โครงการ

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ชื่อโครงการ สมบัติทางทอพอโลยีของปริภูมิ n แบบที่บริบูรณ์และไม่แย้งกัน
Topological Properties of Spaces of Complete and Consistent n-Types

ชื่อนิสิต นายธนาธิป โพธิ์ขาว 603 35167 23

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์
สาขาวิชา คณิตศาสตร์

ปีการศึกษา 2563

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

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นายธนาธิป โพธิ์ขาว

โครงงานนี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรวิทยาศาสตรบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2563

TOPOLOGICAL PROPERTIES OF SPACES OF COMPLETE AND CONSISTENT N-TYPES

Mr. Thanathip Phokhaw

A Project Submitted in Partial Fulfillment of the Requirements
for the Degree of Bachelor of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2020

หัวข้อโครงงาน	สมบัติทางทอพอโลยีของปริภูมิ n แบบที่บริบูรณ์และไม่แย้งกัน
โดย	นายธนาธิป โพธิ์ขาว
สาขาวิชา	คณิตศาสตร์
อาจารย์ที่ปรึกษาโครงงาน	อาจารย์ ดร.อธิปัตย์ ธำรงธัญลักษณ์

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์ คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย อนุมัติให้นับโครงงานฉบับนี้เป็นส่วนหนึ่ง ของการศึกษาตามหลักสูตรปริญญาบัณฑิต ในรายวิชา 2301499 โครงงานวิทยาศาสตร์ (Senior Project)

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ธนาธิป โพธิ์ขาว: สมบัติทางทอพอโลยีของปริภูมิ n แบบที่บริบูรณ์และไม่แย้งกัน. (TOPOLOGICAL PROPERTIES OF SPACES OF COMPLETE AND CONSISTENT N-TYPES) อ.ที่ปรึกษาโครงงาน: อ.ดร.อธิปัตย์ ธำรงธัญลักษณ์, 27 หน้า.

ในโครงงานชิ้นนี้เราศึกษาสมบัติทางทอพอโลยีของปริภูมิสโตนบนเซต $S_n^{\mathfrak{M}}(A)$ ซึ่งเป็นเซตของ n แบบบนเซต A ที่บริบูรณ์และไม่แย้งกันในโครงสร้าง $\mathfrak{M}$ ใน $\mathcal{L}$ โดยที่ $\mathfrak{M}$ เป็นโครงสร้างขยายตัวของ $(\mathbb{Q})$ หรือ $(\mathbb{R})$ และ A เป็นสับเซตของเซตพื้นฐานของ $\mathfrak{M}$ จากการศึกษาเราพบว่า $S_1^{\mathfrak{M}}(\emptyset)$ เป็นเซตวิยุตเมื่อ $\mathfrak{M} = (\mathbb{Q}, <, \mathrm{S}, \mathrm{P})$ ซึ่ง S และ P เป็นฟังก์ชันจาก $\mathbb{Q}$ ไปยัง $\mathbb{Z}$ ที่นิยามโดย $\mathrm{S}(x) = \min\{n \in \mathbb{Z}: x < n\}$ และ $\mathrm{P}(x) = \max\{n \in \mathbb{Z}: n < x\}$ ทุก ๆ $x \in \mathbb{Q}$ เราพบอีกว่าปริภูมิสโตนบนเซต $S_1^{\mathfrak{M}}(\mathbb{Q})$ นั้นนับได้ลำดับแรกและลำดับที่สอง และปริภูมิสโตนบนเซต $S_1^{\mathfrak{N}}(\mathbb{R})$ นั้นนับได้ลำดับแรกแต่ไม่นับได้ลำดับที่สองเมื่อ $\mathfrak{M} = (\mathbb{Q}, <)$ และ $\mathfrak{N} = (\mathbb{R}, <)$

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาโครงงานหลัก ....

ปีการศึกษา 2563

6033516723: MAJOR MATHEMATICS
KEYWORDS: TOPOLOGICAL PROPERTIES, STONE SPACE, COMPLETE AND CONSISTENT N-TYPES

THANATHIP PHOKHAW: TOPOLOGICAL PROPERTIES OF SPACES OF COMPLETE AND CONSISTENT N-TYPES. ADVISOR: ATHIPAT THAMRONGTHANAYALAK, PH.D, 27 PP.

In this project, we study topological properties of the Stone space over the set $S_n^{\mathfrak{M}}(A)$ of complete and consistent n-types over A in an $\mathcal{L}$-structure $\mathfrak{M}$ where $\mathfrak{M}$ is an expansion of the $(\mathbb{Q})$ or $(\mathbb{R})$; and A is a subset of the underlying set of $\mathfrak{M}$. We found that $S_1^{\mathfrak{M}}(\emptyset)$ is discrete when $\mathfrak{M} = (\mathbb{Q}, <, \mathtt{S}, \mathtt{P})$ and $\mathtt{S}, \mathtt{P}$ are functions from $\mathbb{Q}$ to $\mathbb{Z}$ defined by $\mathtt{S}(x) = \min\{n \in \mathbb{Z} : x < n\}$ and $\mathtt{P}(x) = \max\{n \in \mathbb{Z} : n < x\}$ for each $x \in \mathbb{Q}$. We also found that the Stone space of $S_1^{\mathfrak{M}}(\mathbb{Q})$ is first-countable and second-countable and the Stone space of $S_1^{\mathfrak{N}}(\mathbb{R})$ is first-countable but not second-countable where $\mathfrak{M} = (\mathbb{Q}, <)$ and $\mathfrak{N} = (\mathbb{R}, <)$.

Department	Mathematics and Computer Science	Student's Signature	....
Field of Study	Mathematics	Advisor's Signature	....
Academic Year	2020		

Acknowledgments

First and foremost, I would like to express my deep and sincere gratitude to my project advisor, Dr. Athipat Thamrongthanayalak, for his inestimable advice, comment and encouragment throughout this project. I also thanks for all knowledges he taught me thoughtout my study, especially during the course 2301421: Mathematical Logic and before the topology course. My project cannot be accomplished without all these knowledges.

Secondly, I would like to express my special thanks to my project committee, Associate Professor Dr. Pimpen Vejjajiva and Associate Professor Dr. Nataphan Kitisin, for their comments and suggestions.

Lastly, I also would like to thank all of my teachers, my friends and my family for various lessons in my entire life.

Thanathip Phokhaw
April 2021

Contents

List of Notations

$\mathbb{N}$	Set of all natural numbers $0, 1, 2, \ldots$
$\mathbb{Z}$	Set of all integers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R}$	Set of all real numbers
Var	Set of all variables $v_1, v_2, \ldots$

Chapter 1

n-Types and the Stone Space

Throughout this project, let $\mathcal{L}$ be a language consisting of a set $\mathcal{R}$ of relation symbols, a set $\mathcal{C}$ of constant symbols and a set $\mathcal{F}$ of function symbols. Here, we assume some backgrounds in first-order logic. For more details, we refer to [1] and [3].

In this chapter, we introduce n-types along with some properties. All results in this chapter can be found in [1] and [3]; therefore we omit the proofs.

1.1 n-Types

Let $\mathfrak{M}$ be an $\mathcal{L}$-structure with the underlying set M and $A \subseteq M$. Let $\mathcal{L}_A$ be the language obtained by adding each element of A to $\mathcal{L}$ as a constant symbol. Note that $\mathfrak{M}$ is an $\mathcal{L}_A$-structure by the trivial interpretation $a^{\mathfrak{M}} = a$ for each $a \in A$. For any $\mathcal{L}_A$-sentence σ, we use $\mathfrak{M} \models \sigma$ to denote the statement "σ is true in $\mathfrak{M}$". Let $\mathrm{Th}_A(\mathfrak{M}) = \{\sigma : \sigma \text{ is an } \mathcal{L}_A\text{-sentence and } \mathfrak{M} \models \sigma\}$. Note that $\mathrm{Th}_A(\mathfrak{M})$ is satisfiable. Let T be an $\mathcal{L}_A$-theory. We say that $\mathfrak{M}$ is a model of T (denoted by $\mathfrak{M} \models T$) if $\mathfrak{M} \models \sigma$ for every $\sigma \in T$; and for any $\mathcal{L}_A$-formula $\phi(v_1, \ldots, v_n)$ we say that T satisfies $\phi(v_1, \ldots, v_n)$ (denoted by $T \models \phi(v_1, \ldots, v_n)$) if for every model $\mathfrak{M}$ of T and $a_1, \ldots, a_n \in \mathfrak{M}$, $\mathfrak{M} \models \phi(a_1, \ldots, a_n)$.

Definition 1.1. Let $p(v_1, \ldots, v_n)$ be a set of $\mathcal{L}_A$-formulas. We call $p(v_1, \ldots, v_n)$ an *n-type over A* if all $\mathcal{L}_A$-formulas in $p(v_1, \ldots, v_n)$ have no free occurrences of variables other than $v_1, \ldots, v_n$. We say that $p(v_1, \ldots, v_n)$ is *consistent* if $p(v_1, \ldots, v_n) \cup \mathrm{Th}_A(\mathfrak{M})$ is consistent. We say that $p(v_1, \ldots, v_n)$ is *complete* if $\phi \in p(v_1, \ldots, v_n)$ or $\neg\phi \in p(v_1, \ldots, v_n)$ for all $\mathcal{L}_A$-formulas ϕ with no free occurrences of variables other than $v_1, \ldots, v_n$. We let $S_n^{\mathfrak{M}}(A)$ be the set of all complete and consistent n-types over A.

Let $p(v_1, \ldots, v_n)$ be an n-type and $c_1, \ldots, c_n$ be fresh constant symbols. By the Completeness Theorem and the Compactness Theorem, it follows that an n-type $p(v_1, \ldots, v_n)$ over A is consistent if and only if $p(c_1, \ldots, c_n) \cup \mathrm{Th}_A(\mathfrak{M})$ is finitely satisfiable.

We simply write p instead of $p(v_1, \ldots, v_n)$ when it causes no confusion.

Here, we provide some examples.

Example 1.2. Let $\mathfrak{M} = (\mathbb{Q}, <)$ and $A = \mathbb{N}$. Let $p(v_1) = \{\text{"}i < v_1\text{"} : i \in \mathbb{N}\}$. Then p is a 1-type. Let Δ be a finite subset of $p \cup \mathrm{Th}_A(\mathfrak{M})$. If $\Delta \cap p = \emptyset$, then

$\Delta \subseteq \mathrm{Th}_A(\mathfrak{M})$; so Δ is satisfiable since $\mathrm{Th}_A(\mathfrak{M})$ is satisfiable. If $\Delta \cap p \neq \emptyset$, we may interprete v_1 as $\max\{i \in \mathbb{N} : \text{“}i < v_1\text{”} \in \Delta\} + 1$; therefore Δ is satisfiable. Thus $p \cup \mathrm{Th}_A(\mathfrak{M})$ is finitely satisfiable so p is consistent. It is obvious that p is not complete.

Example 1.3. Let $\mathfrak{M} = (\mathbb{Q}, <)$ and $A = \mathbb{N}$. Let $p(v_1) = \left\{\phi(v_1) : \mathfrak{M} \models \phi(\frac{1}{2})\right\}$. Then p is a 1-type. By replacing v_1 by $\frac{1}{2}$, it follows that p is satisfiable; so it is consistent. For any $\mathcal{L}_A$-formula $\phi(v_1)$, either $\mathfrak{M} \models \phi\left(\frac{1}{2}\right)$ or $\mathfrak{M} \models \neg\phi\left(\frac{1}{2}\right)$. Thus $\phi(v_1) \in p$ or $\neg\phi(v_1) \in p$; therefore p is complete.

The latter example shows a way to generate a complete and consistent n-types.

Definition 1.4. Let $\mathfrak{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and $\bar{a} = (a_1, \ldots, a_n) \in M^n$. Define

$$tp^{\mathfrak{M}}(\bar{a}/A) = \{\phi(v_1, \ldots, v_n) : \phi \text{ is an } \mathcal{L}_A\text{-formula and } \mathfrak{M} \models \phi(a_1, \ldots, a_n)\}.$$

We denote $tp^{\mathfrak{M}}(\bar{a}/\emptyset)$ by $tp^{\mathfrak{M}}(\bar{a})$.

Then $tp^{\mathfrak{M}}(\bar{a}/A)$ is complete and consistent. In [3], $tp^{\mathfrak{M}}(\bar{a}/A)$ is called the *type of $\bar{a}$ in $\mathfrak{M}$ over A* or the *relative type of $\bar{a}$ in $\mathfrak{M}$ over A*. We provide another impotant definition.

Definition 1.5. Suppose p is a consistent n-types over A. We say that $\bar{a} \in M^n$ *realizes* p (or p *is realized by* $\bar{a}$) if $\mathfrak{M} \models \phi(\bar{a})$ for all $\phi \in p$. We say that p *is realized in* $\mathfrak{M}$ if p is realized by some $\bar{a} \in M^n$.

If p is not realized in $\mathfrak{M}$, we say that $\mathfrak{M}$ *omits* p (or p *is omitted in* $\mathfrak{M}$).

In Definition 1.5., it follows that a consistent n-type p over A is realized in $\mathfrak{M}$ if $p \subseteq tp^{\mathfrak{M}}(\bar{a}/A)$ for some $\bar{a} \in M^n$. In [3], this property is used to define the realizability of a consistent n-types over A which is equivalent to Definition 1.5.

Definition 1.6. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures. An *$\mathcal{L}$-homomorphism* $h : \mathfrak{M} \to \mathfrak{N}$ is a function $h : M \to N$ such that

i) for every $R \in \mathcal{R}$ and $a_1, \ldots, a_{\mathbf{arity}(R)} \in M$, we have

$$\text{if } \left(a_1, \ldots, a_{\mathbf{arity}(R)}\right) \in R^{\mathfrak{M}} \text{ then } \left(h(a_1), \ldots, h(a_{\mathbf{arity}(R)})\right) \in R^{\mathfrak{N}};$$

ii) $h(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$ for every $c \in \mathcal{C}$; and

iii) for every $f \in \mathcal{F}$ and $a_1, \ldots, a_{\mathbf{arity}(f)} \in M$, we have

$$h\left(f^{\mathfrak{M}}\left(a_1, \ldots, a_{\mathbf{arity}(f)}\right)\right) = f^{\mathfrak{N}}\left(h(a_1), \ldots, h(a_{\mathbf{arity}(f)})\right).$$

We say that an $\mathcal{L}$-homomorphism $h : \mathfrak{M} \to \mathfrak{N}$ is a strong $\mathcal{L}$-homomorphism if for every $R \in \mathcal{R}$ and $a_1, \ldots, a_{\mathbf{arity}(R)} \in M$, we have

$$\left(a_1, \ldots, a_{\mathbf{arity}(R)}\right) \in R^{\mathfrak{M}} \text{ if and only if } \left(h(a_1), \ldots, h(a_{\mathbf{arity}(R)})\right) \in R^{\mathfrak{N}}.$$

An $\mathcal{L}$-*embedding* from $\mathfrak{M}$ to $\mathfrak{N}$ is an injective strong $\mathcal{L}$-homomorphism; and an $\mathcal{L}$-*isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ is a surjective $\mathcal{L}$-embedding.

We say that $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{L}$-*isomorphic* if there is an $\mathcal{L}$-isomorphism from $\mathfrak{M}$ to $\mathfrak{N}$.

We call an $\mathcal{L}$-isomorphism $\mathfrak{M}$ to $\mathfrak{M}$ itself an $\mathcal{L}$-*automorphism on* $\mathfrak{M}$.

Definition 1.7. Let $h : \mathfrak{M} \to \mathfrak{N}$. We say that h is an *elementary* $\mathcal{L}$-*embedding* from $\mathfrak{M}$ to $\mathfrak{N}$ if h is an $\mathcal{L}$-embedding and for every $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n)$ and $a_1, \ldots, a_n \in M$,

$$\mathfrak{M} \models \phi(a_1, \ldots, a_n) \text{ if and only if } \mathfrak{N} \models \phi(h(a_1), \ldots, h(a_n)).$$

Theorem 1.8. *Every* $\mathcal{L}$-*isomorphism is an elementary* $\mathcal{L}$-*embedding.*

Definition 1.9. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures. We say that $\mathfrak{M}$ is a *substructure* of $\mathfrak{N}$ (or $\mathfrak{N}$ is an *extension* of $\mathfrak{M}$), denoted by $\mathfrak{M} \subseteq \mathfrak{N}$, if

i) $M \subseteq N$;

ii) $R^{\mathfrak{M}} = R^{\mathfrak{N}} \cap M^{\mathbf{arity}(R)}$ for every $R \in \mathcal{R}$;

iii) $c^{\mathfrak{M}} = c^{\mathfrak{N}}$ for every $c \in \mathcal{C}$; and

iv) for every $f \in \mathcal{F}$, $f^{\mathfrak{M}}$ is the restriction of $f^{\mathfrak{N}}$ to $M^{\mathbf{arity}(f)}$.

Definition 1.10. Suppose $\mathfrak{M} \subseteq \mathfrak{N}$. We say that $\mathfrak{M}$ is an *elementary substructure* of $\mathfrak{N}$ or $\mathfrak{N}$ is an *elementary extension* of $\mathfrak{M}$, (denoted by $\mathfrak{M} \preccurlyeq \mathfrak{N}$) if for all $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n)$, and $a_1, \ldots, a_n \in M$, $\mathfrak{M} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathfrak{N} \models \phi(a_1, \ldots, a_n)$.

We provide some properties.

Proposition 1.11. *Let* $\mathfrak{M}$ *be an* $\mathcal{L}$-*structure,* $A \subseteq M$ *and* p *be a consistent* n-*type over* A. *There is an elementary extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *such that* p *is realized in* $\mathfrak{N}$.

Corollary 1.12. $p \in S_n^{\mathfrak{M}}(A)$ *if and only if there are an elementary extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *and* $\bar{a} \in N^n$ *such that* $p = tp^{\mathfrak{N}}(\bar{a}/A)$.

1.2 Stone Space of Complete and Consistent n-Types

In this section, we provide the definition of the stone space of complete and consistent n-types and some facts about the space.

Before we begin, we will provide the definition of bases of a topological space. For more backgrounds in topology, we refer to [2].

Definition 1.13. Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements or basis open sets*) such that

i) for each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$; and

ii) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there is $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$ and $x \in B_3$.

If $\mathcal{B}$ satisfies these two conditions, then we define the *topology τ generated by $\mathcal{B}$* as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

Definition 1.14. A topology τ on the set of complete and consistent n-types $S_n^{\mathfrak{M}}(A)$ can be generated by the basis $\mathcal{B}$ consisting of

$$[\phi] = \left\{p \in S_n^{\mathfrak{M}}(A) : \phi \in p\right\}$$

for all $\mathcal{L}_A$-formulas ϕ with no free occurrences of variables other than $v_1, \dots, v_n$. We call τ *the Stone topology on* $S_n^{\mathfrak{M}}(A)$ and $\left(S_n^{\mathfrak{M}}(A), \tau\right)$ *the Stone space of complete and consistent n-types (over A).* It follows that $[\top] = S_n^{\mathfrak{M}}(A)$ and $[\bot] = \emptyset$.

We simply write $S_n^{\mathfrak{M}}(A)$ as a topological space equipped with a topology τ generated by $\mathcal{B}$.

We recall some properties of the Stone space of complete and consistent n-types.

Proposition 1.15. *Let $\phi(v_1, \dots, v_n)$ and $\psi(v_1, \dots, v_n)$ be $\mathcal{L}_A$-formulas with no free occurrences of variables other than $v_1, \dots, v_n$. Then*

i) $[\phi \vee \psi] = [\phi] \cup [\psi]$,

ii) $[\phi \wedge \psi] = [\phi] \cap [\psi]$,

iii) $[\neg\phi] = S_n^{\mathfrak{M}}(A) \setminus [\phi]$.

Proposition 1.16. *$S_n^{\mathfrak{M}}(A)$ is compact; that is every collection of basis open set covering $S_n^{\mathfrak{M}}(A)$ has a finite subcollection covering $S_n^{\mathfrak{M}}(A)$.*

Proposition 1.17. *$S_n^{\mathfrak{M}}(A)$ is Hausdoff and totally disconected (that is, if $p, q \in S_n^{\mathfrak{M}}(A)$ and $p \neq q$, then there is a clopen set X such that $p \in X$ and $q \notin X$).*

Proposition 1.18. *$S_n^{\mathfrak{M}}(A)$ is zero-dimensional, i.e. the topology has a clopen base.*

Proposition 1.19. *If $\mathcal{L}$ and A are countable, then $S_n^{\mathfrak{M}}(A)$ is metrizable (that is, there exists a metric d on $S_n^{\mathfrak{M}}(A)$ that induces the Stone topology).*

Proposition 1.20. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures and A, B be subsets of M.*

i) Suppose $A \subseteq B$. For each $p \in S_n^{\mathfrak{M}}(B)$, let $p|A$ be the set of $\mathcal{L}_A$-formulas in p. Then $p|A \in S_n^{\mathfrak{M}}(A)$ and the map $p \mapsto p|A$ is a continuous map from $S_n^{\mathfrak{M}}(B)$ onto $S_n^{\mathfrak{M}}(A)$.

ii) Suppose $f : \mathfrak{M} \to \mathfrak{N}$ is an elementary embedding and $p \in S_n^{\mathfrak{M}}(A)$. Let

$$f(p) = \{\phi(v_1, \dots, v_n, f(a_1), \dots, f(a_k)) : \phi(v_1, \dots, v_n, a_1, \dots, a_k) \in p\}.$$

Then $f(p) \in S_n^{\mathfrak{N}}(f(A))$ and the map $p \mapsto f(p)$ is continuous.

Next, we provide the definition of isolated points in $S_n^{\mathfrak{M}}(A)$ along with its properties.

Definition 1.21. We say that $p \in S_n^{\mathfrak{M}}(A)$ is *isolated* if $\{p\}$ is an open subset of $S_n^{\mathfrak{M}}(A)$.

Proposition 1.22. *Let $p \in S_n^{\mathfrak{M}}(A)$ and ϕ be an $\mathcal{L}_A$-formula with no free occurrences of variables other than $v_1, \ldots, v_n$. Then the following are equivalent.*

i) $\{p\} = [\phi]$.

ii) *For any $\mathcal{L}_A$-formula $\psi(v_1, \ldots, v_n)$, $\psi \in p$ if and only if $\mathfrak{M} \models \forall v_1 \ldots \forall v_n(\phi \to \psi)$.*

If one of the above holds, we say that ϕ *isolates* p.

Proposition 1.23. *Let $p \in S_n^{\mathfrak{M}}(A)$. Then the following are equivalent.*

i) *p is isolated.*

ii) *$\{p\} = [\phi]$ for some $\mathcal{L}_A$-formula ϕ which is no free occurrences of variables other than $v_1, \ldots, v_n$.*

iii) *There exists an $\mathcal{L}_A$-formula $\phi(v_1, \ldots, v_n)$ with no free occurrences of variables other than $v_1, \ldots, v_n$ such that for any $\mathcal{L}_A$-formula $\psi(v_1, \ldots, v_n)$, $\psi \in p$ if and only if $\mathfrak{M} \models \forall v_1 \ldots \forall v_n(\phi \to \psi)$.*

Chapter 2

Topological Properties of The Stone Space of Complete Consistent 1-types over $(\mathbb{Q}, <, \mathtt{S}, \mathtt{P})$

Let $\mathcal{L} = \{<\}$ and $\mathcal{L}' = \{<, \mathtt{S}, \mathtt{P}\}$ where $\mathtt{S}$ and $\mathtt{P}$ are unary function symbols. Let $\mathfrak{M} = (\mathbb{Q}, <, \mathtt{S}, \mathtt{P})$ be an $\mathcal{L}'$-structure which is an expansion of an $\mathcal{L}$-stucture $(\mathbb{Q}, <)$; the interpretation $\mathtt{S}^{\mathfrak{M}}$ and $\mathtt{P}^{\mathfrak{M}}$ are functions from $\mathbb{Q}$ to $\mathbb{Z}$ defined by

$$\mathtt{S}^{\mathfrak{M}}(x) = \mathtt{S}(x) = \min\{n \in \mathbb{Z} : x < n\} \text{ and}$$
$$\mathtt{P}^{\mathfrak{M}}(x) = \mathtt{P}(x) = \max\{n \in \mathbb{Z} : n < x\}$$

for each $x \in \mathbb{Q}$.

It follows immediately that $x < \mathtt{S}(x)$ and $\mathtt{P}(x) < x$. If $x \in \mathbb{Z}$, we have $\mathtt{S}(x) = x + 1$ and $\mathtt{P}(x) = x - 1$. For each $x \in \mathbb{Q}$, we denote $\mathtt{S}(\mathtt{P}(x))$ and $\mathtt{P}(\mathtt{S}(x))$ by $\mathtt{S} \circ \mathtt{P}(x)$ and $\mathtt{P} \circ \mathtt{S}(x)$, respectively.

Next, we will show some properties about $\mathtt{P}$ and $\mathtt{S}$.

Lemma 2.1. *For any $n \in \mathbb{Z}$ and $x \in \mathbb{Q}$, $\mathtt{S}(n + x) = n + \mathtt{S}(x)$ and $\mathtt{P}(n + x) = n + \mathtt{P}(x)$.*

Proof. Let $n \in \mathbb{Z}$ and $x \in \mathbb{Q}$. Since $x < \mathtt{S}(x)$ and $\mathtt{P}(x) < x$, it follows that $n + x < n + \mathtt{S}(x)$ and $n + \mathtt{P}(x) < n + x$. Thus $\mathtt{S}(n + x) \leq n + \mathtt{S}(x)$ and $n + \mathtt{P}(x) \leq \mathtt{P}(n + x)$. Then we have

$$\begin{aligned} n + \mathtt{S}(x) &= n + \mathtt{S}(-n + (n + x)) \\ &\leq n - n + \mathtt{S}(n + x) \\ &= \mathtt{S}(n + x) \end{aligned}$$

and

$$\begin{aligned} \mathtt{P}(n + x) &= n - n + \mathtt{P}(n + x) \\ &\leq n + \mathtt{P}(-n + n + x) \\ &= n + \mathtt{P}(x). \end{aligned}$$

Hence $\mathtt{S}(n + x) = n + \mathtt{S}(x)$ and $\mathtt{P}(n + x) = n + \mathtt{P}(x)$. □

Lemma 2.2. $\mathtt{S}$ *and* $\mathtt{P}$ *are incresing functions.*

Proof. Let $x, y \in \mathbb{Q}$ be such that $x < y$. Then $x < y < \mathtt{S}(y)$ and $\mathtt{P}(x) < x < y$. Thus $\mathtt{S}(x) \leq \mathtt{S}(y)$ and $\mathtt{P}(x) \leq \mathtt{P}(y)$ by the minimality of $\mathtt{S}(x)$ and the maximality of $\mathtt{P}(x)$. Hence the lemma holds. □

Lemma 2.3. *For all* $x \in \mathbb{Q}$*, the following are equivalent:*

i) $x \in \mathbb{Z}$,

ii) $\mathtt{P} \circ \mathtt{S}(x) = x$,

iii) $\mathtt{S} \circ \mathtt{P}(x) = x$.

Proof. Let $x \in \mathbb{Q}$.

(i)⇒(ii) Assume $x \in \mathbb{Z}$. It follows that $\mathtt{P} \circ \mathtt{S}(x) = \mathtt{P}(x+1) = x$.

(ii)⇒(iii) Suppose $\mathtt{P} \circ \mathtt{S}(x) = x$. Then $\mathtt{S} \circ \mathtt{P}(x) = \mathtt{P}(x) + 1 = \mathtt{P}(x+1) = \mathtt{P} \circ \mathtt{S}(x)$.

(iii)⇒(i) Suppose $\mathtt{S} \circ \mathtt{P}(x) = x$. It follows immediately that $x \in \mathbb{Z}$.

Hence the lemma holds. □

Lemma 2.4. *For all* $x \in \mathbb{Q}$*, the following are equivalent:*

i) $x \notin \mathbb{Z}$,

ii) $\mathtt{P} \circ \mathtt{S}(x) < x$,

iii) $x < \mathtt{S} \circ \mathtt{P}(x)$.

Proof. Let $x \in \mathbb{Q}$.

(i)⇒(ii) Assume $x \notin \mathbb{Z}$. Then $\mathtt{P} \circ \mathtt{S}(x) \neq x$. Since $\mathtt{P} \circ \mathtt{S}(x) < \mathtt{S}(x)$, $\mathtt{S}(x) \nleq \mathtt{P} \circ \mathtt{S}(x)$ so $x \nless \mathtt{P} \circ \mathtt{S}(x)$. Thus $\mathtt{P} \circ \mathtt{S}(x) < x$.

(ii)⇒(iii) Suppose $\mathtt{P} \circ \mathtt{S}(x) < x$. Then $\mathtt{P} \circ \mathtt{S}(x) \leq \mathtt{P}(x)$ by the maximality of $\mathtt{P}(x)$. Thus $\mathtt{S} \circ \mathtt{P} \circ \mathtt{S}(x) \leq \mathtt{S} \circ \mathtt{P}(x)$. Since $\mathtt{S}(x)$ is an integer, we have $x < \mathtt{S}(x) = \mathtt{S} \circ \mathtt{P}(\mathtt{S}(x)) = \mathtt{S} \circ \mathtt{P} \circ \mathtt{S}(x)$. Then $x < \mathtt{S} \circ \mathtt{P}(x)$.

(iii)⇒(i) Suppose $x < \mathtt{S} \circ \mathtt{P}(x)$. By Lemma 2.3, $x \notin \mathbb{Z}$.

Hence the lemma holds. □

Lemma 2.5. *Let* $a \in \mathbb{Q}$*. If* $a \in \mathbb{Z}$*, then* $\mathtt{P}(x) = \mathtt{P}(a)$ *and* $\mathtt{S}(x) = a$ *for all* $x \in (\mathtt{P}(a), a)$*. If* $a \notin \mathbb{Z}$*, then* $\mathtt{P}(x) = \mathtt{P}(a)$ *and* $\mathtt{S}(x) = \mathtt{S}(a)$ *for all* $x \in (\mathtt{P}(a), \mathtt{S}(a))$*.*

Proof. First, assume $a \in \mathbb{Z}$. Let $x \in (\mathtt{P}(a), a)$. Then $a - 1 = \mathtt{P}(a) < x < a$, so $\mathtt{P}(x) = a - 1 = \mathtt{P}(a)$ and $\mathtt{S}(x) = a$.

Lastly, suppose $a \notin \mathbb{Z}$. Let $x \in (\mathtt{P}(a), \mathtt{S}(a))$.

If $x = a$, we are done.

Assume $x < a$. Then $\mathtt{P}(x) \leq \mathtt{P}(a)$. Since $\mathtt{P}(a) < x$, $\mathtt{P}(a) \leq \mathtt{P}(x)$ by the maximality of $\mathtt{P}(x)$. Thus $\mathtt{P}(x) = \mathtt{P}(a)$. Since $\mathtt{P}(a) < x < a$, $\mathtt{S} \circ \mathtt{P}(a) \leq \mathtt{S}(x) \leq \mathtt{S}(a)$. Since $a \notin \mathbb{Z}$, $a < \mathtt{S} \circ \mathtt{P}(a)$. So $\mathtt{S}(a) \leq \mathtt{S} \circ \mathtt{P}(a)$ by the minimality of $\mathtt{S}(a)$. Then $\mathtt{S}(x) = \mathtt{S}(a)$.

Assume $a < x$. Then $\mathtt{S}(a) \leq \mathtt{S}(x)$. Since $x < \mathtt{S}(a)$, $\mathtt{S}(x) \leq \mathtt{S}(a)$ by the minimality of $\mathtt{S}(x)$. Thus $\mathtt{S}(x) = \mathtt{S}(a)$. Since $a < x < \mathtt{S}(a)$, $\mathtt{P}(a) \leq \mathtt{P}(x) \leq \mathtt{P} \circ \mathtt{S}(a)$. Since a is not an integer, $\mathtt{P} \circ \mathtt{S}(a) < a$. So $\mathtt{P} \circ \mathtt{S}(a) \leq \mathtt{P}(a)$ by the maximality of $\mathtt{P}(a)$. Then $\mathtt{P}(x) = \mathtt{P}(a)$. □

Next, we will state the definition of discrete topologies and discrete sets.

Definition 2.6. Let X be a nonempty set. Then $\mathcal{P}(X)$ is the *discrete topology* on X. We say that a subset Y of the topological space X is a *discrete set* if the subspace topology of Y is the discrete topology; that is for all $y \in Y$, there is an open set G of X such that $G \cap Y = \{y\}$.

From the previous definition, it follows that the basis consisting of all singletons generates the discrete topology on a nonempty set.

We provide the result.

Theorem 2.7. *$S_1^{\mathfrak{M}}(\emptyset)$ is discrete.*

Proof. By Lemmas 2.3 and 2.4,

$$S_1^{\mathfrak{M}}(\emptyset) = [\mathtt{P} \circ \mathtt{S}(v_1) < v_1] \cup [\mathtt{P} \circ \mathtt{S}(v_1) = v_1]$$

and $[\mathtt{P} \circ \mathtt{S}(v_1) < v_1] \cap [\mathtt{P} \circ \mathtt{S}(v_1) = v_1] = \emptyset$. It is enough to show that both $[\mathtt{P} \circ \mathtt{S}(v_1) < v_1]$ and $[\mathtt{P} \circ \mathtt{S}(v_1) = v_1]$ are singletons.

By proposition 1.21, it suffices to show that "$\mathtt{P} \circ \mathtt{S}(v_1) = v_1$" isolates any $p \in [\mathtt{P} \circ \mathtt{S}(v_1) = v_1]$ and "$\mathtt{P} \circ \mathtt{S}(v_1) < v_1$" isolates any $p \in [\mathtt{P} \circ \mathtt{S}(v_1) < v_1]$.

Let $p \in [\mathtt{P} \circ \mathtt{S}(v_1) = v_1]$.

Let $\psi(v_1) \in p(v_1)$. Since $\{\mathtt{P} \circ \mathtt{S}(v_1) = v_1, \psi(v_1)\} \subseteq p(v_1)$ and $p(v_1)$ is consistent, there exists $r \in \mathbb{Q}$ such that $\mathfrak{M} \models (\mathtt{P} \circ \mathtt{S}(r) = r) \wedge \psi(r)$. Then $r \in \mathbb{Z}$. Next, it suffices to show that $\mathfrak{M} \models \forall v_1 ((\mathtt{P} \circ \mathtt{S}(v_1) = v_1) \rightarrow \psi(v_1))$.

Suppose not. Then $\mathfrak{M} \models \exists v_1 ((\mathtt{P} \circ \mathtt{S}(v_1) = v_1) \wedge (\neg\psi(v_1)))$. Thus there exists $c \in \mathbb{Q}$ such that $\mathfrak{M} \models (\mathtt{P} \circ \mathtt{S}(c) = c) \wedge \neg\psi(c)$. Then $c \in \mathbb{Z}$

Define $f : \mathbb{Q} \to \mathbb{Q}$ by $f(x) = x - c + r$. We will show that f is an $\mathcal{L}$-isomorphism. (This will imply that $\mathfrak{M} \models \psi(c)$ if and only if $\mathfrak{M} \models \psi(f(c))$.)

It is obvious that f is a strictly increasing bijection and we have

$$f \circ \mathtt{S}(x) = \mathtt{S}(x) + r - c = \mathtt{S}(x + r - c) = \mathtt{S} \circ f(x)$$

and

$$f \circ \mathtt{P}(x) = \mathtt{P}(x) + r - c = \mathtt{P}(x + r - c) = \mathtt{P} \circ f(x).$$

Therefore f is an $\mathcal{L}$-isomorphism.

Since $\mathfrak{M} \models \neg\psi(c)$, $\mathfrak{M} \models \neg\psi(f(c))$. Since $f(c) = r$, $\mathfrak{M} \models \neg\psi(r)$ which contradicts $\mathfrak{M} \models \psi(r)$. Hence $\mathfrak{M} \models \forall v_1 ((\mathtt{P} \circ \mathtt{S}(v_1) = v_1) \rightarrow \psi(v_1))$.

Let $p \in [\mathtt{P} \circ \mathtt{S}(v_1) < v_1]$.

Let $\psi(v_1) \in p(v_1)$. Since $\{\mathtt{P} \circ \mathtt{S}(v_1) < v_1, \psi(v_1)\} \subseteq p(v_1)$ and $p(v_1)$ is consistent, there exists $r \in \mathbb{Q}$ such that $\mathfrak{M} \models (\mathtt{P} \circ \mathtt{S}(r) < r) \wedge \psi(r)$. Then $r \notin \mathbb{Z}$.

Similarly, it suffices to show that $\mathfrak{M} \models \forall v_1 ((\mathtt{P} \circ \mathtt{S}(v_1) < v_1) \rightarrow \psi(v_1))$.

Suppose not. Then $\mathfrak{M} \models \exists v_1 ((\mathtt{P} \circ \mathtt{S}(v_1) < v_1) \wedge (\neg\psi(v_1)))$. Thus there exists $c \in \mathbb{Q}$ such that $\mathfrak{M} \models (\mathtt{P} \circ \mathtt{S}(c) < c) \wedge \neg\psi(c)$. Then $c \notin \mathbb{Z}$.

Define $g : \mathbb{Q} \to \mathbb{Q}$ by

$$g(x) = \begin{cases} (\mathtt{P}(r) - \mathtt{P}(c)) + x, & \text{if } x \leq \mathtt{P}(c); \\ \dfrac{r - \mathtt{P}(r)}{c - \mathtt{P}(c)}(x - \mathtt{P}(c)) + \mathtt{P}(r), & \text{if } \mathtt{P}(c) < x \leq c; \\ \dfrac{\mathtt{S}(r) - r}{\mathtt{S}(c) - c}(x - c) + r, & \text{if } c < x < \mathtt{S}(c) \text{ and} \\ (\mathtt{S}(r) - \mathtt{S}(c)) + x, & \text{if } \mathtt{S}(c) \leq x. \end{cases}$$

We will show that g is an $\mathcal{L}$-isomorphism. (This will imply that $\mathfrak{M} \models \psi(c)$ if and only if $\mathfrak{M} \models \psi(g(c))$.)

First, we will show that g is surjective.

Let $y \in \mathbb{Q}$.

Suppose $y \leq \mathtt{P}(r)$. Choose $x = y - (\mathtt{P}(r) - \mathtt{P}(c))$. Since

$$\begin{aligned} x &= y - (\mathtt{P}(r) - \mathtt{P}(c)) \\ &\leq \mathtt{P}(r) - (\mathtt{P}(r) - \mathtt{P}(c)) \\ &= \mathtt{P}(c), \end{aligned}$$

it follows that $g(x) = (\mathtt{P}(r) - \mathtt{P}(c)) + y - (\mathtt{P}(r) - \mathtt{P}(c)) = y$.

Suppose $\mathtt{P}(r) < y \leq r$. Choose $x = \frac{c - \mathtt{P}(c)}{r - \mathtt{P}(r)}(y - \mathtt{P}(r)) + \mathtt{P}(c)$. Since $0 < \frac{c - \mathtt{P}(c)}{r - \mathtt{P}(r)}$, we have

$$\begin{aligned} \mathtt{P}(c) &= \frac{c - \mathtt{P}(c)}{r - \mathtt{P}(r)}(\mathtt{P}(r) - \mathtt{P}(r)) + \mathtt{P}(c) \\ &< \frac{c - \mathtt{P}(c)}{r - \mathtt{P}(r)}(y - \mathtt{P}(r)) + \mathtt{P}(c) \\ &= x \end{aligned}$$

and

$$\begin{aligned} x &= \frac{c - \mathtt{P}(c)}{r - \mathtt{P}(r)}(y - \mathtt{P}(r)) + \mathtt{P}(c) \\ &\leq \frac{c - \mathtt{P}(c)}{r - \mathtt{P}(r)}(r - \mathtt{P}(r)) + \mathtt{P}(c) \\ &= c. \end{aligned}$$

So $\mathtt{P}(c) < x \leq c$.

Then $g(x) = \frac{r - \mathtt{P}(r)}{c - \mathtt{P}(c)}\left(\frac{c - \mathtt{P}(c)}{r - \mathtt{P}(r)}(y - \mathtt{P}(r)) + \mathtt{P}(c) - \mathtt{P}(c)\right) + \mathtt{P}(r) = y$.

Suppose $r < y < \mathtt{S}(r)$. Choose $x = \frac{\mathtt{S}(c) - c}{\mathtt{S}(r) - r}(y - r) + c$. Since $0 < \frac{\mathtt{S}(c) - c}{\mathtt{S}(r) - r}$, we have

$$\begin{aligned} c &= \frac{\mathtt{S}(c) - c}{\mathtt{S}(r) - r}(r - r) + c \\ &< \frac{\mathtt{S}(c) - c}{\mathtt{S}(r) - r}(y - r) + c \\ &= x \end{aligned}$$

and

$$\begin{aligned} x &= \frac{\mathtt{S}(c) - c}{\mathtt{S}(r) - r}(y - r) + c \\ &< \frac{\mathtt{S}(c) - c}{\mathtt{S}(r) - r}(\mathtt{S}(r) - r) + c \\ &= \mathtt{S}(c). \end{aligned}$$

So $c < x < \mathtt{S}(c)$.

Then $g(x) = \frac{\mathtt{S}(r)-r}{\mathtt{S}(c)-c}\left(\frac{\mathtt{S}(c)-c}{\mathtt{S}(r)-r}(y-r)+c-c\right)+r = y$.
Assume $\mathtt{S}(r) \le y$. Choose $x = y - (\mathtt{S}(r) - \mathtt{S}(c))$.
Since

$$\begin{aligned} s(c) &= \mathtt{S}(r) - (\mathtt{S}(r) - \mathtt{S}(c)) \\ &\le y - (\mathtt{S}(r) - (c)) \\ &= x, \end{aligned}$$

it follows that $g(x) = (\mathtt{S}(r) - \mathtt{S}(c)) + y - (\mathtt{S}(r) - \mathtt{S}(c)) = y$.

Thus g is surjective.

Next, we will show that g is strictly increasing which implies that g is injective.

Let

$$\begin{aligned} D_1 &= (-\infty, \mathtt{P}(c)], & D_2 &= (\mathtt{P}(c), c], \\ D_3 &= (c, \mathtt{S}(c)) \text{ and } & D_4 &= [\mathtt{S}(c), \infty). \end{aligned}$$

Then

$$\begin{aligned} g[D_1] &= (-\infty, \mathtt{P}(r)], & g[D_2] &= (\mathtt{P}(r), r], \\ g[D_3] &= (r, \mathtt{S}(r)) \text{ and } & g[D_4] &= [\mathtt{S}(r), \infty). \end{aligned}$$

By linearity of $<$, it is enough to show that if $x < y$, then $g(x) < g(y)$.

Let $x, y \in \mathbb{Q}$. Assume $x < y$. If $x \in D_i$ and $y \in D_j$ where $i, j \in \{1,2,3,4\}$ and $i < j$, then $g(x) < g(y)$. Assume $x, y \in D_i$ where $i \in \{1,2,3,4\}$.

If $i = 1$, we have

$$\begin{aligned} g(x) &= (\mathtt{P}(r) - \mathtt{P}(c)) + x \\ &< (\mathtt{P}(r) - \mathtt{P}(c)) + y \\ &= g(y). \end{aligned}$$

If $i = 2$, we have

$$\begin{aligned} g(x) &= \frac{r - \mathtt{P}(r)}{c - \mathtt{P}(c)}(x - \mathtt{P}(c)) + \mathtt{P}(r) \\ &< \frac{r - \mathtt{P}(r)}{c - \mathtt{P}(c)}(y - \mathtt{P}(c)) + \mathtt{P}(r) \\ &= g(y). \end{aligned}$$

If $i = 3$, we have

$$\begin{aligned} g(x) &= \frac{\mathtt{S}(r) - r}{\mathtt{S}(c) - c}(x - c) + r \\ &< \frac{\mathtt{S}(r) - r}{\mathtt{S}(c) - c}(y - c) + r \\ &= g(y). \end{aligned}$$

If $i = 4$, we have

$$\begin{aligned} g(x) &= (\mathtt{S}(r) - \mathtt{S}(c)) + x \\ &< (\mathtt{S}(r) - \mathtt{S}(c)) + y \\ &= g(y). \end{aligned}$$

Thus $g(x) < g(y)$.

Then g is strictly incresing. Hence g is strictly incresing bijection.

Lastly, we will show that $g \circ \mathtt{S}(x) = \mathtt{S} \circ g(x)$ and $g \circ \mathtt{P}(x) = \mathtt{P} \circ g(x)$ for all $x \in \mathbb{Q}$.

Let $x \in \mathbb{Q}$. We now have 3 cases to consider.

Case 1: $x \in (-\infty, \mathtt{P}(c)]$. We have

$$\begin{aligned} \mathtt{S} \circ g(x) &= \mathtt{S}((\mathtt{P}(r) - \mathtt{P}(c)) + x) \\ &= (\mathtt{P}(r) - \mathtt{P}(c)) + \mathtt{S}(x) \\ &= g \circ \mathtt{S}(x) \end{aligned}$$

and

$$\begin{aligned} \mathtt{P} \circ g(x) &= \mathtt{P}((\mathtt{P}(r) - \mathtt{P}(c)) + x) \\ &= (\mathtt{P}(r) - \mathtt{P}(c)) + \mathtt{P}(x) \\ &= g \circ \mathtt{P}(x). \end{aligned}$$

Case 2: $x \in [\mathtt{S}(c), \infty)$. We have

$$\begin{aligned} \mathtt{S} \circ g(x) &= \mathtt{S}((\mathtt{S}(r) - \mathtt{S}(c)) + x) \\ &= (\mathtt{S}(r) - \mathtt{S}(c)) + \mathtt{S}(x) \\ &= g \circ \mathtt{S}(x) \end{aligned}$$

and

$$\begin{aligned} \mathtt{P} \circ g(x) &= \mathtt{P}((\mathtt{P}(r) - \mathtt{P}(c)) + x) \\ &= (\mathtt{P}(r) - \mathtt{P}(c)) + \mathtt{P}(x) \\ &= g \circ \mathtt{P}(x). \end{aligned}$$

Case 3: $x \in (\mathtt{P}(c), \mathtt{S}(c))$. Since $c \notin \mathbb{Z}$, $\mathtt{P}(x) = \mathtt{P}(c)$ and $\mathtt{S}(x) = \mathtt{S}(c)$. Then we have

$$\begin{aligned} g \circ \mathtt{P}(x) &= (\mathtt{P}(r) - \mathtt{P}(c)) + \mathtt{P}(x) \\ &= (\mathtt{P}(r) - \mathtt{P}(c)) + \mathtt{P}(c) \\ &= \mathtt{P}(r) \end{aligned}$$

and

$$\begin{aligned} g \circ \mathtt{S}(x) &= (\mathtt{S}(r) - \mathtt{S}(c)) + \mathtt{S}(x) \\ &= (\mathtt{S}(r) - \mathtt{S}(c)) + \mathtt{S}(c) \\ &= \mathtt{S}(r). \end{aligned}$$

Since $g[(\mathtt{P}(c), \mathtt{S}(c))] = (\mathtt{P}(r), \mathtt{S}(r))$ and $r \notin \mathbb{Z}$, $\mathtt{P} \circ g(x) = \mathtt{P}(r)$ and $\mathtt{S} \circ g(x) = \mathtt{S}(r)$. Then $g \circ \mathtt{P}(x) = \mathtt{P} \circ g(x)$ and $g \circ \mathtt{S}(x) = \mathtt{S} \circ g(x)$.

Thus $g \circ \mathtt{P}(x) = \mathtt{P} \circ g(x)$ and $g \circ \mathtt{S}(x) = \mathtt{S} \circ g(x)$.

Therefore g is an $\mathcal{L}$-isomorphism.

Since $\mathfrak{M} \models \neg\psi(c)$, $\mathfrak{M} \models \neg\psi(g(c))$. Since $g(c) = r$, $\mathfrak{M} \models \neg\psi(r)$ which contradicts $\mathfrak{M} \models \psi(r)$. Hence $\mathfrak{M} \models \forall v_1 ((\mathtt{P} \circ \mathtt{S}(v_1) < v_1) \to \psi(v_1))$. $\square$

Chapter 3

First Countability and Second Countability of the Stone Spaces of Complete and Consistent 1-type over $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$

In this section, we consider the countability axioms of $S_1^{\mathfrak{M}}(\mathbb{Q})$ and $S_1^{\mathfrak{N}}(\mathbb{R})$ where $\mathfrak{M} = (\mathbb{Q}, <)$ and $\mathfrak{N} = (\mathbb{R}, <)$ which are $\mathcal{L}$-structures with $\mathcal{L} = \{<\}$.

First, we introduce the definition of quantifier elimination and a properties which can be seen in [1] and [3].

Definition 3.1. Let T be an $\mathcal{L}$-theory. We say that T has *quantifier elimination* if for every formula $\phi(v_1, \dots, v_n)$ there is a quantifier-free $\mathcal{L}$-formula $\psi(v_1, \dots, v_n)$ such that

$$T \models \forall v_1 \dots \forall v_n (\phi(v_1, \dots, v_n) \leftrightarrow \psi(v_1, \dots, v_n)).$$

Proposition 3.2. *Suppose that $\mathcal{L}$ contains a constant symbol c, T is an $\mathcal{L}$-theory, and $\phi(v_1, \dots, v_n)$ is an $\mathcal{L}$-formula. The following are equivalent:*

i) there is a quantifier-free $\mathcal{L}$-formula $\psi(v_1, \dots, v_n)$ such that

$$T \models \forall v_1 \dots \forall v_n (\phi(v_1, \dots, v_n) \leftrightarrow \psi(v_1, \dots, v_n)),$$

ii) if $\mathfrak{M}$ and $\mathfrak{N}$ are models of T, $\mathfrak{A}$ is an $\mathcal{L}$-structure, $\mathfrak{A} \subseteq \mathfrak{M}$ and $\mathfrak{A} \subseteq \mathfrak{N}$, then $\mathfrak{M} \models \phi(a_1, \dots, a_n)$ if and only if $\mathfrak{N} \models \phi(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in A$.

Proposition 3.3. *Let DLO_∞ be the $\mathcal{L}$-theory of dense linear orders without end-points. Then DLO_∞ has quantifier elimination.*

From Proposition 3.3, since $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are models of DLO_∞, we have that both $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ admit quantifier elimination.

Let $\mathfrak{M} \models \mathrm{DLO}_\infty$ and $A \subseteq M$. Define L_p and U_p by

$$L_p = \{a \in A : \text{“}a < v_1\text{”} \in p\} \text{ and}$$
$$U_p = \{b \in A : \text{“}v_1 < b\text{”} \in p\}$$

where $p \in S_1^{\mathfrak{M}}(A)$.

Proposition 3.4. *Let* $\mathfrak{M} \models \mathrm{DLO}_\infty$ *and* $A \subseteq M$. *Then for any* $p \in S_1^{\mathfrak{M}}(A)$, *if* p *is not realized by elements in* A, *then* $L_p \cup U_p = A$ *and* $a < b$ *for all* $a \in L_p$ *and* $b \in U_p$.

Proof. Trivial. □

Next, we provide the definitions of the first countability axiom and the second countability axiom. We refer to [2] for more details.

Definition 3.5. A topological space X is said to have a *countable basis at* x if there is a countable collection $\mathcal{B}_x$ of open neighborhoods of x such that each open neighborhood of x contains at least one of the elements of $\mathcal{B}_x$. A topological space that has a countable basis at each of its points is said to satisfy the *first countability axiom*, or to be *first-countable.*

Definition 3.6. If a topological space X has a countable basis for its topology, then X is said to satisfy the *second countability axiom*, or to be *second-countable.*

We provide the result.

Theorem 3.7. $S_1^{\mathfrak{M}}(\mathbb{Q})$ *is both first-countable and second-countable but* $S_1^{\mathfrak{N}}(\mathbb{R})$ *is first-countable but not second-countable where* $\mathfrak{M} = (\mathbb{Q}, <)$ *and* $\mathfrak{N} = (\mathbb{R}, <)$.

Proof. Since the set of words on $\mathbf{Var} \cup \mathbb{Q} \cup \{<\} \cup \{\neg, \wedge, \vee, \forall, \exists, \top, \bot\}$ is countable, the set of all $\mathcal{L}_{\mathbb{Q}}$-formulas on $\mathfrak{M}$ is also countable. Hence $S_1^{\mathfrak{M}}(\mathbb{Q})$ is second-countable. Thus it is also first-countable.

For $S_1^{\mathfrak{N}}(\mathbb{R})$, since $tp^{\mathfrak{N}}(a/\mathbb{R})$ is isolated by "$v_1 = a$" for all $a \in \mathbb{R}$, the basis genarating $S_1^{\mathfrak{N}}(\mathbb{R})$ must contain $\{[v = a] : a \in \mathbb{R}\}$ which is an uncountable set. Thus $S_1^{\mathfrak{N}}(\mathbb{R})$ is not second-countable.

We will show that $S_1^{\mathfrak{N}}(\mathbb{R})$ is first-countable.

Let $p \in S_1^{\mathfrak{N}}(\mathbb{R})$.

In the case that p is realized by $a \in \mathbb{R}$, p is isolated by "$v_1 = a$"; so $\{p\} = [v_1 = a]$. Therefore every open neighborhood G of $p \in S_1^{\mathfrak{N}}(\mathbb{R})$, $[v_1 = a] \subseteq G$. Choose $\mathcal{B}_p = \{[v_1 = a]\}$. Then we are done.

Suppose p is not realized by any $a \in \mathbb{R}$. By Proposition 3.4, we have 4 cases to consider.

<u>Case 1:</u> Assume $L_p = \mathbb{R}$ and $U_p = \emptyset$.

Choose $\mathcal{B}_p = \{[d < v_1] : d \in \mathbb{Z}\}$. Observe that $\mathcal{B}_p$ is countable.

Let G be an open neighborhood of p. Then there is an $\mathcal{L}_{\mathbb{R}}$-formula $\phi(v_1)$ such that $p \in [\phi(v_1)] \subseteq G$.

Observe that $(\mathbb{R}, <) \models \neg (x < y) \leftrightarrow (x = y \vee x < y)$ and $(\mathbb{R}, <) \models \neg (x = y) \leftrightarrow (x < y \vee y < x)$. Since $(\mathbb{R}, <)$ admits quantifier elimination, we may assume that

$$\phi(v_1) = \bigvee_{i=1}^{k} \bigwedge_{j=1}^{l_i} \theta_{ij}(v_1)$$

where $\theta_{ij}(v_1)$ is of the form $v_1 = a$, $v_1 < a$ and $a < v_1$ for some $a \in \mathbb{R}$. We can see that $\left[\bigvee_{i=1}^{k} \bigwedge_{j=1}^{l_i} \theta_{ij}(v_1)\right] = \bigcup_{i=1}^{k} \left[\bigwedge_{j=1}^{l_i} \theta_{ij}(v_1)\right]$. Therefore, we may assume further that $\phi(v_1) = \bigwedge_{j=1}^{l} \theta_j(v_1)$. Since $p \in [\phi(v_1)]$, we have that $\theta_j(v_1) \in p$ for all $j \in \{1, \ldots, l\}$. Since p is not realized in $(\mathbb{R}, <)$ and $U_p = \emptyset$, we have that for each $j \in \{1, \ldots, l\}$, $\theta_j(v_1) =$ "$a_j < v_1$" for some $a_j \in \mathbb{R}$.

Let $d \in \mathbb{Z}$ be such that $\max\{a_1, \ldots, a_l\} \leq d$. Then $(\mathbb{R}, <) \models d < v_1 \rightarrow \bigwedge_{j=1}^{l} \theta_j(v_1)$. Therefore $p \in [d < v_1] \subseteq [\phi(v_1)] \subseteq G$.

Case 2: Assume $L_p = \emptyset$ and $U_p = \mathbb{R}$.

Choose $\mathcal{B}_p = \{[v_1 < d] : d \in \mathbb{Z}\}$. Observe that $\mathcal{B}_p$ is countable.

Let G be an open neighborhood of p. Then there is an $\mathcal{L}_{\mathbb{R}}$-formula $\phi(v_1)$ such that $p \in [\phi(v_1)] \subseteq G$.

Similar to Case 1, we may assume that $\phi(v_1) = \bigwedge_{j=1}^{l} \theta_j(v_1)$ where $\theta_j(v_1)$ is of the form $v_1 = a$, $v_1 < a$ and $a < v_1$ for some $a \in \mathbb{R}$. Since $p \in [\phi(v_1)]$, we have that $\theta_j(v_1) \in p$ for all $j \in \{1, \ldots, l\}$. Since p is not realized in $(\mathbb{R}, <)$ and $U_p = \emptyset$, we have that for each $j \in \{1, \ldots, l\}$, $\theta_j(v_1) =$ "$v_1 < b_j$" for some $b_j \in \mathbb{R}$.

Let $d \in \mathbb{Z}$ be such that $d \leq \min\{b_1, \ldots, b_l\}$. Then $(\mathbb{R}, <) \models v_1 < d \rightarrow \bigwedge_{j=1}^{l} \theta_j(v_1)$. Therefore $p \in [v_1 < d] \subseteq [\phi(v_1)] \subseteq G$.

Case 3: Assume $L_p = (-\infty, c)$ and $U_p = [c, \infty)$ for some $c \in \mathbb{R}$.

Choose $\mathcal{B}_p = \{[d < v_1 \wedge v_1 < c] : d \in L_p \cap \mathbb{Q}\}$. Observe that $\mathcal{B}_p$ is countable.

Let G be an open neighborhood of p. Then there is an $\mathcal{L}_{\mathbb{R}}$-formula $\phi(v_1)$ such that $p \in [\phi(v_1)] \subseteq G$.

Similar to Case 1, we may assume that $\phi(v_1) = \bigwedge_{j=1}^{l} \theta_j(v_1)$ where $\theta_j(v_1)$ is of the form $v_1 = a$, $v_1 < a$ and $a < v_1$ for some $a \in \mathbb{R}$. Since $p \in [\phi(v_1)]$, we have that $\theta_j(v_1) \in p$ for all $j \in \{1, \ldots, l\}$. Since p is not realized in $(\mathbb{R}, <)$, we have that for each $j \in \{1, \ldots, l\}$, $\theta_j(v_1) =$ "$a_j < v_1$" for some $a_j \in L_p$ or $\theta_j(v_1) =$ "$v_1 < a_j$" for some $a_j \in U_p$. Then $\{a_1, \ldots, a_l\} = \{a_i\}_{i \in I} \cup \{a_k\}_{k \in K}$ where $\theta_i(v_1) =$ "$a_i < v_1$" and $\theta_k(v_1) =$ "$v_1 < a_k$" for each $i \in I$ and $k \in K$.

If $I = \emptyset$, then we have $(\mathbb{R}, <) \models v_1 < c \rightarrow \bigwedge_{j=1}^{l} \theta_j(v_1)$. Since $(\mathbb{R}, <) \models (d < v_1 \wedge v_1 < c) \rightarrow v_1 < c$ for any $d \in L_p \cap \mathbb{Q}$, we have $(\mathbb{R}, <) \models (d < v_1 \wedge v_1 < c) \rightarrow \bigwedge_{j=1}^{l} \theta_j(v_1)$. Therefore $p \in [d < v_1 \wedge v_1 < c] \subseteq [\phi(v_1)] \subseteq G$ for any $d \in L_p \cap \mathbb{Q}$.

If $I \neq \emptyset$, then there exists $d \in \mathbb{Q}$ such that $\max\{a_i\}_{i \in I} \leq d < c$. Then $(\mathbb{R}, <) \models (d < v_1 \wedge v_1 < c) \rightarrow \bigwedge_{j=1}^{l} \theta_j(v_1)$. Therefore $p \in [d < v_1 \wedge v_1 < c] \subseteq [\phi(v_1)] \subseteq G$.

Case 4: Assume $L_p = (-\infty, c]$ and $U_p = (c, \infty)$ for some $c \in \mathbb{R}$.

Choose $\mathcal{B}_p = \{[c < v_1 \wedge v_1 < d] : d \in U_p \cap \mathbb{Q}\}$. Observe that $\mathcal{B}_p$ is countable.

Let G be an open neighborhood of p. Then there is an $\mathcal{L}_{\mathbb{R}}$-formula $\phi(v_1)$ such that $p \in [\phi(v_1)] \subseteq G$.

Similar to Case 1, we may assume that $\phi(v_1) = \bigwedge_{j=1}^{l} \theta_j(v_1)$ where $\theta_j(v_1)$ is of the form $v_1 = a$, $v_1 < a$ and $a < v_1$ for some $a \in \mathbb{R}$. Since $p \in [\phi(v_1)]$, we have that $\theta_j(v_1) \in p$ for all $j \in \{1, \ldots, l\}$. Since p is not realized in $(\mathbb{R}, <)$, we have that for each $j \in \{1, \ldots, l\}$, $\theta_j(v_1) =$ "$a_j < v_1$" for some $a_j \in L_p$ or $\theta_j(v_1) =$ "$v_1 < a_j$" for some $a_j \in U_p$. Then $\{a_1, \ldots, a_l\} = \{a_i\}_{i \in I} \cup \{a_k\}_{k \in K}$ where $\theta_i(v_1) =$ "$a_i < v_1$" and $\theta_k(v_1) =$ "$v_1 < a_k$" for each $i \in I$ and $k \in K$.

If $K = \emptyset$, then we have $(\mathbb{R}, <) \models c < v_1 \rightarrow \bigwedge_{j=1}^{l} \theta_j(v_1)$. Since $(\mathbb{R}, <) \models (c < v_1 \wedge v_1 < d) \rightarrow c < v_1$ for any $d \in U_p \cap \mathbb{Q}$, we have $(\mathbb{R}, <) \models (c < v_1 \wedge v_1 < d) \rightarrow \bigwedge_{j=1}^{l} \theta_j(v_1)$. Therefore $p \in [c < v_1 \wedge v_1 < d] \subseteq [\phi(v_1)] \subseteq G$ for any $d \in U_p \cap \mathbb{Q}$.

If $K \neq \emptyset$, then there exists $d \in \mathbb{Q}$ such that $c < d \leq \min\{a_k\}_{k \in K}$. Then $(\mathbb{R}, <) \models (c < v_1 \wedge v_1 < d) \rightarrow \bigwedge_{j=1}^{l} \theta_j(v_1)$. Therefore $p \in [c < v_1 \wedge v_1 < d] \subseteq [\phi(v_1)] \subseteq G$.

Hence $S_1^{\mathfrak{M}}(\mathbb{R})$ is first-countable. $\square$

References

[1] D. Marker. **Model Theory : An Introduction.** Graduate texts in mathematics vol. 217. New York: Springer-Verlag, 2002.

[2] J. Munkres. **Topology.** 2^{nd}ed. Essex: Pearson Education Limited, 2014.

[3] H. Sarbadhikari and S. M. Srivastava. **A Course on Basic Model Theory.** Singapore: Springer, 2017.

Appendix A
Project Proposal

กลุ่มที่ 24

เอกสารนี้ได้รับการอนุมัติจากอาจารย์ที่ปรึกษาโครงงานแล้ว

ลงชื่อ ....

(วันที่ ....)

The Project Proposal of Course 2301399 Project Proposal
Academic Year 2020

Project Tittle (Thai)	สมบัติทางทอพอโลยีของปริภูมิ n แบบที่บริบูรณ์และไม่แย้งกัน
Project Tittle (English)	Topological Properties of Spaces of Complete and Consistent n-Types
Project Advisor:	Dr. Athipat Thamrongthanayalak
By	1. Thanathip Phokhaw ID 6033516723
	Mathematics, Department of Mathematics and Computer Science,
	Faculty of Science, Chulalongkorn University

Background and Rationale

Suppose $\mathfrak{M}$ is an $\mathcal{L}$-structure and A is a subset of an underlying set M of $\mathfrak{M}$. An n-type over A is a set of $\mathcal{L}_A$-formulas with no free occurrences of variables other than $v_1, \dots, v_n$. Let p be an n-type over A and let $\mathrm{Th}_A(\mathfrak{M})$ be the set of all $\mathcal{L}_A$-sentences true in $\mathfrak{M}$. We say that p is consistent if $p \cup \mathrm{Th}_A(\mathfrak{M})$ does not prove a contradiction; and complete if ϕ or $\neg\phi$ is in p for all $\mathcal{L}_A$-formula ϕ with no free occurrences of variables other than $v_1, \dots, v_n$.

Let $S_n^{\mathfrak{M}}(A)$ be the set of all complete and consistent n-types over A. Then $S_n^{\mathfrak{M}}(A)$ can be equipped with a topology τ generated by basic open sets $[\phi] = \{p \in S_n^{\mathfrak{M}}(A) | \phi \in p\}$ for all $\mathcal{L}_A$-formula ϕ with no free occurrences of variables other than $v_1, \dots, v_n$. We call the topological space $(S_n^{\mathfrak{M}}(A), \tau)$ the Stone space of complete and consistent n-types over A. It is known that $(S_n^{\mathfrak{M}}(A), \tau)$ is a compact totally disconnected Hausdoff space.[1] It is natural to study other topological properties of these spaces.

Objectives

To study topological properties of the Stone spaces of complete and consistent n-types over A in $\mathfrak{M}$ where $\mathfrak{M}$ is an expansion of the rationals; or the reals.

Scope

In this project, we are interested in the case that $\mathfrak{M}$ is an expansion of $(\mathbb{Q})$, or $(\mathbb{R})$.

Project Activities

Project activities are consisting of:

1) Studying the background of the Stone space of complete and consistent n-types.
2) Studying $\mathbb{Q}$ and its expansions
3) Studying $\mathbb{R}$ and its expansions
4) Writing a project report.

	2020						2021		
	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar
Study the background of the Stone space									
Study $\mathbb{Q}$ and its expansions									
Study $\mathbb{R}$ and its expansions									
Write a project report									

Benefits

To understand topological properties of the Stone spaces of complete and consistent n-types over A in $\mathfrak{M}$ where $\mathfrak{M}$ is an expansion of ($\mathbb{Q}$), or ($\mathbb{R}$).

Equipment

LaTeX

Budget

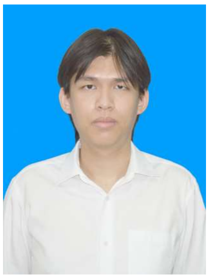
1) Books 5,000 Baht

References

[1] D. Marker. **Model theory: an introduction.** Graduate texts in mathematics vol. 217 New York: Springer-Verlag, 2002.

[2] J. Munkres. **Topology.** 2nd ed. Essex: Pearson Education Limited, 2014.

Appendix B
Author's Profile



Mr. Thanathip Phokhaw
Student ID 6033516723
Department of Mathematics and Computer Sciences
Faculty of Science, Chulalongkorn University