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ชื่อโครงการ การศึกษาริงซึ่งไอดีลทางขวาเฉพาะเป็นไอดีลสองทาง
Study on rings whose prime right ideals are two-sided


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# การศึกษาริงซึ่งไอดีลทางขวาเฉพาะเป็นไอดีลสองทาง

นางสาวชนากานต์ ละออเอี่ยม

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STUDY ON RINGS WHOSE PRIME RIGHT IDEALS ARE TWO-SIDED

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Department of Mathematics and Computer Science
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ชนากานต์ ละออเอี่ยม: การศึกษาริงซึ่งไอดีลทางขวาเฉพาะเป็นไอดีลสองทาง. (STUDY ON RINGS WHOSE PRIME RIGHT IDEALS ARE TWO-SIDED) อ.ที่ปรึกษาโครงงานหลัก : รศ. ดร.ศจี เพียรสกุล, อ.ที่ปรึกษาโครงงานร่วม : รศ.ดร.เอื้อมพร ฟักสุวรรณ, 32 หน้า.

ในโครงงานนี้เราศึกษาริงซึ่งไอดีลทางขวาเฉพาะเป็นไอดีลสองทางและตรวจสอบสมบัติบางประการของริงนี้ โดยเฉพาะอย่างยิ่ง เราได้ว่าไอดีลทางขวาเฉพาะของริงที่ศึกษานี้มีความเกี่ยวข้องกับไอดีลดี แต่อย่างไรก็ตาม ข้อจำกัดบางอย่างของริงนี้ผลักดันให้เราศึกษาริงซึ่งไอดีลทางขวาเฉพาะบริบูรณ์เป็นไอดีลสองทาง เราตรวจสอบสมบัติเกี่ยวกับริงผลหาร และผลตัดของทุกไอดีลทางขวาเฉพาะบริบูรณ์ซึ่งมีความเกี่ยวข้องกับสมาชิกนิรพลของริงนี้

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CHANAGARN LAOIAM: STUDY ON RINGS WHOSE PRIME RIGHT IDEALS ARE TWO-SIDED. ADVISOR : ASSOC. PROF. SAJEE PIANSKOOL, PH.D., CO-ADVISOR : ASSOC. PROF. OUAMPORN PHUKSUWAN, PH.D., 32 PP.

In this project, we study on rings whose prime right ideals are two-sided and investigate some properties of such rings. In particular, we obtain that prime right ideals of these rings relate to good ideals. However, some restriction on this ring motivates us to study the ring whose completely prime right ideals are two-sided. We investigate some of their properties about a quotient ring; moreover, about the intersection of all completely prime right ideals which relates to nilpotent elements in this ring.

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# Contents

# Chapter 1

# Introduction

Let $R$ be an associative ring with identity. Any subgroup $I$ of $R$ is called a *right* (*left*) *ideal* of $R$ if $Ir \subseteq I$ ($rI \subseteq I$) for any $r \in R$. A *two-sided ideal* (or simply called an *ideal*) of $R$ is both a left ideal and a right ideal of $R$. For any nonempty subsets $I$ and $J$ of $R$, define

$$IJ = \left\{ \sum_{k=1}^{n} i_k j_k \mid n \in \mathbb{N},\ i_k \in I,\ j_k \in J \text{ for all } k = 1, \ldots, n \right\}$$

$$\text{and} \quad I \circ J = \{ ij \mid i \in I \text{ and } j \in J \}.$$

It is obvious that $I \circ J \subseteq IJ$ but $IJ$ is not necessarily a subset of $I \circ J$. It is interesting to consider the case when $IJ = I \circ J$. Notice here that if $I$ and $J$ are ideals, then $IJ$ is also an ideal but $I \circ J$ may be not. An ideal $I$ of $R$ is called a *good ideal* if $(r_1 + I)(r_2 + I) = (r_1 + I) \circ (r_2 + I)$ for any $r_1,\ r_2 \in R$. A ring $R$ is called a *good ring* if all ideals of $R$ are good ideals. A ring $R$ is called a *right duo ring* if any right ideal of $R$ is two-sided. A ring $R$ is called a *von Neuman regular* (or simply called *regular*) ring if each principal right ideal of $R$ is generated by an idempotent, which is an element $e \in R$ such that $e^2 = e$. A ring $R$ is called a *strongly regular* ring if it is a regular ring and a right duo ring. In 1992, L. Huang and W. Xue [1] studied some properties of right duo rings relating to good ideals. Moreover, they obtained a relation between strongly regular rings and good rings.

An element $x$ in $R$ is called a *nilpotent element* of $R$ if $x^n = 0$ for some $n \in \mathbb{N}$. A proper ideal $I$ of $R$ is called a *prime ideal* if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for any ideals $A$ and $B$ of $R$. The *prime radical* of $R$ is the intersection of all prime ideals of $R$. In 1971, K. Koh [2] studied some properties of nilpotent elements relating to the prime radical.

A proper right ideal $I$ of $R$ is called a *prime right ideal* if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for any right ideals $A$ and $B$ of $R$. A proper right ideal $I$ of $R$ is called a *completely prime right ideal* if for any $a, b \in R$ such that $aI \subseteq I$, $ab \in I$ implies that either $a \in I$ or $b \in I$. In a commutative ring $R$, prime right ideals of $R$ and completely prime right ideals of $R$ coincide but they may not be in a noncommutative ring. For example, let $F$ be a field,

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{bmatrix} \mid a, b, c, d, e \in F \right\} \quad \text{and} \quad I = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix} \mid x, y \in F \right\}.$$

We can prove that $I$ is a completely prime right ideal of $R$ but is not a prime right ideal. To show this, let

$$m = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{bmatrix}, n = \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & 0 \\ 0 & 0 & e_2 \end{bmatrix} \in R$$

where $a_1, b_1, c_1, d_1, e_1, a_2, b_2, c_2, d_2, e_2 \in F$ be such that $mn \in I$ and $mI \subseteq I$ and assume that $m \notin I$. Then we must show that $n \in I$. Let $p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where 1 is the identity of $F$. Then $p \in I$ and so

$$\begin{bmatrix} 0 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = mp \in mI \subseteq I.$$

Then $b_1 = c_1 = 0$. Thus, $m = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{bmatrix}$ and so $a_1 \neq 0$ because $m \notin I$. Then

$$\begin{bmatrix} a_1a_2 & a_1b_2 & a_1c_2 \\ 0 & d_1d_2 & 0 \\ 0 & 0 & e_1e_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & 0 \\ 0 & 0 & e_2 \end{bmatrix} = mn \in I.$$

Then $a_1a_2 = 0, a_1b_2 = 0$ and $a_1c_2 = 0$, so $a_2 = b_2 = c_2 = 0$ since $a_1 \neq 0$. Thus, $n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & e_2 \end{bmatrix} \in I$. Then $I$ is a completely prime right ideal of $R$. Moreover, we consider right ideals $J$ and $K$ of $R$ such that

$$J = \left\{ \begin{bmatrix} 0 & p & q \\ 0 & r & 0 \\ 0 & 0 & s \end{bmatrix} \mid p, q, r, s \in F \right\} \quad \text{and} \quad K = \left\{ \begin{bmatrix} u & v & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid u, v, w \in F \right\}.$$

Then $JK \subseteq I$ but $J \nsubseteq I$ and $K \nsubseteq I$. Hence, $I$ is not a prime right ideal of $R$.

In this project, we are interested in rings which have similar idea as right duo rings. To be precise, we are interested in rings whose prime right ideals are two-sided and focus on noncommutative associative rings with identity. We investigate some properties of such rings. Since prime right ideals and completely prime right ideals in noncommutative rings may be different, we also study rings whose completely prime right ideals are two-sided and investigate some of their properties.

# Chapter 2

# Background and Preliminaries

In this project, a "ring" means an associative ring with identity. First, we gather the definitions of a right duo ring, a good ideal of a ring and a good ring given by L. Huang and W. Xue in [1]. Then we provide some relations between a finitely generated right ideal and a good ideal of a right duo ring that are mentioned in [1].

**Definition 2.1.** A ring $R$ is called a *right duo ring* if any right ideal of $R$ is two-sided.

**Definition 2.2.** An ideal $I$ of a ring $R$ is called a *good ideal* if for each $r_1,\ r_2 \in R$,

$$(r_1 + I)\,(r_2 + I) = \ (r_1 + I) \circ (r_2 + I).$$

Let $R$ be a ring and $I$ be an ideal of $R$. Recall that the product of cosets of $I$ in $R/I$ is defined as

$$(r_1 + I)\,(r_2 + I) = r_1 r_2 + I \qquad \text{where } r_1, r_2 \in R.$$

Moreover, $(r_1 + I) \circ (r_2 + I) \subseteq (r_1 + I)\,(r_2 + I)$ for any $r_1,\ r_2 \in R$. As a result, to verify that $I$ is a good ideal of $R$ is enough to show only that $r_1 r_2 + I \subseteq\ (r_1 + I) \circ (r_2 + I)$ for all $r_1, r_2 \in R$.

**Definition 2.3.** A ring $R$ is called a *good ring* if any ideal of $R$ is a good ideal.

**Proposition 2.4.** [1, Lemma 1] *Let $R$ be a right duo ring. If $I$ is a finitely generated right ideal, then the following are equivalent:*

*(i)* $I = eR$ *for some idempotent* $e \in R$*;*

*(ii)* $I$ *is a good ideal;*

*(iii)* $I = I \circ I$*;*

*(iv)* $I = I^2$.

**Proposition 2.5.** [1, Lemma 4] *The following are equivalent for a right duo ring $R$:*

*(i)* $R$ *is a good ring;*

*(ii) each finitely generated right ideal of* $R$ *is a good ideal;*

*(iii) each principal right ideal of $R$ is a good ideal.*

Moreover, we introduce the definitions of a von Neuman regular ring and a strongly regular ring given in [1]. L. Huang and W. Xue [1] found that strongly regular rings and good rings coincide in right duo rings.

**Definition 2.6.** A ring $R$ is called a *von Neuman regular* (or simply called a *regular*) ring if each principal right (or left) ideal of $R$ is generated by an idempotent of $R$.

**Definition 2.7.** A ring $R$ is a *strongly regular ring* if it is a regular ring and a right duo ring.

**Theorem 2.8.** [1, Theorem 5] *Let $R$ be a right duo ring. Then $R$ is a (strongly) regular ring if and only if it is a good ring.*

K. Koh [2] showed that the *prime radical* of an associative ring $R$ with identity, denoted by $\text{rad}(R)$, which is the intersection of all prime ideals of $R$, has a relation with nilpotent elements in $R$. The author investigated some properties of nilpotent elements in $R$ under the condition that $p(R) \subseteq m(R)$, where $p(R)$ is the set of all prime right ideals of $R$ and $m(R)$ is the set of all maximal right ideals of $R$. In general, $m(R) \subseteq p(R)$, so the condition $p(R) \subseteq m(R)$ means that all prime right ideals of $R$ are maximal right ideals of $R$. Moreover, they studied the ring $R/\text{rad}(R)$ under the condition $p(R) \subseteq m(R)$ as well.

**Proposition 2.9.** [2, Lemma 2.4] *Let $R$ be a ring. If $p(R) \subseteq m(R)$, then every nilpotent element of $R$ is contained in* $\text{rad}(R)$.

**Proposition 2.10.** [2, Lemma 2.5] *Let $R$ be a ring and $R' = R/\text{rad}(R)$. If $p(R) \subseteq m(R)$, then for any $b \in R'$*

*(i) $br = 0$ if and only if $rb = 0$ for any $r \in R'$, and*

*(ii) $(b)^{\perp} = \{r \in R' \mid br = 0\}$ is a two-sided ideal of $R'$.*

As mentioned at the end of Chapter 1, we study on specific rings; namely, rings whose prime right ideals are two-sided and rings whose completely prime right ideals are two-sided, so we would like to recall the definitions of a prime right ideal and a completely prime right ideal. Also, a simple relation between prime right ideals and completely prime right ideals is provided.

**Definition 2.11.** [3] A proper right ideal $I$ of a ring $R$ is called a *prime right ideal* if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for any right ideals $A$ and $B$ of $R$.

**Definition 2.12.** [4] A proper right ideal $I$ of a ring $R$ is called a *completely prime right ideal* if for any $a, b \in R$ such that $aI \subseteq I$, $ab \in I$ implies that either $a \in I$ or $b \in I$.

From the above definitions, our work focuses on noncommutative rings with identity.

**Proposition 2.13.** *Let $R$ be a ring whose completely prime right ideals are two-sided. If $I$ is a completely prime right ideal of $R$, then $I$ is a prime right ideal of $R$.*

*Proof.* Assume that $I$ is a completely prime right ideal of $R$. Let $A$ and $B$ be right ideals of $R$ such that $AB \subseteq I$. Suppose that $A \nsubseteq I$ and $B \nsubseteq I$. Then there exist $a \in A \setminus I$ and $b \in B \setminus I$. Thus, $ab \in AB \subseteq I$ where $a \notin I$ and $b \notin I$. Since $R$ is a ring whose completely prime right ideals are two-sided, $I$ is a two-sided ideal. Then $aI \subseteq I$. Since $I$ is a completely prime right ideal, $a \in I$ or $b \in I$. This is a contradiction. Then $A \subseteq I$ or $B \subseteq I$. Hence, $I$ is a prime right ideal of $R$. □

We are interested to investigate the properties of finitely generated prime right ideals and finitely generated completely prime right ideals relating to good ideals as in [1]. Since strongly regular rings and good rings are the same in right duo rings, we would like to explore what kind of the relation between good rings and rings whose prime right ideals are two-sided and what kind of the relation between good rings and rings whose completely prime right ideals are two-sided. In addition, we are interested in studying about a quotient ring. However, there is some restriction to say about this in the ring whose prime right ideals are two-sided. With this reason, we change to use the concept of completely prime right ideal instead. In the ring $R$ whose completely prime right ideals are two-sided, we find that the quotient ring $R/I$, where $I$ is a completely prime ring ideal is a domain. Moreover, we study the relation between nilpotent elements and the intersection of all completely prime right ideals as in [2], but without the condition $p(R) \subseteq m(R)$. Finally, we consider the structure of some ideals of $R$ that relates to completely prime right ideals.

# Chapter 3

# Main results

## 3.1 Rings Whose Prime Right Ideals Are Two-Sided

Our purpose here is to provide some properties of rings that are analogous to the properties of right duo rings. First, we give the definition of a central element of a ring $R$ and some basic properties of an idempotent element in $R$ and a principal right ideal of $R$ which is also a left ideal.

**Definition 3.1.1.** The set of all elements in a ring $R$ that commute every element in $R$ is denoted by $Z(R)$; that is $Z(R) = \{x \in R \mid xy = yx \text{ for all } y \in R\}$. Moreover, an element in $Z(R)$ is called a *central element* of $R$.

**Definition 3.1.2.** Let $I$ and $J$ be subrings of a ring $R$. Then $R$ is the *direct sum* of $I$ and $J$, denoted by $R = I \oplus J$, if $R = I + J$ and $I \cap J = \{0\}$.

**Lemma 3.1.3.** *Let $e$ be an idempotent element in a ring $R$. Then*

*(i)* $eR \cap (1-e)R = \{0\}$*; moreover,* $eR \oplus (1-e)R = R$;

*(ii)* $e \in Z(R)$ *if and only if* $(1-e)xe = 0$ *and* $ex(1-e) = 0$ *for all* $x \in R$.

*Proof.* Note that $e^2 = e$ since $e$ is an idempotent element in $R$.

(i) Let $a \in eR \cap (1-e)R$. Then there exist $x, y \in R$ such that $a = ex$ and $a = (1-e)y$. Then $ex = (1-e)y$. Thus, $a = ex = e(ex) = e(1-e)y = (e-e^2)y = (0)y = 0$. Hence, $eR \cap (1-e)R \subseteq \{0\}$. It follows that $eR \cap (1-e)R = \{0\}$ because $0 \in eR \cap (1-e)R$. Moreover, let $r \in R$. Then $r = e(r) + (1-e)r \in eR + (1-e)R$. Thus, $R \subseteq eR + (1-e)R$ so $eR + (1-e)R = R$. As a result, $eR \oplus (1-e)R = R$.

(ii) Assume that $e \in Z(R)$. Then $xe = ex$ for all $x \in R$. Thus,

$$(1-e)xe = xe - exe = xee - exe = (xe - ex)e = (0)e = 0$$

$$\text{and} \quad ex(1-e) = ex - exe = eex - exe = e(ex - xe) = e(0) = 0$$

for all $x \in R$.

Conversely, assume that $(1-e)xe = 0$ and $ex(1-e) = 0$ for all $x \in R$. Then $xe = exe$ and $ex = exe$ for all $x \in R$. Hence, $xe = ex$ for all $x \in R$. □

**Remark 3.1.4.** Let $e$ be an element in a ring $R$. Then $e$ is an idempotent element in $R$ if and only if $1-e$ is an idempotent element in $R$.

**Lemma 3.1.5.** *Let $a$ be an element in a ring $R$. If $aR$ is also a left ideal of $R$, then $Ra \subseteq aR$.*

*Proof.* Assume that $aR$ is a left ideal of $R$. Then $a = a(1) \in aR$. Thus, $Ra \subseteq aR$ since $Ra$ is the smallest left ideal of $R$ containing $a$. □

Now, we focus on one of the target rings.

**Lemma 3.1.6.** *Let $R$ be a ring whose prime right ideals are two-sided. If $eR$ and $(1-e)R$ are prime right ideals of $R$ where $e$ is an idempotent in $R$, then $e$ is a central element.*

*Proof.* Assume that $eR$ and $(1-e)R$ are prime right ideals of $R$ where $e$ is an idempotent in $R$. Suppose that $e \notin Z(R)$. By Lemma 3.1.3(ii), there exists $x \in R$ such that $(1-e)xe \neq 0$ or $ex(1-e) \neq 0$.

Case $(1-e)xe \neq 0$: Then $0 \neq (1-e)xe \in (1-e)R$. Since $eR$ is a prime right ideal of $R$, it is also a left ideal of $R$. Then $(1-e)xe \in Re \subseteq eR$ by Lemma 3.1.5. This shows that $0 \neq (1-e)xe \in eR \cap (1-e)R$ contradicting to Lemma 3.1.3(i).

Case $ex(1-e) \neq 0$: Then $0 \neq ex(1-e) \in eR$. Since $(1-e)R$ is a prime right ideal of $R$, it is also a left ideal of $R$. Then $ex(1-e) \in R(1-e) \subseteq (1-e)R$ by Lemma 3.1.5. This shows that $0 \neq ex(1-e) \in eR \cap (1-e)R$ contradicting to Lemma 3.1.3(i).

Thus, $e$ is a central element of $R$. □

**Lemma 3.1.7.** *Let $R$ be a ring whose prime right ideals are two-sided. If $I$ is a prime right ideal of $R$ such that $I = eR$ for some idempotent $e \in R$, then $I = I \circ I$.*

*Proof.* Assume that $I$ is a prime right ideal of $R$ such that $I = eR$ for some idempotent $e \in R$. It is obvious that $I \circ I \subseteq I$. Then it is enough to show that $I \subseteq I \circ I$. Let $i \in I = eR$. Then there exists $r \in R$ such that $i = er$. Since $e = e(1) \in eR = I$, it follows that $i = er = (e)(er) \in I \circ I$. □

Next, we are ready to prove our main result.

**Theorem 3.1.8.** *Let $R$ be a ring whose prime right ideals are two-sided. If $I$ is a prime right ideal of $R$, then the following properties hold.*

*(i) If $I = eR$ for some idempotent $e \in R$ and $(1-e)R$ is a prime right ideal of $R$, then $I = eR$ and $(1-e)R$ are good ideals of $R$.*

*(ii) If $I$ is a good ideal of $R$, then $I = I \circ I$.*

*(iii)* *If* $I = I \circ I$, *then* $I = I^2$.

*(iv)* *If* $I$ *is a finitely generated right ideal of* $R$ *such that* $I = I^2$ *and every principal right ideal of* $R$ *is a prime right ideal, then* $I = eR$ *for some idempotent* $e \in R$.

*Proof.* Assume that $I$ is a prime right ideal of $R$.

(i) Let $e$ be an idempotent of $R$ such that $I = eR$ and assume that $(1-e)R$ is a prime right ideal of $R$. Let $r_1, r_2 \in R$ and $i \in I$. We show that $r_1r_2 + i \in (r_1 + I) \circ (r_2 + I)$. By Lemma 3.1.3, $R = I \oplus (1-e)R$. Then there exist $j_1, j_2 \in I$ and $l_1, l_2 \in R$ such that $r_1 = j_1 + (1-e)l_1$ and $r_2 = j_2 + (1-e)l_2$. Thus, $er_1 = ej_1 + e(1-e)l_1 = ej_1$ and $er_2 = ej_2 + e(1-e)l_2 = ej_2$. We consider $j_1j_2 + i$. By Lemma 3.1.7, we obtain

$$j_1j_2 + i \in I = I \circ I = (j_1 + I) \circ (j_2 + I).$$

Then there exist $j_1'$, $j_2' \in I$ such that $j_1j_2 + i = (j_1 + j_1')(j_2 + j_2')$, so $i = j_1j_2' + j_1'j_2 + j_1'j_2'$. Since $j_1'$, $j_2' \in I = eR$, there exist $j_1''$, $j_2'' \in R$ such that $j_1' = ej_1''$ and $j_2' = ej_2''$. By Lemma 3.1.6, $e$ is a central element in $R$. Then we obtain

$$\begin{aligned} i &= j_1j_2' + j_1'j_2 + j_1'j_2' \\ &= j_1ej_2'' + ej_1''j_2 + j_1'j_2' \\ &= ej_1j_2'' + j_1''ej_2 + j_1'j_2' \\ &= er_1j_2'' + j_1''er_2 + j_1'j_2' \\ &= r_1ej_2'' + ej_1''r_2 + j_1'j_2' \\ &= r_1j_2' + j_1'r_2 + j_1'j_2'. \end{aligned}$$

Then $r_1r_2 + i = r_1r_2 + r_1j_2' + j_1'r_2 + j_1'j_2' = (r_1 + j_1')(r_2 + j_2') \in (r_1 + I) \circ (r_2 + I)$. Hence, $I = eR$ is a good ideal of $R$.

Since $1 - e$ is also an idempotent, the result follows.

(ii) Assume that $I$ is a good ideal of $R$. Then $r_1r_2 + I = (r_1 + I) \circ (r_2 + I)$ for all $r_1, r_2 \in R$. Since $0 \in R$, $I = (0)(0) + I = (0 + I) \circ (0 + I) = I \circ I$. Hence, $I = I \circ I$.

(iii) Assume that $I = I \circ I$. Then $I = I \circ I \subseteq I^2 \subseteq I$. Hence, $I = I^2$.

(iv) Assume that $I$ is a finitely generated right ideal of $R$ such that $I = I^2$ and every principal right ideal of $R$ is a prime right ideal. Since $I$ is a finitely generated right ideal of $R$, there exist $x_1, x_2, \ldots, x_n \in I$ such that $I = \sum_{k=1}^{n} x_kR$

where $n \in \mathbb{N}$. For each $\alpha \in \{1, 2, \ldots, n\}$, $x_\alpha \in I = I^2 = (\sum_{k=1}^{n} x_k R)I \subseteq \sum_{k=1}^{n} x_k I$ because $I$ is a prime right ideal of $R$ and then $I$ is an ideal of $R$. It follows that there exist $i_{k,m} \in I$ for all $k, m \in \{1, 2, \ldots, n\}$ which satisfy the following equations:

$$x_1 i_{1,1} + x_2 i_{1,2} + \cdots + x_n i_{1,n} = x_1,$$
$$x_1 i_{2,1} + x_2 i_{2,2} + \cdots + x_n i_{2,n} = x_2,$$
$$\vdots$$
$$x_1 i_{n,1} + x_2 i_{n,2} + \cdots + x_n i_{n,n} = x_n.$$

Then we obtain

$$x_1(1 - i_{1,1}) - x_2 i_{1,2} - \cdots - x_n i_{1,n} = 0, \quad (1)$$
$$-x_1 i_{2,1} + x_2(1 - i_{2,2}) - x_3 i_{2,3} - \cdots - x_n i_{2,n} = 0, \quad (2)$$
$$\vdots$$
$$-x_1 i_{n,1} - x_2 i_{n,2} - \cdots - x_{n-1} i_{n,n-1} + x_n(1 - i_{n,n}) = 0, \quad (n)$$

which are considered as a system of $n$ equations and variables $x_1, x_2, \ldots, x_n$ (the generators of $I$). Since every principal right ideal of $R$ is a prime right ideal, $Ra$ is a prime right ideal of $R$ and so $Ra$ is a left ideal of $R$ for any $a \in R$. Applying Lemma 3.1.5, $R(1 - i_{n,n}) \subseteq (1 - i_{n,n})R$, then for each $l \in \{1, 2, \ldots, n\}$ there exist $j_{l,n} \in R$ such that $i_{l,n}(1 - i_{n,n}) = (1 - i_{n,n})j_{l,n}$. Moreover, $j_{l,n} \in I$ for all $l \in \{1, 2, \ldots, n\}$.

Next, we want to show that for any $k \in \{1, 2, \ldots, n\}$, there exists $u_k \in I$ such that $x_k(1 - u_k) = 0$. Let $k \in \{1, 2, \ldots, n\}$.

Step 1: This process is to eliminate the variables $x_n, x_{n-1}, \ldots, x_{k+2}$, respectively, from the system so that the system consisting of $k + 1$ equations with variables $x_1, x_2, \ldots, x_{k+1}$ is obtained.

Step 1.1: We consider how $x_n$ is eliminated.
For each $l \in \{1, 2, \ldots, n - 1\}$, we use the equations $(l)$ and $(n)$ and the fact that $i_{l,n}(1 - i_{n,n}) = (1 - i_{n,n})j_{l,n}$ where $j_{l,n} \in I$. Multiplying $(l)$ by $(1 - i_{n,n})$ and $(n)$ by $j_{l,n}$ and then adding both of them, we obtain the new equation $(l')$; however, we use the same name "the equation $(l)$". We obtain the following system ($n - 1$ equations with

variables $x_1, x_2, \dots, x_{n-1}$):

$$x_1(1 - j_{1,1}) - x_2 j_{1,2} - \cdots - x_{n-1} j_{1,n-1} = 0, \tag{1}$$

$$-x_1 j_{2,1} + x_2(1 - j_{2,2}) - x_3 j_{2,3} \cdots - x_{n-1} j_{2,n-1} = 0, \tag{2}$$

$$\vdots$$

$$-x_1 j_{n-1,1} - x_2 j_{n-1,2} - \cdots - x_{n-2} j_{n-1,n-2} + x_{n-1}(1 - j_{n-1,n-1}) = 0, \tag{$n-1$}$$

where $j_{l,m} \in I$ for all $l, m \in \{1, 2, \dots, n-1\}$.

Step 1.2: We consider how $x_{n-1}$ is eliminated.
For each $l \in \{1, 2, \dots, n-2\}$, we use the equations $(l)$ and $(n-1)$ and the fact that $j_{l,n-1}(1 - j_{n-1,n-1}) = (1 - j_{n-1,n-1}) j'_{l,n-1}$ for some $j'_{l,n-1} \in I$. Multiplying $(l)$ by $(1 - j_{n-1,n-1})$ and $(n-1)$ by $j'_{l,n-1}$ and then adding both of them, we obtain the new equation say $(l)$. We obtain the following system ($n-2$ equations with variables $x_1, x_2, \dots, x_{n-2}$):

$$x_1(1 - j'_{1,1}) - x_2 j'_{1,2} - \cdots - x_{n-2} j'_{1,n-2} = 0, \tag{1}$$

$$-x_1 j'_{2,1} + x_2(1 - j'_{2,2}) - x_3 j'_{2,3} \cdots - x_{n-2} j'_{2,n-2} = 0, \tag{2}$$

$$\vdots$$

$$-x_1 j_{n-2,1} - x_2 j_{n-2,2} - \cdots - x_{n-3} j_{n-2,n-3} + x_{n-2}(1 - j'_{n-2,n-2}) = 0, \tag{$n-2$}$$

where $j'_{l,m} \in I$ for all $l, m \in \{1, 2, \dots, n-2\}$.

Continue this process so that the following equations are obtained.

$$x_1(1 - s_{1,1}) - x_2 s_{1,2} - \cdots - x_{k+1} s_{1,k+1} = 0, \tag{1}$$

$$-x_1 s_{2,1} + x_2(1 - s_{2,2}) - x_3 s_{2,3} \cdots - x_{k+1} s_{2,k+1} = 0, \tag{2}$$

$$\vdots$$

$$-x_1 s_{k+1,1} - x_2 s_{k+1,2} - \cdots - x_k s_{k+1,k} + x_{k+1}(1 - s_{k+1,k+1}) = 0, \tag{$k+1$}$$

where $s_{l,m} \in I$ for all $l, m \in \{1, 2, \dots, k+1\}$.

Step 2: This process is to eliminate the variables $x_1, x_2, \dots, x_{k-1}$, respectively, from the last system obtained from Step 1 so that we reach the system consisting of 2 equations with variables $x_k$ and $x_{k+1}$.

Step 2.1: We consider how $x_1$ is eliminated.
For each $l \in \{2, 3, \ldots, k+1\}$, we use the equations $(l)$ and $(1)$ and the fact that $s_{l,1}(1-s_{1,1}) = (1-s_{1,1})t_{l,1}$ for some $t_{l,1} \in I$. Multiplying $(l)$ by $(1-s_{1,1})$ and $(1)$ by $t_{l,1}$ and then adding both of them, we obtain the new equation $(l)$. We obtain the following system ($k$ equations with variables $x_2, x_3, \ldots, x_{k+1}$):

$$x_2(1-t_{2,2}) - x_3t_{2,3} - \cdots - x_{k+1}t_{2,k+1} = 0, \quad (2)$$
$$-x_2t_{3,2} + x_3(1-t_{3,3}) - x_4t_{3,4} - \cdots - x_{k+1}t_{3,k+1} = 0, \quad (3)$$
$$\vdots$$
$$-x_2t_{k+1,2} - x_3t_{k+1,3} - \cdots - x_kt_{k+1,k} + x_{k+1}(1-t_{k+1,k+1}) = 0, \quad (k+1)$$

where $t_{l,m} \in I$ for all $l, m \in \{2, 3, \ldots, k+1\}$.

Step 2.2: We consider how $x_2$ is eliminated.
For each $l \in \{3, 4, \ldots, k+1\}$, we use the equations $(l)$ and $(2)$ and the fact that $t_{l,2}(1-t_{2,2}) = (1-t_{2,2})t'_{l,2}$ for some $t'_{l,2} \in I$. Multiplying $(l)$ by $(1-t_{2,2})$ and $(2)$ by $t'_{l,2}$ and then adding both of them, we obtain the new equation $(l)$. We obtain the following system ($k-1$ equations with variables $x_3, x_4, \ldots, x_{k+1}$):

$$x_3(1-t'_{3,3}) - x_4t'_{3,4} - \cdots - x_{k+1}t'_{3,k+1} = 0, \quad (3)$$
$$-x_3t'_{4,3} + x_4(1-t'_{4,4}) - x_5t'_{4,5} - \cdots - x_{k+1}t'_{4,k+1} = 0, \quad (4)$$
$$\vdots$$
$$-x_3t'_{k+1,3} - x_4t'_{k+1,4} - \cdots - x_kt'_{k+1,k} + x_{k+1}(1-t'_{k+1,k+1}) = 0, \quad (k+1)$$

where $t'_{l,m} \in I$ for all $l, m \in \{3, 4, \ldots, k+1\}$.

Continue this process so that the following equations are obtained.

$$x_k(1-\beta_{k,k}) - x_{k+1}\beta_{k,k+1} = 0, \quad (k)$$
$$-x_k\beta_{k+1,k} + x_{k+1}(1-\beta_{k+1,k+1}) = 0, \quad (k+1)$$

where $\beta_{l,m} \in I$ for all $l, m \in \{k, k+1\}$.

Step 3: From Step 2,

$$x_k(1-\beta_{k,k}) - x_{k+1}\beta_{k,k+1} = 0,$$
$$-x_k\beta_{k+1,k} + x_{k+1}(1-\beta_{k+1,k+1}) = 0,$$

where $\beta_{l,m} \in I$ for all $l, m \in \{k, k+1\}$.
Then $x_{k+1}$ is eliminated from both equations by using the fact that there exists $\gamma_{k+1,k+1} \in I$ such that $\beta_{k,k+1}(1-\beta_{k+1,k+1}) = (1-\beta_{k+1,k+1})\gamma_{k,k+1}$. Thus, $x_k(1-\beta_{k,k})(1-\beta_{k+1,k+1}) - x_k\beta_{k+1,k}\gamma_{k,k+1} = 0$, i.e, $x_k(1-u_k) = 0$ for some $u_k \in I$.

Now, we obtain $x_k(1-u_k) = 0$ for some $u_k \in I$ where $k \in \{1, 2, \ldots, n\}$.

Next, let $k \in \{1, 2, \ldots, n\}$ and $r_k \in R$. Then

$$x_k(r_k(1-u_1)(1-u_2)\cdots(1-u_{k-1})) = r'_k x_k \quad \text{for some } r'_k \in R.$$

Since $x_k(1-u_k) = 0$,

$$x_k r_k(1-u_1)(1-u_2)\cdots(1-u_n) = r'_k x_k(1-u_k)\cdots(1-u_n) = 0.$$

Moreover, $I(1-u_1)\cdots(1-u_n) = 0$ since $I$ is generated by $x_1, x_2, \ldots, x_n$. Let $(1-u_1)(1-u_2)\cdots(1-u_n) = 1-e$. Then $e \in I$. Thus, $I(1-e) = 0$. In particular, $e(1-e) = 0$. Thus, $e = e^2$. Since $eR \subseteq IR \subseteq I$, it is enough to show that $I \subseteq eR$. Note that the principal ideal $eR$ is a prime right ideal so that $eR$ is a two-sided ideal. Let $i \in I$. Then $i(1-e) = 0$, so $i = ie \in Ie \subseteq Re \subseteq eR$ by Lemma 3.1.5. Hence, $I = eR$ for some idempotent $e \in I$. □

**Theorem 3.1.9.** *Let $R$ be a ring whose prime right ideals are two-sided. Then the following statements hold.*

*(i) If each finitely generated prime right ideal of $R$ is a good ideal, then each principal prime right ideal of $R$ is a good ideal.*

*(ii) If each principal prime right ideal of $R$ is a good ideal and every principal right ideal of $R$ is a prime right ideal, then every prime right ideal of $R$ is a good ideal.*

*Proof.*

(i) The result follows from the fact that principal right ideals of $R$ are finitely generated prime right ideals of $R$.

(ii) Assume that each principal prime right ideal of $R$ is a good ideal and every principal right ideal of $R$ is a prime right ideal. Let $I$ be a prime right ideal of $R$. Let $r_1, r_2 \in R$ and $i \in I$. Since $iR$ is a principal right ideal, $iR$ is a prime right ideal. Then $iR$ is a good ideal of $R$. Hence, $r_1r_2 + i \in (r_1 + iR) \circ (r_2 + iR) \subseteq (r_1 + I) \circ (r_2 + I)$. □

Finally, we give the definition of a $P$-regular ring which is analogous to a strongly regular ring. Since a good ring (all ideals are good ideals) and a strongly regular ring coincide in a right duo ring, we consider similar relation between a $P$-regular ring and their good ideals.

**Definition 3.1.10.** A ring $R$ is called a $P$*-regular* ring if $R$ is a regular ring and every prime right ideal of $R$ is two-sided.

Recall that $R$ is a regular ring if each principal right ideal of $R$ is generated by an idempotent of $R$.

**Theorem 3.1.11.** *Let $R$ be a $P$-regular ring. If $eR$ is a prime right ideal of $R$ for all idempotent $e \in R$, then each principal prime right ideal of $R$ is a good ideal. Moreover, if every principal right ideal of $R$ is a prime right ideal, then every prime right ideal of $R$ is a good ideal.*

*Proof.* Assume that $eR$ is a prime right ideal of $R$ for all idempotent $e \in R$. First, let $a \in R$ be such that $aR$ is a principal prime right ideal of $R$. Since $R$ is a $P$-regular ring, $R$ is a regular ring so there exists an idempotent $e \in R$ such that $aR = eR$. Then $1 - e$ is an idempotent element in $R$ by Remark 3.1.4 and so $(1 - e)R$ is a prime right ideal of $R$. By Theorem 3.1.8 (i), $aR$ is a good ideal of $R$.

In addition, suppose that every principal right ideal of $R$ is a prime right ideal. By Theorem 3.1.9 (ii), each prime right ideal of $R$ is a good ideal. □

**Theorem 3.1.12.** *Let $R$ be a ring whose prime right ideals are two-sided. If every prime right ideal of $R$ is a good ideal and every principal right ideal of $R$ is a prime right ideal, then $R$ is a $P$-regular ring.*

*Proof.* Assume that every prime right ideal of $R$ is a good ideal and every principal right ideal of $R$ is a prime right ideal. Let $b \in R$ be such that $bR$ is a principal right ideal of $R$. Then $bR$ is a principal prime right ideal of $R$, so $bR$ is a good ideal of $R$. By Theorem 3.1.8 (ii), $bR = bR \circ bR$. By Theorem 3.1.8 (iii), $bR = (bR)^2$. By Theorem 3.1.8 (iv), there exists an idempotent $f \in R$ such that $bR = fR$. Hence, $R$ is a $P$-regular ring. □

## 3.2 Rings Whose Completely Prime Right Ideals Are Two-Sided

First, we inspect some properties of a completely prime right ideal of $R$ relating to a good ideal of $R$. Since rings in this section have the same common character as rings in Section 3.1, we get results similar to Theorems 3.1.8 – 3.1.9 and Theorems 3.1.11 – 3.1.12 by using completely prime right ideals instead of prime right ideals.

**Theorem 3.2.1.** *Let $R$ be a ring whose completely prime right ideals are two-sided. If $I$ is a completely prime right ideal of $R$, then the following properties hold.*

*(i) If $I = eR$ for some idempotent $e \in R$ and $(1-e)R$ is a completely prime right ideal of $R$, then $I = eR$ and $(1-e)R$ are good ideals of $R$.*

*(ii) If $I$ is a good ideal of $R$, then $I = I \circ I$.*

*(iii) If $I = I \circ I$, then $I = I^2$.*

*(iv) If $I$ is a finitely generated right ideal of $R$ such that $I = I^2$ and every principal right ideal of $R$ is a completely prime right ideal, then $I = eR$ for some idempotent $e \in R$.*

**Theorem 3.2.2.** *Let $R$ be a ring whose completely prime right ideals are two-sided. Then the following statements hold.*

*(i) If each finitely generated completely prime right ideal of $R$ is a good ideal, then each principal completely prime right ideal of $R$ is a good ideal.*

*(ii) If each principal completely prime right ideal of $R$ is a good ideal and every principal right ideal of $R$ is a completely prime right ideal, then every completely prime right ideal of $R$ is a good ideal.*

Analogously to the properties of a $P$-regular ring, we give the definition of a $CP$-regular ring and investigate some properties of a $CP$-regular ring.

**Definition 3.2.3.** A ring $R$ is called a *$CP$-regular* ring if $R$ is a regular ring and every completely prime right ideal of $R$ is two-sided.

**Theorem 3.2.4.** *Let $R$ be a $CP$-regular ring. If $eR$ is a completely prime right ideal of $R$ for all idempotent $e \in R$, then each principal completely prime right ideal of $R$ is a good ideal. Moreover, if every principal right ideal of $R$ is a completely prime right ideal, then every completely prime right ideal of $R$ is a good ideal.*

**Theorem 3.2.5.** *Let $R$ be a ring whose completely prime right ideals are two-sided. If every completely prime right ideal of $R$ is a good ideal and every principal right ideal of $R$ is a completely prime right ideal, then $R$ is a $CP$-regular ring.*

Next, we give some relation between a domain, which is a nonzero associative ring with identity such that $ab = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in R$, and a completely prime right ideal and some relation between a nilpotent element of a quotient ring and a zero divisor of a quotient ring.

**Proposition 3.2.6.** *Let $R$ be a ring whose completely prime right ideals are two-sided and $I$ be a right ideal of $R$. Then $R/I$ is a domain if and only if $I$ is a completely prime right ideal of $R$.*

*Proof.* Assume that $R/I$ is a domain. Then $R/I$ is a nonzero ring and so $I$ is a proper ideal of $R$. Let $a, b \in R$ be such that $ab \in I$ and $aI \subseteq I$. Suppose that $a \notin I$. Then $(a+I)(b+I) = ab + I = I$. Since $a + I \neq I$ and $R/I$ is a domain, $b + I = I$, so $b \in I$. Then $I$ is a completely prime right ideal of $R$.

Conversely, assume that $I$ is a completely prime right ideal of $R$. Then $I$ is a proper two-sided ideal of $R$. Then $R/I$ is a nonzero quotient ring with identity. To show that $R/I$ is a domain, let $x, y \in R/I$ be such that $xy = I$. Then there exist $a, b \in R$ such that $x = a + I$ and $y = b + I$. Assume that $x \neq I$. Then $a \notin I$. Thus, $I = xy = (a+I)(b+I) = ab + I$, so $ab \in I$. Since $I$ is a completely prime right ideal of $R$, $aI \subseteq I$ and $a \notin I$, it follows that $b \in I$. Then $y = I$. Hence, $R/I$ is a domain. □

Moreover, we study the structure of $R/I$ where $I$ is a completely prime right ideal of $R$.

**Definition 3.2.7.** An element $r$ of an associative ring $R$ is called a *zero divisor* of $R$ if there exist nonzero $x, y \in R$ such that $rx = 0 = yr$.

**Lemma 3.2.8.** *Let $I$ be a two-sided ideal of an associative ring $R$ with identity 1. If $x + I$ is a nilpotent element of $R/I$, then $x + I$ is a zero divisor of $R/I$.*

*Proof.* Assume that $x + I$ is a nilpotent element of $R/I$. Then there exists the least positive integer $m$ such that $(x+I)^m = x^m + I = I$. If $m = 1$, then $(x+I)(1+I) = x + I = I$ and $(1+I)(x+I) = x + I = I$, so we are done. Now, we consider $m > 1$. Then $x^{m-1} \notin I$, so $x^{m-1} + I \neq I$. Thus, $(x+I)(x^{m-1}+I) = I$ and $(x^{m-1}+I)(x+I) = I$. Hence, $x + I$ is a zero divisor of $R/I$. □

From Proposition 2.9, K. Koh obtained a relation between nilpotent elements of a ring $R$ and the prime radical of $R$. He showed that the prime radical of $R$ is the intersection of all prime right ideals of $R$. Thus, we define the *completely prime radical* of $R$, denoted by rad*$(R)$, to be the intersection of all completely prime right ideals of $R$. Our first aim is to examine some properties of rad*$(R)$ and nilpotent elements of $R$.

**Lemma 3.2.9.** *Let $R$ be a ring whose completely prime right ideals are two-sided. Then every nilpotent element of $R$ is contained in* rad*$(R)$.

*Proof.* Let $x$ be a nilpotent element of $R$. Then $x^n = 0$ for some positive integer $n$. If $n = 1$, then $x = x^1 = 0 \in$ rad*$(R)$. Next, we consider $n > 1$. Suppose that $x \notin$

rad*$(R)$. Then there exists a completely prime right ideal $I$ of $R$ such that $x \notin I$. Then $I$ is a two-sided ideal of $R$. By Proposition 3.2.6, $R/I$ is a domain. Thus, $(x+I)^n = x^n + I = I$, i.e., $x+I$ is a nilpotent element of $R/I$. By Lemma 3.2.8, $x+I$ is a zero divisor. This is a contradiction because $R/I$ is a domain. □

Since rings that we are interested in are rings whose completely prime right ideals are two-sided and any intersection of ideals is an ideal, rad*$(R)$ must be an ideal of $R$. As a result, $R/\text{rad*}(R)$ is a quotient ring.

**Theorem 3.2.10.** *Let $R$ be a ring whose completely prime right ideals are two-sided and $R' = R/I$ where $I = \text{rad*}(R)$. Then for any $b+I \in R'$,*

*(i)* $(b+I)(r+I) = I$ *if and only if* $(r+I)(b+I) = I$ *for any* $r+I \in R'$, *and*

*(ii)* $(b+I)^{\perp} = \{r+I \in R/I \mid r \in R \text{ and } (b+I)(r+I) = I\}$ *is a two sided ideal of* $R'$.

*Proof.* First, we will show that if $x \notin I$, then $x+I$ is not a nilpotent element of $R/I$. Assume that $x \notin I$. There exists a completely prime right ideal $J$ of $R$ such that $x \notin J$. Then $J$ is a two-sided ideal and $R/J$ is a domain by Proposition 3.2.6. Hence, $x+J \in R/J$ is not a zero divisor. By Lemma 3.2.8, $x+J$ is not a nilpotent element of $R/J$. Then $x^n \notin J$ for all $n \in \mathbb{N}$. Since $I = \text{rad*}(R) \subseteq J$, it follows that $x^n \notin I$ for all $n \in \mathbb{N}$. Thus, $x+I$ is not a nilpotent element of $R/I$. Let $b+I \in R/I$ where $b \in R$.

(i) Let $r+I \in R/I$ where $r \in R$. Assume that $(b+I)(r+I) = I$.
If $rb \in I$, then we are done. Suppose that $rb \notin I$. Then $(r+I)(b+I) = rb+I$ is not a nilpotent element of $R/I$. Note that

$$\begin{aligned}\big((r+I)(b+I)\big)^2 &= (r+I)(b+I)(r+I)(b+I)\\ &= (r+I)I(b+I)\\ &= I.\end{aligned}$$

This forces $(r+I)(b+I) = I$.

Conversely, assume that $(r+I)(b+I) = I$. With the same above arguement, $(b+I)(r+I) = I$.

(ii) Let $p+I, q+I \in (b+I)^{\perp}$ where $p, q \in R$. Then $(b+I)(p+I) = I$ and $(b+I)(q+I) = I$. We show that $(p+I)-(q+I) \in R/I$. Note that

$$\begin{aligned}I = (b+I)(p+I) - (b+I)(q+I) &= (bp - bq) + I\\ &= b(p-q) + I = (b+I)\big((p-q)+I\big).\end{aligned}$$

Then $(p+I)-(q+I)=(p-q)+I \in (b+I)^{\perp}$.

Next, let $a+I \in (b+I)^{\perp}$ and let $r+I \in R/I$ where $a, r \in R$. Then

$$(b+I)(a+I)=I \text{ and } (a+I)(b+I)=I.$$

We show that $(a+I)(r+I) \in (b+I)^{\perp}$ and $(r+I)(a+I) \in (b+I)^{\perp}$. These are obtained from $(b+I)(a+I)(r+I)=I(r+I)=I$ and $(r+I)(a+I)(b+I)=(r+I)I=I$. □

Next, we give the definitions of a maximal right ideal of a ring and a right quasi-duo ring. In addition, we refer here the S. Safaeeyan's result shown in [5] that, in a right quasi-duo ring $R$, if $M$ is its maximal right ideal and $a \in R$, then $a^2 \in M$ leads to $a \in M$.

**Definition 3.2.11.** A proper right ideal $I$ of a ring $R$ is called a *maximal right ideal* if for any right ideal $J$ of $R$, $I \subseteq J \subseteq R$ implies that either $J=I$ or $J=R$.

**Definition 3.2.12.** [5] A ring $R$ is called a *right quasi-duo ring* if every maximal right ideal of $R$ is a two-sided ideal.

**Lemma 3.2.13.** [5, Lemma 3.2] *Let $R$ be a right quasi-duo ring and $M$ be a maximal right ideal of $R$. Then for each $a \in R$, $a^2 \in M$ implies that $a \in M$.*

To generate similar result as above, we consider the set

$$(I:x)=\{r \in R \mid \ xr \in I\} \quad \text{where } x \in R \text{ and } I \text{ is an ideal of } R.$$

It can be proved that $(I:x)$ is an ideal of $R$ containing $I$ for all ideals $I$ of $R$ and $x \in R$. However, we would like to study more about its structure for rings whose completely prime right ideals are two-sided.

**Proposition 3.2.14.** *Let $R$ be a ring whose completely prime right ideals are two-sided and $I$ be a completely prime right ideal of $R$.*

*(i) If $x \notin I$, then $(I:x)=I$.*

*(ii) If $x \in I$, then $(I:x)=R$.*

*Proof.*

(i) Assume that $x \notin I$. It suffices to show that $(I:x) \subseteq I$. Let $i \in (I:x)$. Then $xi \in I$. Moreover, $xI \subseteq I$ because $I$ is an ideal of $R$. Since $I$ is a completely prime right ideal of $R$, $xI \subseteq I$ and $x \notin I$, we conclude that $i \in I$. Hence, $(I:x)=I$.

(ii) Assume that $x \in I$. It is enough to show that $R \subseteq (I : x)$. Let $r \in R$. Then $xr \in I$. Thus, $r \in (I : x)$. □

**Corollary 3.2.15.** *Let $R$ be a ring whose completely prime right ideals are two-sided, $I$ be a completely prime right ideal of $R$ and $x \in R$. Then either $(I : x) = R$ or $(I : x)$ is a completely prime right ideal of $R$.*

**Theorem 3.2.16.** *Let $R$ be a ring whose completely prime right ideals are two-sided and $I$ be a completely prime right ideal of $R$. Then for each $x \in R$, $x^n \in I$ implies that $x \in I$ for any $n \in \mathbb{N}$.*

*Proof.* Let $x \in R$. Moreover, let $P(n)$ be the statement "$x^n \in I$ implies that $x \in I$" for any $n \in \mathbb{N}$.

**Basis Step** It is obvious that $x = x^1 \in I$. This shows that $P(1)$ holds.

**Induction Step** Let $k \in \mathbb{N}$ be such that $P(k)$ is true. Then $x^k \in I$ implies that $x \in I$. We will show that $P(k+1)$ is true. Assume that $x^{k+1} \in I$. Suppose that $x \notin I$. Then $(I : x) = I$ by Proposition 3.2.14(i). Then $xx^k = x^{k+1} \in I$, so $x^k \in (I : x) = I$. Since $P(k)$ is valid, we obtain $x \in I$. This is a contradiction. Hence, $x \in I$. This shows that $P(k+1)$ holds.

By mathematical induction, the statement $P(n)$ is true for any $n \in \mathbb{N}$. □

# References

[1] Huang, L. and Xue, W., An internal characterisation of strongly regular rings, *Bulletin of the Australian Mathematical Society*, Vol.46, No.3 (1992), pp.525-528.

[2] Koh, K., On one sided ideals of a prime type, *Proceedings of the American Mathematical Society*, Vol.28, No.2 (1971), pp.321-329.

[3] Kuzucuoglu, F., On one sided strongly prime ideals, *Hacettepe Journal of Mathematics and Statistics*, Vol.39, No.4 (2010), pp.577-582.

[4] Reyes, M.L., A one-sided prime ideal principle for noncommutative rings, *Journal of Algebra and Its Applications*, Vol.9, No.6 (2010), pp.877-919.

[5] Safaeeyan, S., Strongly quasi-duo rings, *Journal of Mathematical Extension*, Vol.5, No.2 (2011), pp.67-74.

# APPENDIX

# APPENDIX A

# The Project Proposal of Course 2301399 Project Proposal

# Academic Year 2020

| | |
|---|---|
| Project Tittle (Thai) | การศึกษาริงซึ่งไอดีลทางขวาเฉพาะเป็นไอดีลสองทาง |
| Project Tittle (English) | Study on rings whose prime right ideals are two-sided |
| Project Advisor | Associate Professor Dr. Sajee Pianskool |
| Project Co-advisor | Associate Professor Dr. Ouamporn Phuksuwan |
| By | Miss Chanagarn Laoiam ID 6033509323<br>Mathematics Program,<br>Department of Mathematics and Computer Science<br>Faculty of Science, Chulalongkorn University |

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## Background and Rationale

Let $R$ be an associative ring with identity. Any subgroup $I$ of $R$ is called a *right* (or *left*) *ideal* if $Ir \subseteq I$ (or $rI \subseteq I$) for any $r \in R$. A *two-sided ideal* is both a left ideal and a right ideal. For any nonempty subsets $I$ and $J$ of $R$, define

$$IJ = \{\sum_{k=1}^{n} i_k j_k \mid n \in \mathbb{N}, i_k \in I, j_k \in J \text{ for all } k = 1, \ldots, n\}$$

$$\text{and } I \circ J = \{ij \mid i \in I, j \in J\}.$$

An ideal $I$ of $R$ is called a *good ideal* if $(r_1 + I)(r_2 + I) = (r_1 + I) \circ (r_2 + I)$ for any $r_1, r_2 \in R$. A ring $R$ is called a *good ring* if all ideals are good ideals. A ring $R$ is called a *right duo ring* if any right ideal is two-sided. A ring $R$ is said to be *von Neuman regular* (or simply called *regular*) if each principal right (or left) ideal of $R$ is generated by an idempotent, which is an element $e \in R$ such that $e^2 = e$. A ring $R$ is said to be *strongly regular* if it is a regular and right duo ring.
In 1992, L. Huang and W. Xue [1] studied some properties of right duo rings related to good ideals. Moreover, they found a relation between strongly regular rings and good rings.

A proper right ideal $I$ of $R$ is called a *prime right ideal* if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for any right ideals $A, B$ of $R$. In this project, we study rings whose prime right ideals are two-sided and investigate some of their properties.

## Objectives

1. Study rings whose prime right ideals are two-sided.
2. Investigate some properties of rings whose prime right ideals are two-sided.

## Scope

All rings in this project are noncommutative associative rings with identity.

## Project Activities

1. Literature reviews on right duo rings.
2. Study rings whose prime right ideals are two-sided.
3. Investigate some properties of rings whose prime right ideals are two-sided.
4. Write a report.

## Activities Table

| Project Activities | August 2020 – April 2021 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Aug | Sep | Oct | Nov | Dec | Jan | Feb | Mar | Apr |
| 1. Literature reviews on right duo rings. | | | | | | | | | |
| 2. Study rings whose prime right ideals are two-sided. | | | | | | | | | |
| 3. Investigate some properties of rings whose prime right ideals are two-sided. | | | | | | | | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4. Write a report. | | | | | | | | |

## Benefits

Obtain some properties of rings whose prime right ideals are two-sided.

## Equipment

1. Computer
2. IPAD
3. Printer

## Budget

| | | |
|---|---|---|
| 1. HDMI cable adapters for ipad | 1,800 | Bahts |
| 2. External harddisk | 1,500 | Bahts |
| 3. Books | 1,700 | Bahts |
| **Total** | **5,000** | **Bahts** |

## References

[1] L. Huang and W. Xue, An internal characterization of strongly regular rings, *Bull. Aust. Math. Soc.*, **46** (1992), pp 525-528.

# Biography

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