)$ must be in H. Next, since all vertice in $\{(1, 4i+1), (7, 4i+2) | 0 \leq i \leq l\}$ and $\{(5, 1)\}$ have only 2 incident edges. Collect $(1, n-1) - (3, n)$ and $(3, n) - (5, n-1)$ which must be in H together with all incident edges from these two sets, it happens to form a cycle $(1, 1)$, $(2, 3)$, $(1, 5)$, ..., $(1, n-5)$, $(2, n-3)$, $(1, n-1)$, $(3, n)$, $(5, n-1)$, $(7, n)$, $(8, n-2)$, $(7, n-4)$,..., $(8, 4)$, $(7, 2)$, $(5, 1)$, $(3, 2)$, $(1, 1)$, see Figure 3.19 for a cycle on $G(8, 14, 2)$. This is a contradiction since this cycle does not contain all vertices of $G(8, 4l+2, 2)$.

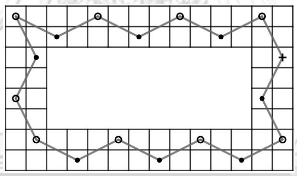

Figure 3.19: A cycle on $G(8, 14, 2)$

Case 15.2: $k \geq 2$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+1), (m-1, 4i+2) | 0 \leq i \leq l\}$, $\{(4i+1, 1) | 0 \leq i \leq k\}$, $\{(m-5, n)\}$ and $\{(4i+1, n-1) | 1 \leq i \leq k-1\}$ instead, see Figure 3.20 for a cycle on $G(16, 18, 2)$.

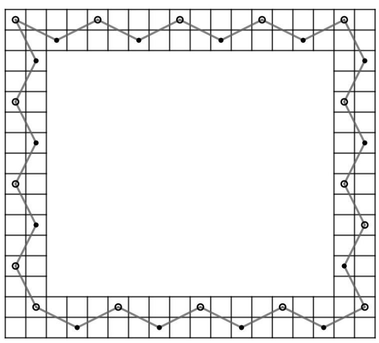

Figure 3.20: A cycle on $G(16, 18, 2)$

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Case 16: $k < l$, $r = 4$ and $q = 3$. We obtain a contradiction similar to Case 1 by considering $\{(1, 4i+1), (2, 4i+1) | 0 \le i \le l-1\}$, $\{(1, n-4), (2, n-4)\}$, $\{(4i+1, 1), (4i+1, n) | 0 \le i \le k\}$, $\{(4i+2, 1), (4i+2, n) | 0 \le i \le k-1\}$, $\{(m-4, 2), (m-4, n-1)\}$, $\{(m, 4i+2) | 0 \le i \le l\}$ and $\{(m, 4i) | 1 \le i \le l\}$ instead, see Figure 3.21 for a cycle on $G(16, 19, 2)$.

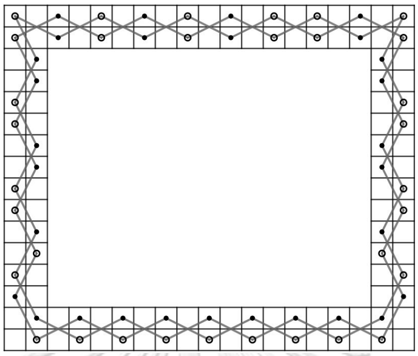

Figure 3.21: A cycle on $G(16, 19, 2)$

This completes the proof. $\square$

Now, we are ready to prove our main theorem about the existence of a CKT on $\mathrm{RB}(m, n, r)$.

Theorem 3.4. *An RB(m, n, r) with $m, n > 2r$ has a CKT if and only if (a) $m = n = 3$ and $r = 1$ or (b) $r \ge 3$.*

Proof. First, for $m, n \ge 3$, $r = 1$ and $(m, n, r) \ne (3, 3, 1)$, the degree of four conner vertices of $G(m, n, 1)$ is at most one. Thus, $\mathrm{RB}(m, n, 1)$ cannot have CKT. For $m, n \ge 5$ and $r = 2$, by Theorem 3.3, an $\mathrm{RB}(m, n, 2)$ has no CKT.

Conversely, for $m = n = 3$ and $r = 1$, it is well-known that an $\mathrm{RB}(3, 3, 1)$ has a CKT. Next, we assume that $r \ge 3$ and $m, n > 2r$.

Case 1: $r = 3$.

Case 1.1: m is odd and n is even, or m is even and n is odd. We partition the $\mathrm{RB}(m, n, 3)$ into $\mathrm{LB}(m, n-3, 3)$ and $7\mathrm{B}(m, n-3, 3)$, see Figure 3.22(a) for $\mathrm{RB}(10, 11, 3)$. Since $m + n - 3$ is even with $m > 6$ and $n - 3 > 3$, by Theorem 2.1(b), the $\mathrm{LB}(m, n-3, 3)$ contains an OKT from $(1, 3)$ to $(2, 2)$ and by Corollary

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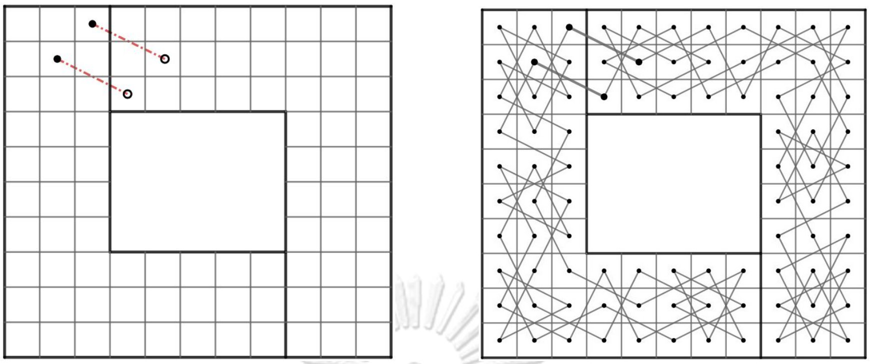
2.2(b), the 7B$(m, n-3, 3)$ contains an OKT from $(3,1)$ to $(2,2)$. By joining $(1,3)$ and $(2,2)$ of LB$(m, n-3, 3)$ to $(2,2)$ and $(3,1)$ of 7B$(m, n-3, 3)$, respectively, we obtain a CKT on RB$(m, n, 3)$ as shown in Figure 3.22(b) for the RB$(10, 11, 3)$.



(a) (b)

Figure 3.22: Two parts of RB$(10, 11, 3)$ and a CKT on RB$(10, 11, 3)$

Case 1.2: m and n are odd or even. We partition the RB$(m, n, 3)$ into LB$(m, n-3, 3)$ and 7B$(m, n-3, 3)$, see Figure 3.23(a) for RB$(11, 13, 3)$. Since $m+n-3$ is odd with $m > 6$ and $n-3 > 3$, by Theorem 2.1(a), the LB$(m, n-3, 3)$ contains an OKT from $(1,2)$ to $(1,3)$ and by Corollary 2.2(a), the 7B$(m, n-3, 3)$ contains an OKT from $(2,1)$ to $(3,1)$. By joining $(1,2)$ and $(1,3)$ of LB$(m, n-3, 3)$ to $(2,1)$ and $(3,1)$ of 7B$(m, n-3, 3)$, respectively, we obtain a CKT on RB$(m, n, 3)$ as shown in Figure 3.23(b) for the RB$(11, 13, 3)$.

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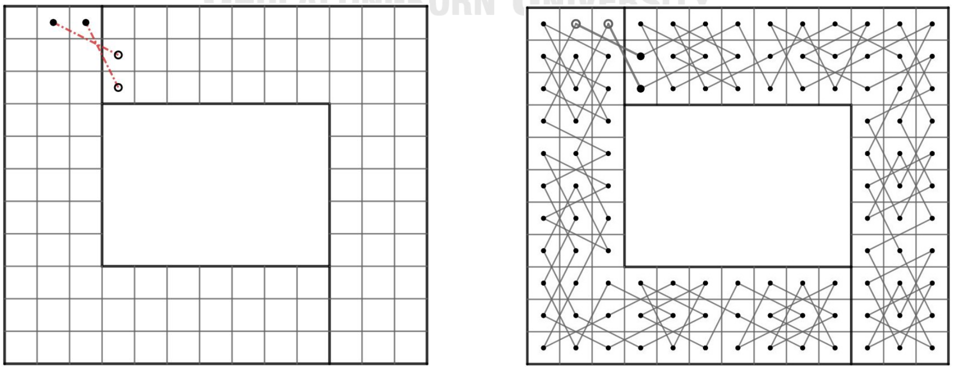
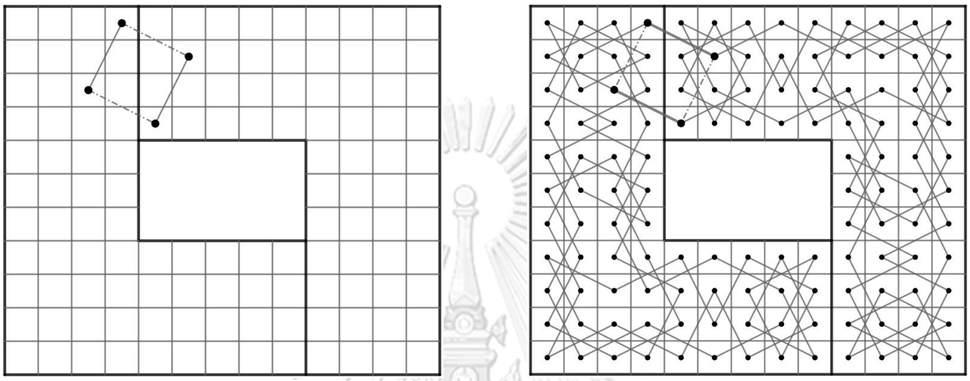


(a) (b)

Figure 3.23: Two parts of RB$(11, 13, 3)$ and a CKT on RB$(11, 13, 3)$

Case 2: $r = 4$. We partition the RB$(m, n, 4)$ into LB$(m, n-4, 4)$ and 7B$(m, n-

4, 4), see Figure 3.24(a) for RB(11, 13, 4). By Theorem 3.1 and Corollary 3.2, the LB$(m, n-4, 4)$ has a CKT that contains an edge $(1,4)-(3,3)$ and 7B$(m, n-4, 4)$ has a CKT that contains an edge $(4,1)-(2,2)$. By deleting $(1,4)-(3,3)$ of LB$(m, n-4, 4)$ and $(4,1)-(2,2)$ of 7B$(m, n-4, 4)$ and joining $(1,4)$ and $(3,3)$ of LB$(m, n-4, 4)$ to $(2,2)$ and $(4,1)$ of 7B$(m, n-4, 4)$, respectively, we obtain a CKT on RB$(m, n, 4)$, as show in Figure 3.24(b) for RB(11, 13, 4).



(a) (b)

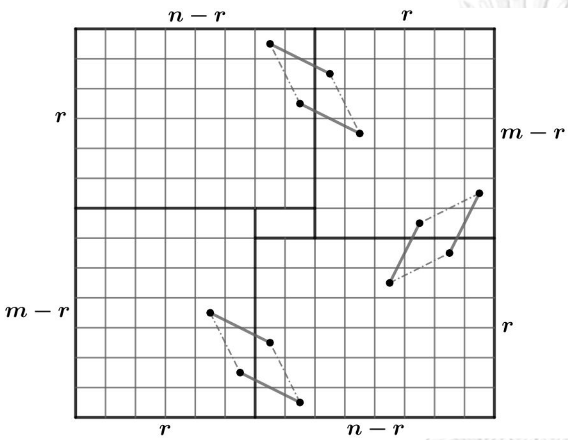
Figure 3.24: Two parts of RB(11, 13, 3) and a CKT on RB(11, 13, 3)

Case 3 $r \geq 5$.

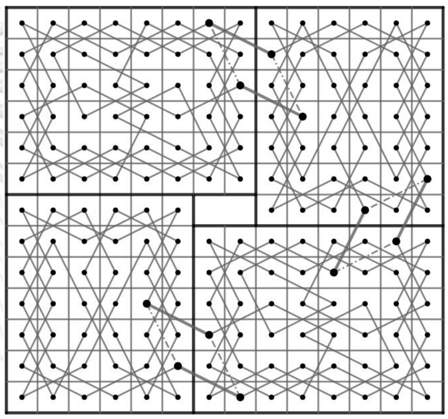
Case 3.1: r is even. We partition the RB(m, n, r) into two CB$(r \times (n-r))$ and two CB$((m-r) \times r)$, see Figure 3.25(a) for RB(13, 14, 6). There are three steps to obtain a CKT having some edges on each partitioned board. First, we consider a CB$(r \times (m-r))$. By Theorem 1.6, it contains a CKT having edges $(1, m-r-1)-(3, m-r)$ and $(r, 2)-(r-1, 4)$. Rotate CB$(r \times (m-r))$ 90 degrees clockwise, we obtain a CKT on CB$((m-r) \times r)$ of the upper right-hand side having edges $(m-r-1, r)-(m-r, r-2)$ and $(2,1)-(4,2)$. Next, rotate CB$(r \times (m-r))$ 90 degrees counterclockwise, we obtain a CKT on CB$((m-r) \times r)$ of the lower left-hand side having edge $(m-r-3, r-1)-(m-r-1, r)$. Finally, we consider a CB$(r \times (n-r))$ on the upper left-hand side. By Theorem 1.6, it contains a CKT having edges $(1, n-r-1)-(3, n-r)$ and $(r, 2)-(r-1, 4)$. Rotate CB$(r \times (n-r))$ 180 degrees clockwise, we obtain a CKT on CB$(r \times (n-r))$ of the lower right-hand side having edges $(r-2, 1)-(r, 2)$ and $(1, n-r-1)-(3, n-r-3)$.

Thus, if we use the position on the RB(m,n,r), there are 4 CKT on each partition having six edges, namely $(1,n-r-1)-(3,n-r)$, $(2,n-r+1)-(4,n-r+2)$, $(m-r-1,n)-(m-r,n-2)$, $(m-r+1,n-1)-(m-r+2,n-3)$, $(m,r+2)-(m-2,r+1)$ and $(m-1,r)-(m-3,r-1)$.

Next, to construct a CKT on RB(m,n,r), we delete these six edges and join these six edges: $(1,n-r-1)-(2,n-r+1)$, $(3,n-r)-(4,n-r+2)$, $(m-r-1,n)-(m-r+1,n-1)$, $(m-r,n-2)-(m-r+2,n-3)$, $(m-1,r)-(m,r+2)$ and $(m-3,r-1)-(m-2,r+1)$ instead, as shown in Figure 3.25(b) for RB$(13,14,6)$.



(a)



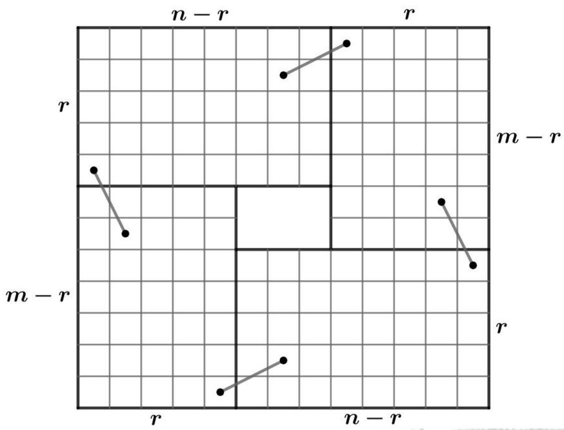
(b)

Figure 3.25: Four parts of RB$(13,14,6)$ and a CKT on RB$(13,14,6)$

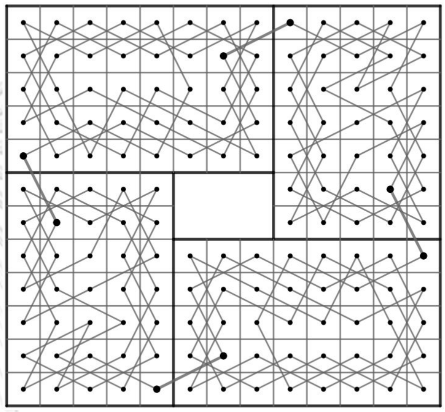
Case 3.2: r is odd. We partition the RB(m,n,r) into two CB$(r\times(n-r))$ and two CB$((m-r)\times r)$, see Figure 3.26(a) for RB$(12,13,5)$. There are three steps to obtain an OKT having two end-points on each partitioned board. First, we consider a CB$(r\times(m-r))$. By Theorem 2.3, it contains an OKT from $(r,1)$ to $(2,m-r-1)$. Rotate CB$(r\times(m-r))$ 90 degrees clockwise, we obtain an OKT from $(1,1)$ to $(m-r-1,r-1)$ on CB$((m-r)\times r)$ of the upper right-hand side. Next, rotate CB$(r\times(m-r))$ 90 degrees counterclockwise, we obtain an OKT from $(m-r,r)$ to $(2,2)$ on CB$((m-r)\times r)$ of the lower left-hand side. Finally, we consider a CB$(r\times(n-r))$ on the upper left-hand side. By Theorem 2.3, it contains an OKT from $(r,1)$ to $(2,n-r-1)$. Rotate CB$(r\times(n-r))$ 180 degrees clockwise, we obtain an OKT from $(1,n-r)$ and $(r-1,2)$ on CB$(r\times(n-r))$ of the lower right-hand side.

Thus, if we use the position on the RB(m,n,r), there are four OKTs on each partition having eight end vertices, namely $(r,1)$, $(2,n-r-1)$, $(1,n-r+1)$, $(m-r-1,n-1)$, $(m-r+1,n)$, $(m-1,r+2)$, (m,r) and $(r+2,2)$.

Next, to construct a CKT on the RB(m,n,r), we join four edges: $(2,n-r-1)-(1,n-r+1)$, $(m-r-1,n-1)-(m-r+1,n)$, $(m-1,r+2)-(m,r)$ and $(r,1)-(r+2,2)$, as shown in Figure 3.26(b) for RB$(12,13,5)$.



(a)



(b)

Figure 3.26: Four parts of RB$(12,13,5)$ and a CKT on RB$(12,13,5)$

This completes the proof. $\square$

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3.3 CKTs on CB($4 \times n$) $- A$

In this section, we consider the existence of CKTs on CB($4 \times n$) $- A$ where $n \geq 3$.

3.3.1 CKTs on CB($4 \times n$) $- A$ where $3 \leq n \leq 6$

For small n such that $3 \leq n \leq 6$, we can only remove some pairs of different colors in the first and the fourth rows.

Lemma 3.5. *There exists a CKT on CB(4×3)$-A$ if and only if* $A = \{(1,2),(4,2)\}$.

Proof. A CKT on CB(4×3) $- \{(1,2),(4,2)\}$ is shown in Figure 3.27.

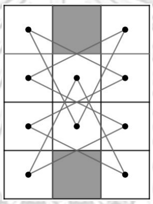

Figure 3.27: A CKT on CB(4×3) $- \{(1,2),(4,2)\}$

Conversely, there are 4 cases, namely (i) $A \in \{\{(1,1),(4,1)\},\{(1,3),(4,3)\}\}$ (ii) $A \in \{\{(1,1),(4,3)\},\{(1,3),(4,1)\}\}$ (iii) $A \in \{\{(1,1),(1,2)\},\{(1,2),(1,3)\},\{(4,1),(4,2)\},\{(4,2),(4,3)\}\}$ and (iv) $A \neq \{(1,2),(4,2)\}$ and is not in cases (i) - (iii). Figure 3.28 from left to right represents each case (i) - (iii) scenario according to their symmetry and also shows components of $(G(4\times 3)-A)-S$, where the shaded squares are elements in A and the crossed squares are elements in S.

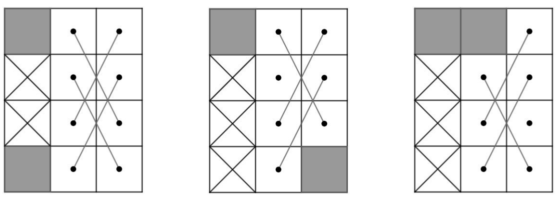

Figure 3.28: Components of $(G(4 \times 3) - A) - S$ in cases (i) - (iii)

It is clear from Theorem 1.5(a) that the CKT does not exist on CB(4×3) $- A$, where A is in cases (i) - (iii) and Proposition 1.11 also implies that if A is in case (iv), then the CKT does not exist on CB$(4 \times 3) - A$. $\square$

Lemma 3.6. *There exists a CKT on CB*$(4{\times}4){-}A$ *if and only if* $A \in \{\{(1,1),(1,4)\},$ $\{(1,1),(4,1)\},$ $\{(1,4),(4,4)\},$ $\{(4,1),(4,4)\}\}$.

Proof. According to the symmetry, a CKT on CB$(4 \times 4) - \{(1,1),(1,4)\}$ is shown in Figure 3.29. If A is not in $\{\{(1,1),(1,4)\},$ $\{(1,1),(4,1)\},$ $\{(1,4),(4,4)\},$ $\{(4,1),$

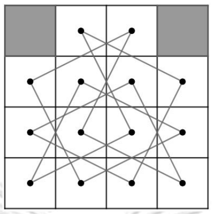

Figure 3.29: A CKT on CB$(4 \times 4) - \{(1,1),(1,4)\}$

$(4,4)\}\}$, then Proposition 1.11 can be apply to conclude the nonexistence of CKTs on CB$(4 \times 4) - A$.

One of A from the first row and one of A from the fourth row which are the same parity color. The Proposition 1.11 can be apply immediately.

One of A is in either the second or the third column. The rotation make this square is on the middle two rows. Then the Proposition 1.11 can be apply. $\square$

Lemma 3.7. *There exists a CKT on CB*$(4{\times}5){-}A$ *if and only if* $A \in \{\{(1,1),(1,2)\},$ $\{(1,4),(1,5)\},$ $\{(4,1),(4,2)\},$ $\{(4,4),(4,5)\},$ $\{(1,1),(4,1)\},$ $\{(1,5),(4,5)\},$ $\{(1,1),$ $(4,5)\},$ $\{(1,5),(4,1)\}\}$.

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Proof. First, we consider 3 cases, namely (i) $A \in \{\{(1,1),(1,2)\},$ $\{(1,4),(1,5)\},$ $\{(4,1),(4,2)\},$ $\{(4,4),(4,5)\}\}$ (ii) $A \in \{\{(1,1),(4,1)\},$ $\{(1,5),(4,5)\}\}$ and (iii) $A \in \{\{(1,1),(4,5)\},$ $\{(1,5),(4,1)\}\}$. A CKT on CB$(4 \times 5) - A$, where $A \in \{\{(1,1),(1,2)\},$ $\{(1,1),(4,1)\},$ $\{(1,1),(4,5)\}\}$, is shown in Figure 3.30. By its symmetry of each of Figure 3.30, A CKT on CB$(4 \times 5) - A$ is obtained for each remaining A of each case, respectively.

Conversely, there are 7 cases, namely (i) $A \in \{\{(1,2),(1,3)\},$ $\{(1,3),(1,4)\},$ $\{(4,2),(4,3)\},$ $\{(4,3),(4,4)\}\}$ (ii) $A = \{(1,3),(4,3)\}$ (iii) $A \in \{\{(1,1),(1,4)\},$ $\{(1,2),(1,5)\},$ $\{(4,1),(4,4)\},$ $\{(4,2),(4,5)\}\}$ (iv) $A \in \{\{(1,2),(4,4)\},$ $\{(1,4),$

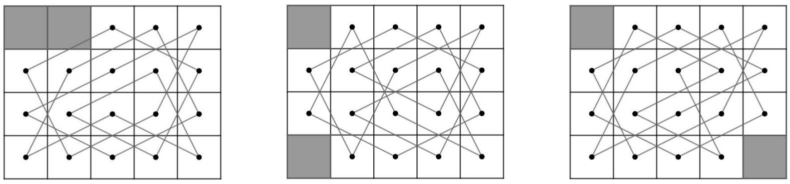

Figure 3.30: CKTs on CB$(4 \times 5) - A$

$(4,2)\}\}$ (v) $A \in \{\{(1,1),(4,3)\}, \{(1,3),(4,1)\}, \{(1,3),(4,5)\}, \{(1,5),(4,3)\}\}$ (vi) $A \in \{\{(1,2),(4,2)\}, \{(1,4),(4,4)\}\}$ and (vii) A is not in $\{\{(1,1),(1,2)\}$, $\{(1,4),(1,5)\}$, $\{(4,1),(4,2)\}$, $\{(4,4),(4,5)\}$, $\{(1,1),(4,1)\}$, $\{(1,5),(4,5)\}$, $\{(1,1),(4,5)\}$, $\{(1,5),(4,1)\}\}$ and is not in cases (i) - (vi). Figure 3.31 from left to right of the first and the second rows represents each case (i) - (vi) scenario according to their symmetry and also shows components of $(G(4 \times 5) - A) - S$, where the shaded squares are elements in A and the crossed squares are elements in S. It is clear from Theorem 1.5(a) that the CKT does not exist on CB$(4 \times 5) - A$, where A is in cases (i) - (vi).

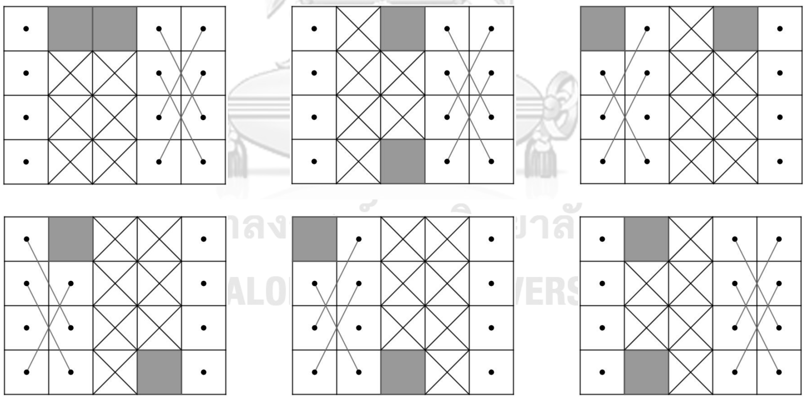

Figure 3.31: Components of $(G(4 \times 5) - A) - S$ in cases (i) - (vi)

In the case (vii), either one of A is from the middle two rows or the two squares of A are the same parity color. Then, Proposition 1.11 also implies the nonexistence of CKTs on CB$(4 \times 5) - A$. $\square$

Lemma 3.8. *There exists a CKT on CB$(4 \times 6) - A$ if and only if* $A \in \{\{(1,1),(4,1)\}$, $\{(1,6),(4,6)\}$, $\{(1,1),(1,2)\}$, $\{(1,5),(1,6)\}$, $\{(4,1),(4,2)\}$, $\{(4,5),(4,6)\}$, $\{(1,2),$

$(4,2)\}$, $\{(1,5),(4,5)\}$, $\{(1,1),(1,6)\}$, $\{(4,1),(4,6)\}$, $\{(1,2),(1,5)\}$, $\{(4,2),(4,5)\}$, $\{(1,1),(4,5)\}$, $\{(1,2),(4,6)\}$, $\{(1,5),(4,1)\}$, $\{(1,6),(4,2)\}\}$.

Proof. First, we consider 6 cases, namely (i) $A \in \{\{(1,1),(4,1)\}, \{(1,6),(4,6)\}\}$ (ii) $A \in \{\{(1,1),(1,2)\}, \{(1,5),(1,6)\}, \{(4,1),(4,2)\}, \{(4,5),(4,6)\}\}$ (iii) $A \in \{\{(1,2),(4,2)\}, \{(1,5),(4,5)\}\}$ (iv) $A \in \{\{(1,1),(1,6)\}, \{(4,1),(4,6)\}\}$ (v) $A \in \{\{(1,2),(1,5)\}, \{(4,2),(4,5)\}\}$ and (vi) $A \in \{\{(1,1),(4,5)\}, \{(1,2),(4,6)\}, \{(1,5),(4,1)\}, \{(1,6),(4,2)\}\}$. A CKT on CB$(4 \times 6) - A$, where $A \in \{\{(1,1),(4,1)\}, \{(1,1),(1,2)\}, \{(1,2),(4,2)\}, \{(1,1),(1,6)\}, \{(1,2),(1,5)\}, \{(1,1),(4,5)\}\}$, is shown in Figure 3.32. By its symmetry of each of Figure 3.32, A CKT on CB$(4 \times 6) - A$ is obtained for each remaining A of each case, respectively.

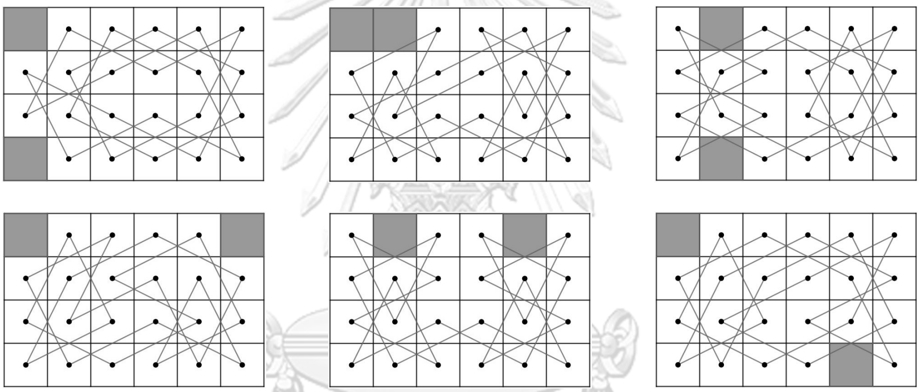

Figure 3.32: CKTs on CB$(4 \times 6) - A$

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Conversely, there are 7 cases, namely (i) $A \in \{\{(1,1),(4,3)\}, \{(1,3),(4,1)\}, \{(1,4),(4,6)\}, \{(1,6),(4,4)\}\}$ (ii) $A \in \{\{(1,2),(4,4)\}, \{(1,3),(4,5)\}, \{(1,4),(4,2)\}, \{(1,5),(4,3)\}\}$ (iii) $A \in \{\{(1,1),(1,4)\}, \{(1,3),(1,6)\}, \{(4,1),(4,4)\}, \{(4,3),(4,6)\}\}$ (iv) $A \in \{\{(1,3),(4,3)\}, \{(1,4),(4,4)\}\}$ (v) $A \in \{\{(1,2),(1,3)\}, \{(1,4),(1,5)\}, \{(4,2),(4,3)\}, \{(4,4),(4,5)\}\}$ (vi) $A \in \{\{(1,3),(1,4)\}, \{(4,3),(4,4)\}\}$ and (vii) A is not in $\{\{(1,1),(4,1)\}, \{(1,6),(4,6)\}, \{(1,1),(1,2)\}, \{(1,5),(1,6)\}, \{(4,1),(4,2)\}, \{(4,5),(4,6)\}, \{(1,2),(4,2)\}, \{(1,5),(4,5)\}, \{(1,1),(1,6)\}, \{(4,1),(4,6)\}, \{(1,2),(1,5)\}, \{(4,2),(4,5)\}, \{(1,1),(4,5)\}, \{(1,2),(4,6)\}, \{(1,5),(4,1)\}, \{(1,6),(4,2)\}\}$ and is not in cases (i) - (vi). Figure 3.33 from left to right of the first and the second rows represents each case (i) - (vi) scenario according to their symmetry and also shows components of $(G(4 \times 6) - A) - S$, where the shaded squares are elements in A and the crossed squares are elements in S. It is clear from Theorem 1.5(a) that the CKT does not exist on CB$(4 \times 6) - A$, where A is in cases (i) - (vi).

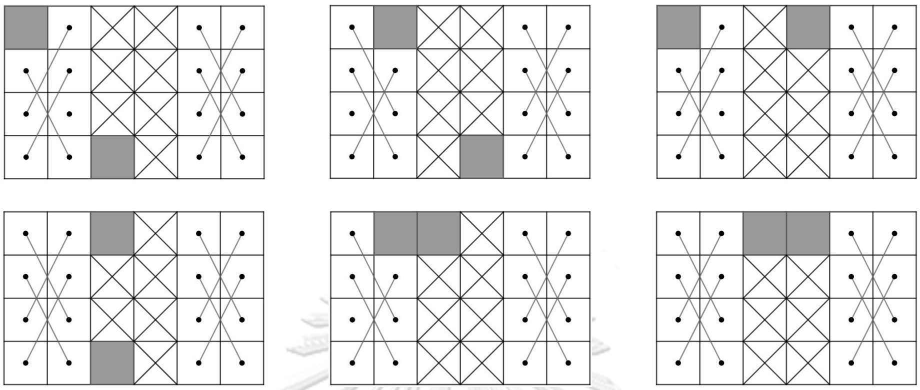

Figure 3.33: Components of $(G(4 \times 6) - A) - S$ in cases (i) - (vi)

In the case (vii), either one of A is from the middle two rows or the two squares of A are the same parity color. Then, Proposition 1.11 also implies the nonexistence of CKTs on CB$(4 \times 6) - A$. □

3.3.2 CKTs on CB$(4 \times n) - A$ where $n \geq 7$

By mainly using the mathematical induction and using special OKTs constructed in Section 2.3 Chapter II in some cases, we can prove our main result which is the Conjecture 1 as follows.

Theorem 3.9. *Consider CB$(4 \times n)$ with $n \geq 7$. For any pair of squares, with one of each parity of color and neither coming from the middle two rows, there is a CKT on the board that avoids only these two squares.*

Proof. Let $n \geq 7$ and

$$S_n = \{\{(x,y),(z,w)\} \mid (x,z \in \{1,4\}, 1 \leq y,w \leq n, (x,y) \neq (z,w)) \text{ and } ((x+y \text{ is odd and } z+w \text{ is even}) \text{ or } (x+y \text{ is even and } z+w \text{ is odd}))\}.$$

Now, we consider CB$(4 \times n) - A$ with $n \geq 7$ and $A \in S_n$. Let $n = a + 3k$ where $a \in \{7, 8, 9\}$ and $k \in \mathbb{N} \cup \{0\}$. We prove by the mathematical induction on k.

First, for $k = 0$, we construct the CKTs on $\mathrm{CB}(4 \times a) - A$ for some $A \in S_a$ as shown in Figures 3.34, 3.35 and 3.36. Note that actually the CKTs on $\mathrm{CB}(4 \times a) - A$ for all $A \in S_a$ can be obtained from the diagrams represented in Figures 3.34, 3.35 and 3.36 according to its symmetry. Note that some $\mathrm{CB}(4 \times a) - A$ in Figures 3.34, 3.35, 3.36 contain edges $(1, a) - (3, a-1)$ and $(2, a-1) - (4, a)$.

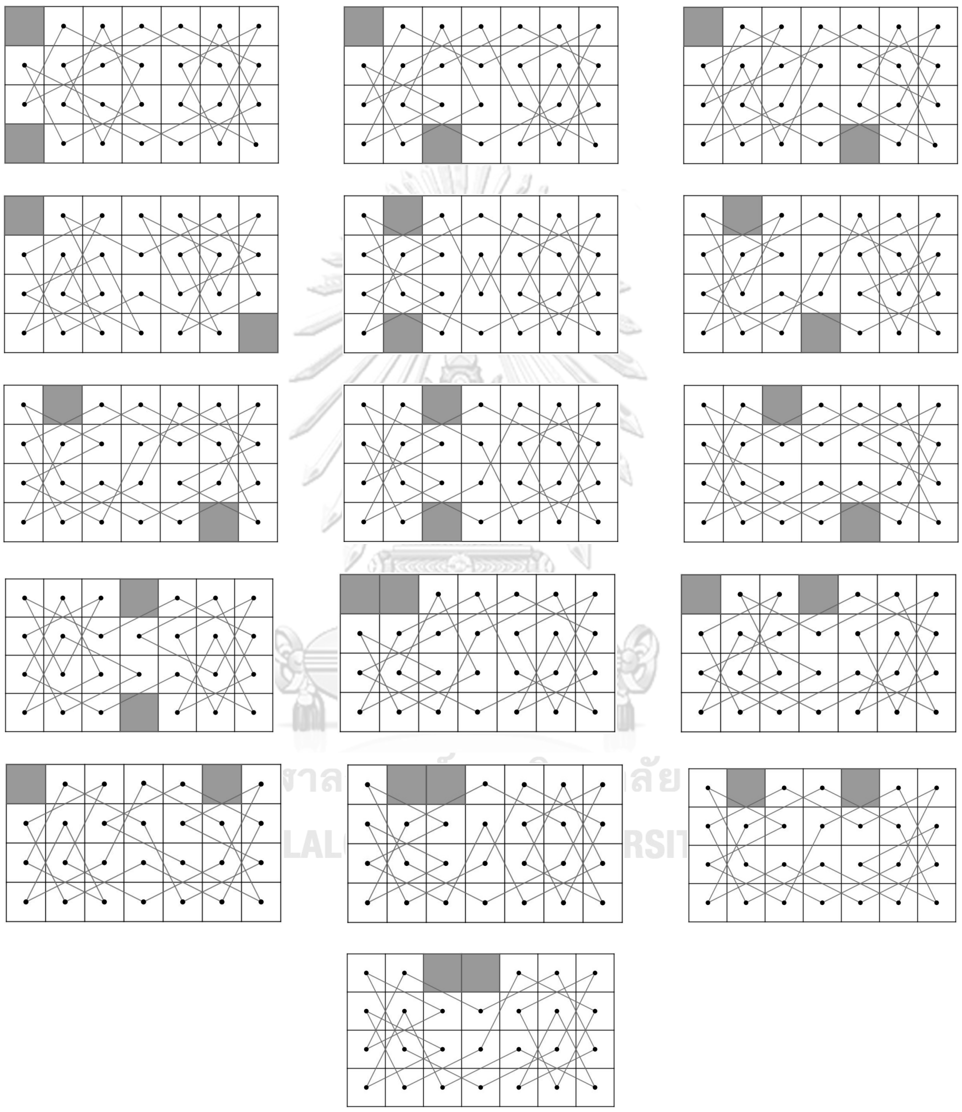

Figure 3.34: CKTs on $\mathrm{CB}(4 \times 7) - A$ for some $A \in S_7$

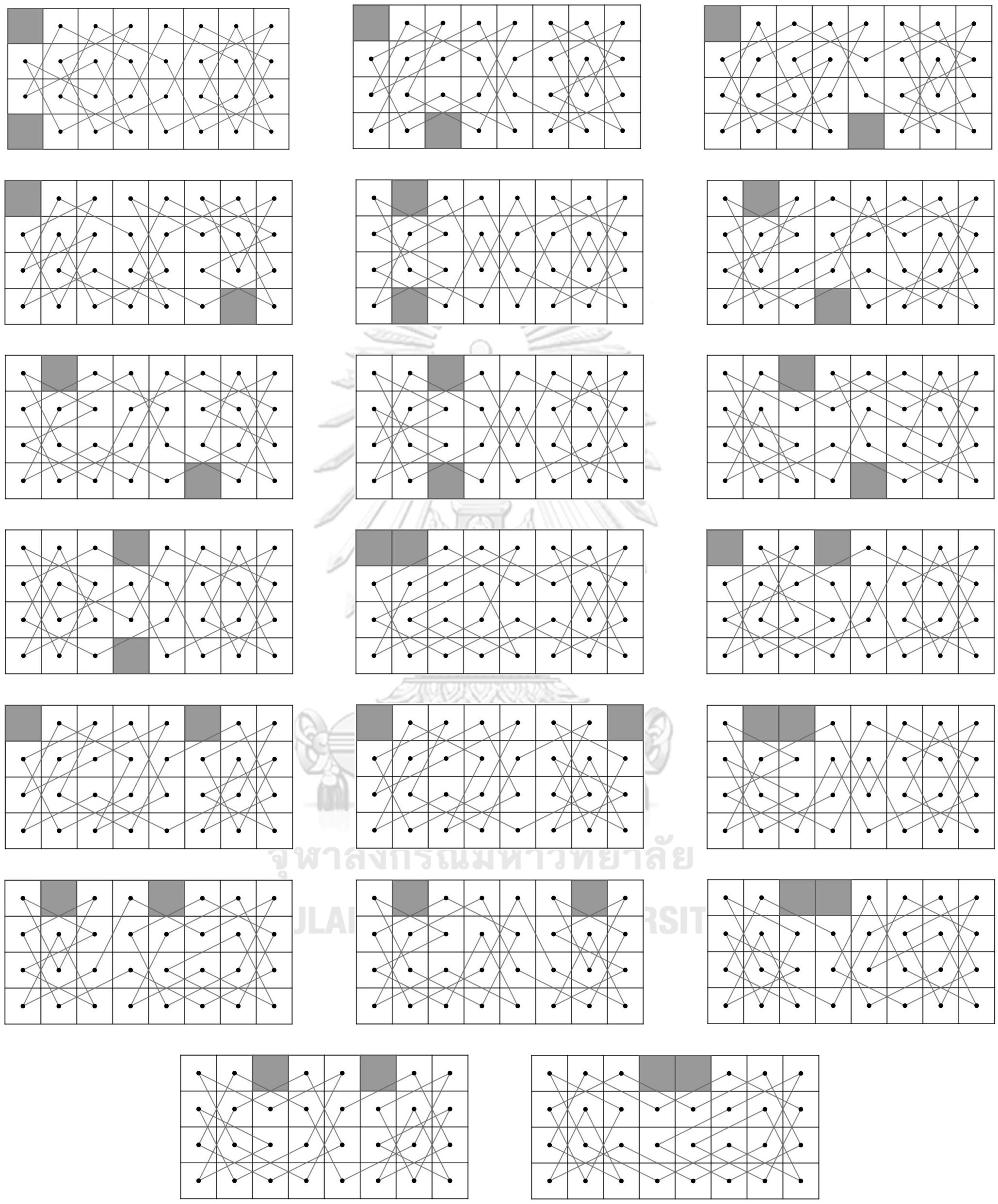

Figure 3.35: CKTs on $\mathrm{CB}(4 \times 8) - A$ for some $A \in S_8$

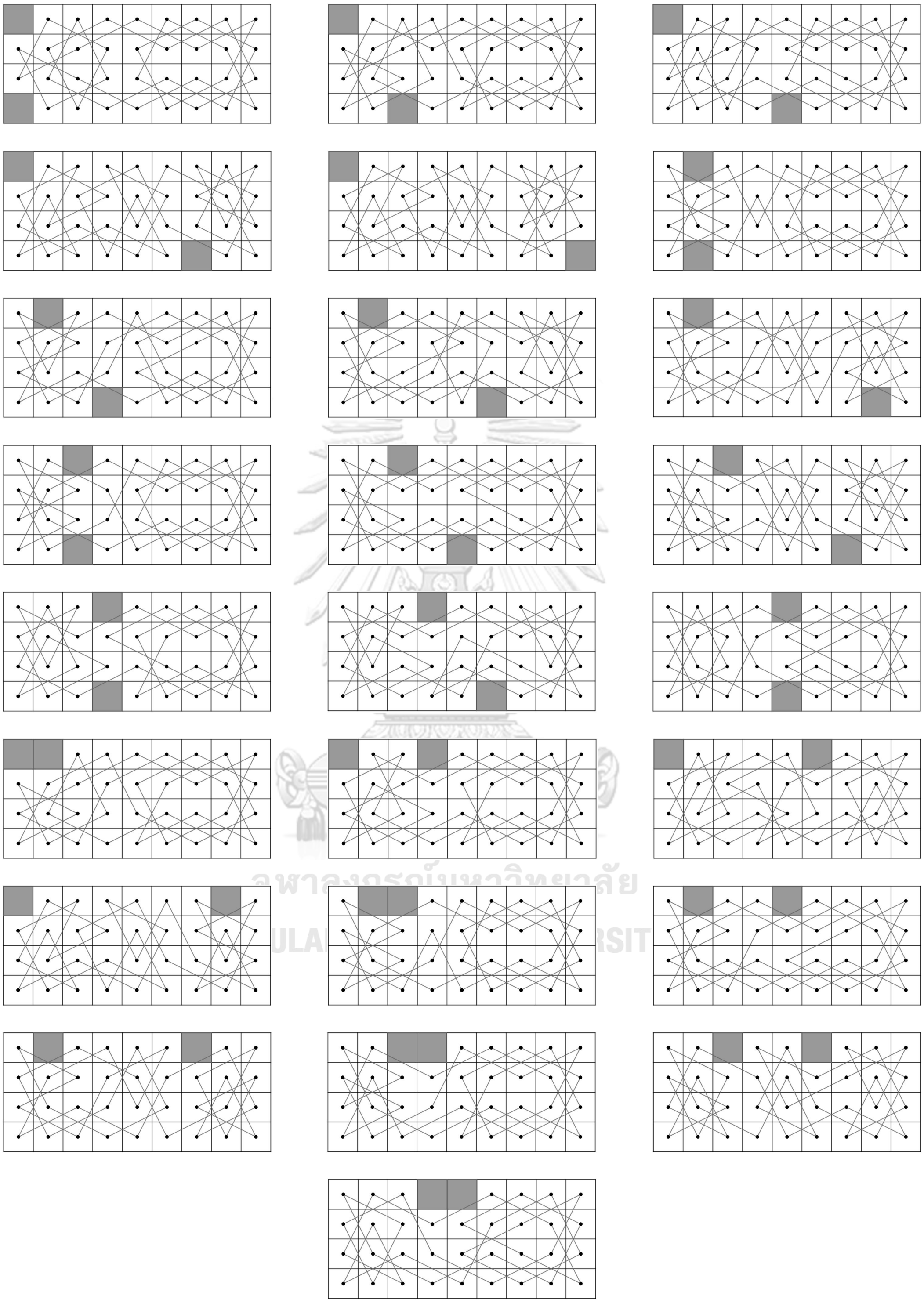

Figure 3.36: CKTs on $\mathrm{CB}(4 \times 9) - A$ for some $A \in S_9$

Next, let $k \geq 0$ be an integer. Assume that $\mathrm{CB}(4 \times (a+3k)) - B$ contains a CKT for all $B \in S_{a+3k}$. Let $A = \{(x,y),(z,w)\} \in S_{a+3(k+1)}$.

Case 1: $1 \leq y, w \leq a+3k-2$.

We separate $\mathrm{CB}(4\times(a+3(k+1)))-A$ into two sub-boards, $\mathrm{CB}(4\times(a+3k))-A$ and $\mathrm{CB}(4\times 3)$ as shown in Figure 3.37 with $A = \{(1,1),(4,1)\}$.

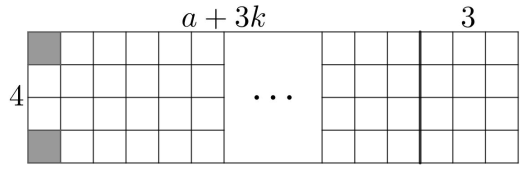

Figure 3.37: $\mathrm{CB}(4 \times (a+3(k+1))) - \{(1,1),(4,1)\}$ with two sub-boards

Since $A \in S_{a+3k}$, by the induction hypothesis, the sub-board $\mathrm{CB}(4\times(a+3k))-A$ contains a CKT. For the sub-board $\mathrm{CB}(4\times 3)$, we construct two cycles C_1 and C_2 as shown in Figure 3.38.

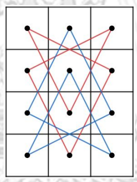

Figure 3.38: Two cycles C_1 and C_2 on $\mathrm{CB}(4 \times 3)$

Since $(1, a+3k)$ and $(4, a+3k)$ have degree 2 in $G(4\times(a+3k)) - A$, $(1, a+3k) - (3, a+3k-1)$ and $(2, a+3k-1) - (4, a+3k)$ are two edges of the CKT on the sub-board $\mathrm{CB}(4\times(a+3k)) - A$.

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Then, we construct the required CKT by

(i) delete $(1, a+3k) - (3, a+3k-1)$ and $(2, a+3k-1) - (4, a+3k)$ of the CKT on the sub-board $\mathrm{CB}(4\times(a+3k)) - A$ and delete $(1,1)-(3,2)$ and $(2,2)-(4,1)$ of C_1 and C_2 on the sub-board $\mathrm{CB}(4\times 3)$, respectively;

(ii) join $(1, a+3k)$ and $(3, a+3k-1)$ of the CKT on the sub-board $\mathrm{CB}(4\times(a+3k))-A$ to $(2,2)$ and $(4,1)$ of C_2 on the sub-board $\mathrm{CB}(4\times 3)$, respectively and join $(2, a+3k-1)$ and $(4, a+3k)$ of the CKT on the sub-board $\mathrm{CB}(4\times(a+3k)) - A$ to $(1,1)$ and $(3,2)$ of C_1 on the sub-board $\mathrm{CB}(4\times 3)$, respectively.

The constructed CKT is shown in Figure 3.39 with $A = \{(1,1),(4,1)\}$.

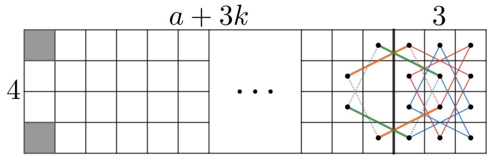

Figure 3.39: A CKT on $\mathrm{CB}(4 \times (a+3(k+1))) - \{(1,1),(4,1)\}$

Case 2: $a + 3k - 1 \leq y, w \leq a + 3(k+1)$.

The required CKT can be obtained by rotating 180 degrees of the suitable CKT on $\mathrm{CB}(4 \times (a+3(k+1))) - A$ in Case 1.

Case 3: $x + y$ is odd, $z + w$ is even, $1 \leq y \leq 5$ and $a + 3k - 1 \leq w \leq a + 3(k+1)$.

We separate $\mathrm{CB}(4 \times (a+3(k+1))) - \{(x,y),(z,w)\}$ into two sub-boards, $\mathrm{CB}(4 \times (a+3k-2)) - (x,y)$ and $\mathrm{CB}(4 \times 5) - (z,w)$ as shown in Figure 3.40 with $(x,y) = (1,2)$ and $(z,w) = (4, a+3(k+1))$.

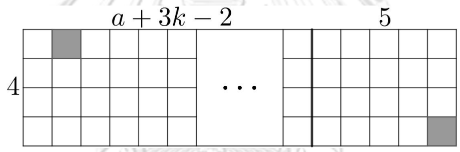

Figure 3.40: $\mathrm{CB}(4 \times (a+3(k+1))) - \{(1,2),(4,a+3(k+1))\}$ with two sub-boards

Case 3.1: (k is even and $a \in \{7,9\}$) or (k is odd and $a = 8$).

In this case, we have $a + 3k - 2 \geq 5$ is odd. Since $x + y$ is odd, by Lemma 2.4(b), the sub-board $\mathrm{CB}(4 \times (a+3k-2)) - \{(x,y)\}$ contains an OKT from $(1, a+3k-2)$ to $(3, a+3k-2)$.

If we regard (z,w) as the square of $\mathrm{CB}(4 \times (a+3(k+1)))$, then $z + w$ is even. However, if we regard (z,w) as the square of the sub-board $\mathrm{CB}(4 \times 5)$, then $z + w$ is odd. By Lemma 2.4(c), the sub-board $\mathrm{CB}(4 \times 5) - \{(z,w)\}$ contains an OKT from $(1,1)$ to $(3,1)$.

Then, as shown in Figure 3.41 with $(x,y) = (1,2)$ and $(z,w) = (4, a+3(k+1))$, we construct the required CKT on $\mathrm{CB}(4 \times (a+3(k+1))) - \{(x,y),(z,w)\}$ by joining $(1, a+3k-2)$ and $(3, a+3k-2)$ of the OKT on the sub-board $\mathrm{CB}(4 \times (a+3k-2)) - \{(x,y)\}$ to $(3,1)$ and $(1,1)$ of the OKT on the sub-board $\mathrm{CB}(4 \times 5) - \{(z,w)\}$, respectively.

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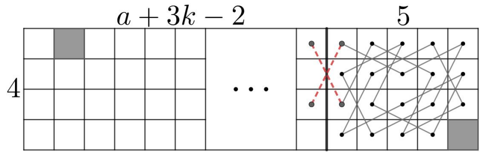

Figure 3.41: The required CKT on $\mathrm{CB}(4\times(a+3(k+1)))-\{(1,2),(4,a+3(k+1))\}$

Case 3.2: (k is odd and $a \in \{7,9\}$) or (k is even and $a = 8$).

In this case, we have $a+3k-2 \geq 6$ is even. Since $x+y$ is odd, by Lemma 2.5(a), the sub-board $\mathrm{CB}(4\times(a+3k-2))-\{(x,y)\}$ contains an OKT from $(2, a+3k-2)$ to $(4, a+3k-2)$.

If we regard (z,w) as the square of $\mathrm{CB}(4\times(a+3(k+1)))$, then $z+w$ is even. Similarly, if we regard (z,w) as the square of the sub-board $\mathrm{CB}(4\times 5)$, then $z+w$ is even. By Lemma 2.4(d), the sub-board $\mathrm{CB}(4\times 5)-\{(z,w)\}$ contains an OKT from $(2,1)$ to $(4,1)$.

Then, as shown in Figure 3.42 with $(x,y) = (1,2)$ and $(z,w) = (1, a+3(k+1))$, we construct the required CKT on $\mathrm{CB}(4\times(a+3(k+1)))-\{(x,y),(z,w)\}$ by joining $(2, a+3k-2)$ and $(4, a+3k-2)$ of the OKT on the sub-board $\mathrm{CB}(4\times(a+3k-2))-\{(x,y)\}$ to $(4,1)$ and $(2,1)$ of the OKT on the sub-board $\mathrm{CB}(4\times 5)-\{(z,w)\}$, respectively.

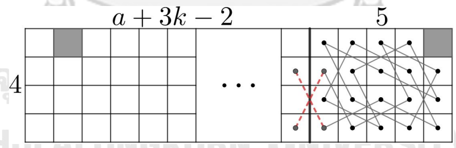

Figure 3.42: The required CKT on $\mathrm{CB}(4\times(a+3(k+1)))-\{(1,2),(1,a+3(k+1))\}$

Case 4: $x+y$ is even, $z+w$ is odd, $1 \leq y \leq 5$ and $a+3k-1 \leq w \leq a+3(k+1)$.

The required CKT can be obtained by horizontally or vertically flipping of the suitable CKT on $\mathrm{CB}(4\times(a+3(k+1))) - A$ in Case 3.

Hence, in every cases, $\mathrm{CB}(4\times(a+3(k+1))) - A$ contains a CKT for all $A \in S_{a+3(k+1)}$. Thus, by the mathematical induction, we obtain the CKT on $\mathrm{CB}(4\times n) - A$ for all $A \in S_n$. This completes the proof.

$\square$

CHAPTER IV
CONCLUSION AND DISCUSSION

There are two main results of this dissertation. First, we obtain the necessary and sufficient conditions for the existence of a CKT on the RB(m, n, r). Next, we find all positions of two squares on CB$(4 \times n)$ so that after deleting these squares, then there exists a CKT on the deficient board. This result proves Conjecture 1.

In addition, this dissertation also includes some results related to the existence of OKTs and CKTs on some boards as follows.

The existence of some special OKTs on LB$(m, n, 3)$, 7B$(m, n, 3)$, CB$(m \times n)$ and CB$(m \times n) - \{(i, j)\}$ are given. Next, the existence of a CKT on LB$(m, n, 4)$, CB$(4 \times 3) - A$, CB$(4 \times 4) - A$, CB$(4 \times 5) - A$ and CB$(4 \times 6) - A$ for some set A containing two squares of CB$(4 \times n)$, $n \in \{3, 4, 5, 6\}$, are given. After that, we prove the extended result of Theorem 1.9 stating that there is no CKT on RB$(m, n, 2)$.

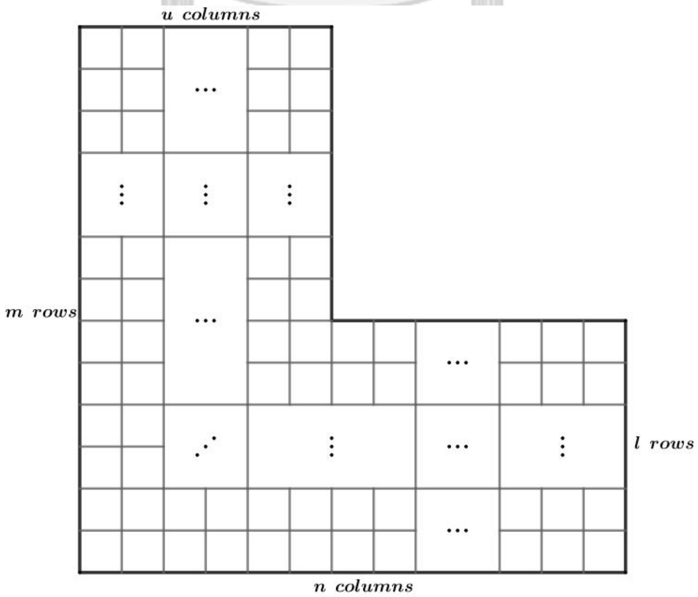

Figure 4.1: LB(m, n, u, l)

In the future, an interesting study is to find necessary and sufficient conditions for the existence of a CKT for the general L-board, namely LB(m, n, l, u), which is the L-board consisting of m rows n with the lower leg of width l and the upper leg of width u, see Figure 4.1. However, there is another interesting study as follows.

We note that the CKT of this dissertation is constructed using the legal knight's move. In 2005, Chia and Ong [5] defined the generalized knight's move or (a, b)-knight's move for which the knight moves a squares vertically or horizontally and then b squares at 90 degrees angle. Especially, they gave the existence of a CKT using the $(2, 3)$-knight's move on some CB$(m \times n)$. After that, there are some researchers [12] studied the nonexistence of CKTs using the (a, b)-knight's move on some CB$(m \times n)$. Therefore, as a future research, if we consider some CB$(m \times n)$ for which a CKT from the generalized knight's move does not exist, then we can investigate the minimum numbers of square to be removed and a CKT from the generalized knight's move exists on the deficient board as well as the exact positions of these squares to be removed.

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