Chapter II

The Logic

2.1 The Type System

Types are symbols used to separate the objects in the universe into different groups. Every nonlogical symbol in this logic will have a fixed type and the type associated with the symbol will not change. A **type system** is a collection of types constructed from some predefined types—called **primitive types**—with new types created from other types by **type operators**. In this thesis we will use only two type operators: the **function type operator** ($\rightarrow$) and the **(Cartesian) product type operator** ($*$).

The formal definition of type system that we give is based on the type system commonly used in functional languages, (see [2]).

Definition 2.1.1 (Type System) *Let $P_1, P_2, ..., P_n$ be distinct symbols, called the primitive types. The* **type system** *generated by $P_1, P_2, ..., P_n$, denoted by $\langle P_1, P_2, ..., P_n \rangle$, is a set of strings of symbols defined inductivly as follows.*

Define $\mathbb{T}_0$ to be the set of all primitive types, i.e., $\mathbb{T}_0 = \{P_1, P_2, ..., P_n\}$.

Assume $\mathbb{T}_k$ is defined. We define $\mathbb{T}_{k+1}$ by

$$\mathbb{T}_{k+1} = \mathbb{T}_k \cup \{(T_1 \rightarrow T_2) \mid T_1, T_2 \in \mathbb{T}_k\} \cup \{(T_1 * ... * T_m) \mid T_1, ..., T_m \in \mathbb{T}_k\}$$

Finally, we define the set of all **types** *to be the set*

$$\mathbb{T} = \langle P_1, ..., P_n \rangle = \bigcup_{k=0}^{\infty} \mathbb{T}_k.$$

A type of the form $(T_1 \rightarrow T_2)$ is called a **function type**, *and a type of the form $(T_1 * \cdots * T_m)$ is called a* **product type**.

Definition 2.1.2 *Let $T \in \mathbb{T}$. The* **order of** *T, denoted by $\mathcal{O}(T)$, is defined inductively as follows.*

If $T \in \mathbb{T}_0$, then $\mathcal{O}(T) = 1$.

Assume $k \in \mathbf{N}$ and that $\mathcal{O}(T_1)$ is defined for every $T_1 \in \mathbb{T}_k$.

If $T \in \mathbb{T}_{k+1} \setminus \mathbb{T}_k$ then T is either a function type or a product type.

Case T is a function type. Then T is of the form $(T_1 \rightarrow T_2)$ where $T_1, T_2 \in \mathbb{T}_k$. Define $\mathcal{O}(T) = \mathcal{O}(T_1) + \mathcal{O}(T_2)$.

Case T is a product type. Then T is of the form $(T_1 * ... * T_m)$ where T_1 , ..., $T_m \in \mathbb{T}_k$. Define $\mathcal{O}(T) = max\{\mathcal{O}(T_1), ..., \mathcal{O}(T_m)\}$.

Our type systems are ω–**order** because they contain types of order k for every $k \in \omega$.

We will leave out parentheses whenever it does not cause any confusion. In addition, we will adopt the conventions that

(i) $T_1 \to T_2 \to \cdots \to T_m$ means $((\cdots(T_1 \to T_2) \to \cdots) \to T_m)$,

(ii) $T_1 \to T_2 * T_3$ means $(T_1 \to (T_2 * T_3))$, and

(iii) $T_1 * T_2 \to T_3$ means $((T_1 * T_2) \to T_3)$.

Example 2.1.3 *Let N and Z be primitive types.*

- *N, Z, $(N \to Z)$, $(N * N * Z)$, and $((N * N) \to Z)$ are types in $\langle N, Z \rangle$.*
- *$((Z \to (Z * N)) \to (Z \to N))$ simplifies to $Z \to Z * N \to (Z \to N)$.*
- *$((((Z * N * Z) \to (N \to (Z * N))) * Z * (N \to Z))$ simplifies to $(Z * N * Z \to (N \to Z * N)) * Z * (N \to Z)$.*

2.2 The syntax of predicate logic with types

Our syntax is similar to the syntax of traditional first-order logic, (see [3]). Thus, the first step is to choose a language to use. For this we must start by choosing a type system $\mathbb{T} = \langle P_1, P_2, ..., P_n \rangle$. As we shall see later, every term will have a fixed type associated with it. For brevity, we will denote the statement "The term t has type T" by t^T.

There is a fixed set of logical symbols, which always remains the same. It consists of an infinite set of variables

$$\mathbf{V} = \{x_{(T,\imath)} \mid T \in \mathbb{T}, \imath \in \mathbb{N}\}$$

together with the set whose members are the symbols

$$\neg \wedge \vee \Rightarrow \Longleftrightarrow \ \exists \forall \ (\ ) \ ,$$

(including the comma) and the set

$$\{=^T \mid T \in \mathbb{T}\}.$$

For each type T the variables $x_{(T,\imath)}$, $\imath \in \mathbb{N}$, are all of type T. Thus for any type T we have infinitely many variables of that type. For convenience we write $x_\imath^T$ for $x_{(T,\imath)}$. Note also that we have an equals sign for each type. Let us denote the set of all logical symbols by $\mathcal{S}$. A language also contains two kinds of nonlogical symbols, the set of constant symbols $\mathcal{C}$ and the set of relation symbols $\mathcal{R}$. The sets $\mathcal{S}, \mathcal{C}, \mathcal{R}$, and $\{P_1, ..., P_n, \rightarrow, *\}$ must be pairwise disjoint, and each symbol in $\mathcal{C} \cup \mathcal{R}$ must have a type associated with it. Note that there are no function symbols. Indeed, there is no need for function symbols, because we can have constant symbols whose types are function types. Thus, all functions are included in $\mathcal{C}$.

The union $\mathcal{L} = \mathcal{R} \cup \mathcal{C} \cup \mathcal{S}$ is called a **language over the type system** $\mathbb{T}$. However, since $\mathcal{S}$ is fixed, to describe $\mathcal{L}$ we only need to say what symbols are in the sets $\mathcal{R}$ and $\mathcal{C}$, and we will usually write, "Let $\mathcal{L} = \mathcal{R} \cup \mathcal{C}$ be a language over the type system $\mathbb{T}$".

We use the symbol $\equiv$ for **syntactic equality**, i.e., for two strings of symbols s_1 and s_2, $s_1 \equiv s_2$ will mean that s_1 and s_2 are exactly the same strings. This symbol is *not* allowed in any of our languages.

2.2.1 Terms and formulas

Let $\mathcal{L}$ be a language over a type system $\mathbb{T}$. An **expression** over $\mathcal{L}$ is a finite sequence of symbols from $\mathcal{L}$. There are two special kinds of expressions we will use in our logic, **terms** and **formulas**, which we will now define. Every term will have a type associated with it, which we will define at the same time we define terms.

Definition 2.2.1 *We start by defining a sequence $\{\mathbb{T}_k\}$ of sets by induction, as follows. Let $\mathbb{T}_0$ be the set $\mathbb{V} \cup C$. The type of $x_i^T \in \mathbb{V}$ is T, and the elements of C already have types associated with them.*

Assume $k \in \mathbb{N}$, and $\mathbb{T}_k$ is defined and the type of every term in $\mathbb{T}_k$ is also defined. We define

$$\mathbb{T}_{k+1} = \mathbb{T}_k \cup \{f^{S\rightarrow T}(s^S) \mid f^{S\rightarrow T}, s^S \in \mathbb{T}_k\} \cup \{(t_1^{T_1}, ..., t_m^{T_m}) \mid t_1^{T_1}, ..., t_m^{T_m} \in \mathbb{T}_k\},$$

and define the type of the term $f^{S\rightarrow T}(s^S)$ to be T, while the type of the term $(t_1^{T_1}, ..., t_m^{T_m})$ is $(T_1 * ... * T_m)$. Finally, we define the set of all **terms** *over $\mathcal{L}$ to be*

$$\mathbb{T} = \bigcup_{k=0}^{\infty} \mathbb{T}_k.$$

To make it easier to write our inductions, for each term $t^T \in \mathbb{T}$ let us define the **rank of** t^T to be the smallest k such that $t^T \in \mathbb{T}_k$, and denote it by $rank(t^T)$. When we say "by induction on t^T" , we will mean "by induction on $rank(t^T)$". Then we have the following lemma.

Lemma 2.2.2 *Let t^T be a term.*

(i) If $rank(t^T) = 0$, then either $t^T \in \mathbb{V}$ or $t^T \in C$.

(ii) If $rank(t^T)$ is k, where $k > 0$, then either $t^T \equiv (f^{S\to T}(s^S))$ for some terms $f^{S\to T}$ and s^S of rank less than k, or $t^T \equiv (t_1^{T_1}, ..., t_m^{T_m})$ for some terms $t_1^{T_1}, ..., t_m^{T_m}$ of rank less than k.

Proof: Let t^T be a term.

(i) If $rank(t^T) = 0$, then $t^T \in \mathbb{T}_0$. Thus either $t^T \in \mathbb{V}$ or $t^T \in \mathcal{C}$.

(ii) If $rank(t^T)$ is k, where $k > 0$, then $t^T \in \mathbb{T}_k$. Since $k > 0$, $\mathbb{T}_k = \mathbb{T}_\ell + 1$, where $\ell \geq 0$. Since k is the smallest number such that $t^T \in \mathbb{T}_k$, $t^T \notin \mathbb{T}_\ell$. Thus $t^T \in \mathbb{T}_k \setminus \mathbb{T}_\ell$. Hence either $t^T \equiv f^{S\to T}(s^S)$ for some terms $f^{S\to T}$ and s^S in $\mathbb{T}_\ell$, or $t^T \equiv (t_1^{T_1}, ..., t_m^{T_m})$ for some terms $t_1^{T_1}, ..., t_m^{T_m}$ in $\mathbb{T}_\ell$. Since $\ell < k$, the ranks of the terms $f^{S\to T}, s^S, t_1^{T_1}, ..., t_m^{T_m}$ are all less than k.

Our definition of formulas is very similar to the one in traditional predicate logic because formulas have no types. (They can be considered as all having the fixed type boolean.)

Definition 2.2.3 *As in our definition of terms, we will define a sequence $\{\mathbb{F}_k\}$. Let*

$$\mathbb{F}_0 = \{(r^{T_1*\dots*T_m}(t_1^{T_1}, ..., t_m^{T_m})) \mid r^{T_1*\dots*T_m} \in \mathcal{R}, t_1^{T_1}, ..., t_m^{T_m} \in \mathbb{T}\}$$
$$\cup\{(t_1^T =^T t_2^T) \mid t_1^T, t_2^T \in \mathbb{T}\}.$$

(Note: The case $m = 1$ of $r^{T_1*\dots*T_m}(t_1^{T_1}, ..., t_m^{T_m})$ is just $r^{T_1}(t_1^{T_1})$.) Assuming $\mathbb{F}_k$ has already been defined, we define $\mathbb{F}_{k+1}$ by

$$\mathbb{F}_{k+1} = \mathbb{F}_k \cup \{(\neg\varphi) \mid \varphi \in \mathbb{F}_k\} \cup \{(\varphi \wedge \psi) \mid \varphi, \psi \in \mathbb{F}_k\}$$
$$\cup\{(\exists x^T \varphi) \mid x^T \in \mathbb{V}, \varphi \in \mathbb{F}_k\}.$$

Then the set of all **formulas** *over $\mathcal{L}$ is the union*

$$\mathbb{F} = \bigcup_{k=0}^{\infty} \mathbb{F}_k.$$

We can give a definition of rank for formulas analogous to the way we gave one for terms: The rank of $\varphi \in \mathbb{F}$, $rank(\varphi)$, is the smallest k such that $\varphi \in \mathbb{F}_k$. Then we have a lemma similar to the lemma for terms.

Lemma 2.2.4 *Let φ be a formula.*

(i) If $rank(\varphi) = 0$, then either $\varphi \equiv (r^{T_1*\dots*T_m}(t_1^{T_1}, ..., t_m^{T_m}))$ for some $r^{T_1*\dots*T_m} \in \mathcal{R}$ and $t_1^{T_1}, ..., t_m^{T_m} \in \mathbb{T}$, or $\varphi \equiv (t_1^T =^T t_2^T)$ for some terms t_1^T and t_2^T.

(ii) If $rank(\varphi)$ is k, where $k > 0$, then φ is in one of the following forms: $(\neg\psi)$, $(\psi \wedge \chi)$, or $(\exists x^T \psi)$, where ψ and χ are formulas of rank less than k ,and $x^T \in \mathbb{V}$

Proof: Similar to the proof of Lemma 2.2.2.

In order to simplify the writing of formulas, we will introduce the following abbreviations. For $\varphi, \psi \in \mathbb{F}$ and $x^T \in \mathbb{V}$,

(a) $(\varphi \vee \psi)$ *will abbreviate* $(\neg((\neg\varphi) \wedge (\neg\psi)))$,

(b) $(\varphi \Rightarrow \psi)$ *will abbreviate* $((\neg\varphi) \vee \psi)$,

(c) $(\varphi \Leftrightarrow \psi)$ *will abbreviate* $((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi))$, *and*

(d) $(\forall x^T \varphi)$ *will abbreviate* $(\neg(\exists x^T(\neg\varphi)))$.

(e) We will leave out parentheses whenever it is clear how to put them back in correctly. In addition, we will adopt the conventions that

  (i) $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_m$ *means* $((\cdots(\varphi_1 \wedge \varphi_2) \wedge \cdots) \wedge \varphi_m)$, *and*

  (ii) $\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_m$ *means* $((\cdots(\varphi_1 \vee \varphi_2) \vee \cdots) \vee \varphi_m)$.

Example 2.2.5 *This example is based on group theory. Our objective is to create a language which we can use to write statements about group homomorphisms — something we cannot do in ordinary (first-order) predicate logic.*

Let $\mathcal{L} = \{e_1^N, e_2^Z, +^{N*N\to N}, \oplus^{Z*Z\to Z}, h^{N\to Z}\}$ *be a language over the type system* $\mathbb{T} = \langle N, Z\rangle$. *Here are some terms and formulas in our language.*

- e_1^N, $+^{N*N\to N}(x^N, e_1^N)$, $\oplus^{Z*Z\to Z}(x_1^Z, x_2^Z)$ *are terms.*
- $h^{N\to Z}(e_1^N)$, $\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e_2^Z)$ *are terms.*
- $\forall x_1^N(+^{N*N\to N}(x_1^N, e_1^N) =^N x_1^N)$ *is a formula.*
- $\forall x_1^Z \exists x_2^Z(\oplus^{Z*Z\to Z}(x_1^Z, x_2^Z) =^Z e_2^Z)$ *is a formula.*
- $\forall x_1^N \forall x_2^N(h^{N\to Z}(+^{N*N\to N}(x_1^N, x_2^N)) =^Z \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(x_2^N)))$ *is a formula.*

2.2.2 Free Variables

Free variables play important roles in substitution, so we will define free variables rigorously. We begin with the set of free variables for terms.

Definition 2.2.6 *Let* t^T *be a term over* $\mathcal{L}$. **The set of free variables** *of* t^T, $\mathbb{FV}(t^T)$, *will be defined by induction on* t^T. *If* $rank(t^T)=0$, *then either* $t^T \in \mathbb{V}$ *or* $t^T \in \mathcal{C}$.

Case $t^T \in \mathbb{V}$: $\mathbb{FV}(t^T) = \{t^T\}$.

Case $t^T \in \mathcal{C}$: $\mathbb{FV}(t^T) = \emptyset$.

Assume $rank(t^T) = k$, *where* $k > 0$, *and for all terms* s^S *with* $rank(s^S) < k$, $\mathbb{FV}(s^S)$ *is defined.*

Case $t^T \equiv (f^{S\to T}(s^S))$ *for some terms* $f^{S\to T}$ *and* s^S *with rank less than* k. *Define*

$$\mathbb{FV}(t^T) = \mathbb{FV}(f^{S\to T}) \cup \mathbb{FV}(s^S)$$

Case $t^T \equiv (t_1^{T_1}, ..., t_m^{T_m})$ *for some terms* $t_1^{T_1}, ..., t_m^{T_m}$ *with rank less than* k. *Define*

$$\mathbb{FV}(t^T) = \mathbb{FV}(t_1^{T_1}) \cup \cdots \cup \mathbb{FV}(t_m^{T_m}).$$

Now we can define the set of free variables for a formula φ by induction on φ, as follows.

Definition 2.2.7 *Let* φ *be a formula over* $\mathcal{L}$. **The set of free variables** *of* φ, $\mathbb{FV}(\varphi)$, *will be defined by induction on* φ. *If* $rank(\varphi) = 0$, *then there are two cases.*

Case $\varphi \equiv (t_1^T =^T t_2^T)$ *for some terms* t_1^T, t_2^T. *Define*

$$\mathbb{FV}(\varphi) = \mathbb{FV}(t_1^T) \cup \mathbb{FV}(t_2^T).$$

Case $\varphi \equiv (r^{T_1 * \cdots * T_m}(t_1^{T_1}, ..., t_m^{T_m}))$ *for some terms* $t_1^{T_1}, ..., t_m^{T_m}$. *Define*

$$\mathbb{FV}(\varphi) = \mathbb{FV}(t_1^{T_1}) \cup \cdots \cup \mathbb{FV}(t_m^{T_m}).$$

Assume that $rank(\varphi) = k$, *where* $k > 0$, *and that for all formulas* ψ *with* $rank(\psi) < k$, $\mathbb{FV}(\psi)$ *is defined.*

Case $\varphi \equiv (\neg\psi)$ *for some formula* ψ *with* $rank(\psi) < k$. *Define*

$$\mathbb{FV}(\varphi) = \mathbb{FV}(\psi).$$

Case $\varphi \equiv (\psi \wedge \chi)$ *for some formulas* ψ *and* χ *with rank less than* k. *Define*

$$\mathbb{FV}(\varphi) = \mathbb{FV}(\psi) \cup \mathbb{FV}(\chi).$$

Case $\varphi \equiv (\exists x^T \psi)$ *for some variable* x^T *and some formula* ψ *with rank less than* k. *Define*

$$\mathbb{FV}(\varphi) = \mathbb{FV}(\psi) \setminus \{x^T\}.$$

2.2.3 Substitution

The definition of substitution that we want to use in this thesis is a little different from the standard one. It is inspired by the definition used in the lambda calculus (see [4]), and will simplify one of our axiom schemes, but complicate our semantics a little bit. First, we need to define what it means to substitute the term t^T for the variable v^T in the term s^S, which is defined by induction as follows.

Definition 2.2.8 *Let t^T and s^S be terms and let v^T be a variable. The result of substituting t^T for v^T in s^S, denoted by $[t^T/v^T]s^S$, is defined by induction on s^S, as follows. If rank$(s^S) = 0$, then either $s^S \in \mathbf{V}$ or $s^S \in \mathcal{C}$.*

Case $s^S \in \mathbf{V}$. Define

$$[t^T/v^T]s^S \equiv \begin{cases} t^T & \text{if } s^S \equiv v^T \\ s^S & \text{if } s^S \not\equiv v^T \end{cases}$$

Case $s^S \in \mathcal{C}$. Define

$$[t^T/v^T]s^S \equiv s^S.$$

Assume rank$(s^S) = k$, where $k > 0$, and for all terms u^U with rank less than k, $[t^T/v^T]u^U$ is defined.

Case $s^S \equiv f^{U\to S}(u^U)$ for some terms $f^{U\to S}$ and u^U with rank less than k. Define

$$[t^T/v^T]s^S \equiv ([t^T/v^T]f^{U\to S})([t^T/v^T]u^U).$$

Case $s^S \equiv (t_1^{T_1}, ..., t_m^{T_m})$ for some terms $t_1^{T_1}, ..., t_m^{T_m}$ with rank less than k. Define

$$[t^T/v^T]s^S \equiv ([t^T/v^T]t_1^{T_1}, ..., [t^T/v^T]t_m^{T_m}).$$

Now we can define substitution into formulas.

Definition 2.2.9 *Let t^T be a term, v^T a variable, and φ a formula. The result of substituting t^T for v^T in φ, denoted by $[t^T/v^T]\varphi$, is defined by induction on φ, as follows. If rank$(\varphi) = 0$, then there are two cases to consider.*

Case $\varphi \equiv (t_1^S =^S t_2^S)$. Define

$$[t^T/v^t]\varphi \equiv ([t^T/v^T]t_1^S =^S [t^T/v^T]t_2^S).$$

Case $\varphi \equiv r^{T_1*\cdots*T_m}(t_1^{T_1}, ..., t_m^{T_m})$. Define

$$[t^T/v^T]\varphi \equiv r^{T_1*\cdots*T_m}([t^T/v^T]t_1^{T_1}, ..., [t^T/v^T]t_m^{T_m}).$$

Assume that rank$(\varphi) = k$, where $k > 0$, and that for all formulas ψ with rank$(\psi) < k$, $[t^T/v^T]\psi$ is defined.

Case $\varphi \equiv (\neg\psi)$ for some formula ψ. Define

$$[t^T/v^T]\varphi \equiv (\neg[t^T/v^T]\psi).$$

Case $\varphi \equiv (\psi \wedge \chi)$ for some formulas ψ and χ. Define

$$[t^T/v^T]\varphi \equiv ([t^T/v^T]\psi \wedge [t^T/v^T]\chi).$$

Case $\varphi \equiv (\exists x^S\psi)$ for some variable x^S and formula ψ. There are three subcases to consider.

Subcase (i): $v^T = x^S$. *Define* $[t^T/v^T]\varphi \equiv \varphi$.

Subcase (ii): $v^T \neq x^S$ *and* $x^S \notin \mathbb{FV}(t^T)$ *or* $v^T \notin \mathbb{FV}(\psi)$. *Define*

$$[t^T/v^T]\varphi \equiv (\exists x^S[t^T/v^T]\psi).$$

Subcase (iii): $v^T \neq x^S$, $v^T \in \mathbb{FV}(\psi)$, *and* $x^S \in \mathbb{FV}(t^T)$. *Choose the variable* y^S *to be the first variable of type* S *not occurring in either* ψ *or* t^T, *and define*

$$[t^T/v^T]\varphi \equiv (\exists y^S[t^T/v^T][y^S/x^S]\psi).$$

Intuitively, we change the x^S *in* φ *to the new variable* y^S *first, then substitute* t^T *for* v^T *as in subcase (ii).*

Example 2.2.10 *Let* $\mathcal{L} = \{e_1^N, e_2^Z, +^{N*N\to N}, \oplus^{Z*Z\to Z}, h^{N\to Z}\}$ *be a language over the type system* $\mathbb{T} = \langle N, Z\rangle$.

- $[+^{N*N\to N}(x^N, e_1^N)/x^N]\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e_2^Z) \equiv$ $\oplus^{Z*Z\to Z}(h^{N\to Z}(+^{N*N\to N}(x_1^N, e_1^N)), e_2^Z)$
- $[+^{N*N\to N}(x_1^N, e_1^N)/x_1^N]\forall x_1^N(+^{N*N\to N}(x_1^N, e_1^N) =^N x_1^N) \equiv$ $\forall x_1^N(+^{N*N\to N}(x_1^N, e_1^N) =^N x_1^N)$.
- $[\oplus^{Z*Z\to Z}(x_1^Z, e_2^Z)/x_2^Z]\exists x_1^Z(\oplus^{Z*Z\to Z}(x_1^Z, x_2^Z) =^Z e_2^Z) \equiv$ $\exists x_3^Z[\oplus^{Z*Z\to Z}(x_1^Z, e_2^Z)/x_3^Z][x_3^Z/x_1^Z](\oplus^{Z*Z\to Z}(x_1^Z, x_2^Z) =^Z e_2^Z) \equiv$ $\exists x_3^Z(\oplus^{Z*Z\to Z}(x_3^Z, \oplus^{Z*Z\to Z}(x_1^Z, e_2^Z)) =^Z e_2^Z)$.

Note. In the definitions above, the type of the term to substitue, t^T, and the type of the variable substituted for, v^T, must always be the same.

Lemma 2.2.11 *Let* v^T *be a variable,* t^T *and* s^S *be terms, and* φ *be a formula. Assume that* $v^T \notin \mathbb{FV}(s^S)$ *and* $v^T \notin \mathbb{FV}(\varphi)$. *Then* $[t^T/v^T]s^S \equiv s^S$ *and* $[t^T/v^T]\varphi \equiv \varphi$.

Proof: We can prove the statement about terms by a simple induction. For formulas, we need to prove something more general than what is stated.

(i) Let v^T be a variable, and let t^T and s^S be terms. Assume that $v^T \notin \mathbb{FV}(s^S)$. If $rank(s^S) = 0$, then either $s^S \in \mathbf{V}$ or $s^S \in \mathcal{C}$.

Case $s^S \in \mathbf{V}$: Then $s^S \equiv x^S$ for some $x^S \in \mathbf{V}$. Since $v^T \notin \mathbb{FV}(s^S)$, $x^S \not\equiv v^T$. So $[t^T/v^T]s^S \equiv s^S$.

Case $s^S \in \mathcal{C}$: By definition, $[t^T/v^T]s^S \equiv s^S$.

Assume that $rank(s^S) = k$, where $k > 0$, and that for all terms u^U with rank less than k, if $v^T \notin \mathbb{FV}(u^U)$, then $[t^T/v^T]u^U \equiv u^U$. There are two cases that we must consider.

Case $s^S \equiv f^{U\to S}(u^U)$ for some terms $f^{U\to S}$ and u^U of rank less than k. Since $v^T \notin \mathbb{FV}(s^S)$, $v^T \notin \mathbb{FV}(f^{U\to S})$ and $v^T \notin \mathbb{FV}(t_1^{T_1})$. By induction, $[t^T/v^T]f^{U\to S} \equiv f^{U\to S}$ and $[t^T/v^T]u^U \equiv u^U$. Hence $[t^T/v^T]f^{U\to S}(u^U) \equiv ([t^T/v^T]f^{U\to S})([t^T/v^T]u^U) \equiv f^{U\to S}(u^U) \equiv s^S$.

Case $s^S \equiv (t_1^{T_1},...,t_m^{T_m})$ for some terms $t_1^{T_1},...,t_m^{T_m}$ of rank less than k. Since $v^T \notin \mathbb{FV}(s^S)$, $v^T \notin \mathbb{FV}(t_i^{T_i})$ for all $i \in \{1,...,m\}$. By induction, $[t^T/v^T]t_i^{T_i} \equiv t_i^{T_i}$ for all $i \in \{1,...,m\}$. Thus $[t^T/v^T]s^S \equiv [t^T/v^T](t_1^{T_1},...,t_m^{T_m}) \equiv (t_1^{T_1},...,t_m^{T_m}) \equiv s^S$.

(ii) The statement that we need to prove is, "For all terms t^T, all formulas φ and all variables v^T, if $v^T \notin \mathbb{FV}(\varphi)$, then $[t^T/v^T]\varphi \equiv \varphi$." We will prove this by induction on the rank of the formula φ.

Let t^T be a term, φ a formula, and v^T a variable such that $v^T \notin \mathbb{FV}(\varphi)$. If $rank(\varphi) = 0$, then there are two cases that we must consider.

Case $\varphi \equiv (t_1^S =^S t_2^S)$ for some terms t_1^S, t_2^S. Since $v^T \notin \mathbb{FV}(\varphi)$, $v^T \notin \mathbb{FV}(t_1^S)$ and $v^T \notin \mathbb{FV}(t_2^S)$. By part (i), $[t^T/v^T]\varphi \equiv ([t^T/v^T]t_1^S =^T [t^T/v^T]t_2^S) \equiv (t_1^S =^T t_2^S) \equiv \varphi$.

Case $\varphi \equiv r^{T_1*\cdots*T_m}(t_1^{T_1},...,t_m^{T_m})$ for some $r^{T_1*\cdots*T_m} \in \mathcal{R}$ and $t_1^{T_1},...,t_m^{T_m} \in \mathbb{T}$. Since $v^T \notin \mathbb{FV}(\varphi)$, for all $i \in \{1,...,m\}$, $v^T \notin \mathbb{FV}(t_i^{T_i})$. By part(i), $[t^T/v^T]\varphi \equiv r^{T_1*\cdots*T_m}([t^T/v^T]t_1^{T_1},...,[t^T/v^T]t_m^{T_m}) \equiv (t_1^{T_1},...,t_m^{T_m}) \equiv \varphi$.

Assume that $rank(\varphi) = k$, where $k > 0$, and for all formulas ψ of rank less than k and all variables y^T, if $y^T \notin \mathbb{FV}(\psi)$, then $[t^T/y^T]\psi \equiv \psi$. There are three case that we must consider.

Case $\varphi \equiv (\neg\psi)$ for some formula ψ of rank less than k. Since $v^T \notin \mathbb{FV}(\varphi)$, $v^T \notin \mathbb{FV}(\psi)$. By induction, $[t^T/v^T]\varphi \equiv (\neg[t^T/v^T]\psi) \equiv (\neg\psi) \equiv \varphi$.

Case $\varphi \equiv (\psi \wedge \chi)$ for some formulas ψ, χ of rank less than k. Since $v^T \notin \mathbb{FV}(\varphi)$, $v^T \notin \mathbb{FV}(\psi)$ and $v^T \notin \mathbb{FV}(\chi)$. By induction, $[t^T/v^T]\varphi \equiv ([t^T/v^T]\psi \wedge [t^T/v^T]\chi) \equiv (\psi \wedge \chi) \equiv \varphi$.

Case $\varphi \equiv (\exists x^S\psi)$ for some $x^S \in \mathbb{V}$ and $\psi \in \mathbb{F}$. Since $v^T \notin \mathbb{FV}(\varphi)$, $v^T \equiv x^S$ or $v^T \not\equiv x^S$ and $v^T \notin \mathbb{FV}(\psi)$.

Subcase $v^T \equiv x^S$: By definition $[t^T/v^T](\exists x^S\psi) \equiv (\exists x^S\psi) \equiv \varphi$.

Subcase $v^T \not\equiv x^S$ and $v^T \notin \mathbb{FV}(\psi)$: By induction, $[t^T/v^T]\psi \equiv \psi$. Thus $[t^T/v^T](\exists x^S\psi) \equiv (\exists x^S[t^T/v^T]\psi) \equiv (\exists x^S\psi) \equiv \varphi$.

Lemma 2.2.12 *Let t^T be term, x^S, v^S be variables, and φ be a formula. Then*

(i) $rank([x^S/v^S]t^T) = rank(t^T)$, and

(ii) $rank([x^S/v^S]\varphi) = rank(\varphi)$

Proof:

(i) Let $k = rank(t^T)$. If $k = 0$, then either $t^T \in \mathcal{C}$ or $t^T \in \mathbf{V}$. If $t^T \in \mathcal{C}$, then $[x^S/v^S]t^T \equiv t^T$, so $rank([x^S/v^S]t^T) = rank(t^T)$. If $t^T \in \mathbf{V}$, then either $t^T \equiv v^S$ or $t^T \not\equiv v^S$. In both cases $[x^S/v^S]t^T \in \mathbf{V}$. So $rank([x^S/v^S]t^T) = rank(t^T)$.

Assume $k > 0$, and that for all terms u^U with $rank(u^U) < k$, $rank([x^S/v^S]u^U) = rank(u^U)$.

Case $t^T \equiv f^{U\to T}(u^U)$ for some terms $f^{U\to T}$ and u^U of rank less than k. We claim that either $rank(f^{U\to T}) = k-1$ or $rank(u^U) = k-1$. Assume for a contradiction that $rank(f^{U\to T}) < k-1$ and $ramk(u^U) < k-1$. Then by definition, $f^{U\to T}(u^U) \in \mathbb{T}_{k-1}$. This contradicts the fact that $rank(f^{U\to T}(u^U) = k$. Without loss of generality, $rank(f^{U\to T}) = k-1$. By induction, $rank([x^S/v^S]f^{U\to T}) = rank(f^{U\to T}) = k-1$ and $rank([x^S/v^S]u^U) = rank(u^U) \leq k-1$. Then $[x^S/v^S]f^{U\to T} \in \mathbb{T}_{k-1}$. This implies that $[x^S/v^S]f^{U\to T}(u^U) \in \mathbb{T}_k$. Clearly $rank([x^S/v^S]f^{U\to T}(u^U))$ $= k$.

Case $t^T \equiv (t_1^{T_1}, ..., t_m^{T_m})$ for some $t_1^{T_1}, ..., t_m^{T_m}$ terms of rank less than k. This is similar to the previous case.

(ii) This is similar to (i).

Lemma 2.2.13 *Let v^T be a variable, t^T and s^S be terms, and φ be a formula. Assume that $v^T \in \mathbb{FV}(s^S)$ and $v^T \in \mathbb{FV}(\varphi)$. Then*

(i) $\mathbb{FV}([t^T/v^T]s^S) = (\mathbb{FV}(s^S) \setminus \{v^T\}) \cup \mathbb{FV}(t^T)$ *and*

(ii) $\mathbb{FV}([t^T/v^T]\varphi) = (\mathbb{FV}(\varphi) \setminus \{v^T\}) \cup \mathbb{FV}(t^T)$.

Proof: As in the previous lemma, for formulas we need to prove something more general than what is stated.

(i) Let v^T be a variable, t^T and s^S be terms. Assume that $v^T \in \mathbb{FV}(s^S)$. If $rank(s^S) = 0$, then either $s^S \in \mathbf{V}$ or $s^S \in \mathcal{C}$. Since we assume that $v^T \in \mathbb{FV}(s^S)$, $s^S \equiv v^T$. Hence $\mathbb{FV}(s^S) = \{v^T\}$ and $[t^T/v^T]s^S \equiv t^T$. Then $\mathbb{FV}([t^T/v^T]s^S) = \mathbb{FV}(t^T) = \mathbb{FV}(s^S) \setminus \{v^T\} \cup \mathbb{FV}(t^T)$.
Assume that $rank(s^S) = k$, where $k > 0$, and that for all terms u^U of rank less than k, $\mathbb{FV}([t^T/v^T]u^U) = (\mathbb{FV}(u^U) \setminus \{v^T\}) \cup \mathbb{FV}(t^T)$.

Case(i) $s^S \equiv f^{U\to S}(u^U)$: Since $v^T \in \mathbb{FV}(s^S)$, $v^T \in \mathbb{FV}(f^{U\to S})$ or $v^T \in \mathbb{FV}(u^U)$.

Subcase $v^T \in \mathbb{FV}(f^{U\to S})$ and $v^T \in \mathbb{FV}(u^U)$: By induction,

$$\begin{aligned}
\mathbb{FV}([t^T/v^T]s^S) &= \mathbb{FV}([t^T/v^T]f^{U\to S}) \cup \mathbb{FV}([t^T/v^T]u^U) \\
&= (\mathbb{FV}(f^{U\to S}) \setminus \{v^T\} \cup \mathbb{FV}(t^T)) \\
&\quad \cup(\mathbb{FV}(u^U) \setminus \{v^T\} \cup \mathbb{FV}(t^T)) \\
&= (\mathbb{FV}(f^{U\to S}) \cup \mathbb{FV}(u^U)) \setminus \{v^T\} \\
&\quad \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(f^{U\to S}(u^U)) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(s^S) \setminus \{v^T\} \cup \mathbb{FV}(t^T).
\end{aligned}$$

Subcase $v^T \in \mathbb{FV}(f^{U\to S})$ and $v^T \notin \mathbb{FV}(u^U)$: By the previous lemma, $[t^T/v^T]u^U \equiv u^U$. Thus $\mathbb{FV}([t^T/v^T]u^U) = \mathbb{FV}(u^U) = \mathbb{FV}(u^U) \setminus \{v^T\}$. So by induction,

$$\begin{aligned}
\mathbb{FV}([t^T/v^T]s^S) &= \mathbb{FV}([t^T/v^T]f^{U\to S}) \cup \mathbb{FV}([t^T/v^T]u^U) \\
&= (\mathbb{FV}(f^{U\to S}) \setminus \{v^T\} \cup \mathbb{FV}(t^T)) \cup (\mathbb{FV}(u^U) \\
&= (\mathbb{FV}(f^{U\to S}) \cup \mathbb{FV}(u^U)) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(f^{U\to S}(u^U)) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(s^S) \setminus \{v^T\} \cup \mathbb{FV}(t^T).
\end{aligned}$$

Subcase $v^T \notin \mathbb{FV}(f^{U\to S})$ and $v^T \in \mathbb{FV}(u^U)$: Similar to the previous subcase.

Case(ii) $s^S \equiv (t_1^{T_1}, ..., t_m^{T_m})$ for some terms $t_1^{T_1}, ..., t_m^{T_m}$. Since $v^T \in \mathbb{FV}(s^S)$, $v^T \in \bigcup_{i=1}^{m} \mathbb{FV}(t_i^{T_i})$. Let $I = \{i \in \{1, ..., m\} \mid v^T \in \mathbb{FV}(t_i^{T_i})\}$. Then $I \neq \emptyset$. By induction, for all $i \in I$, $\mathbb{FV}([t^T/v^T]t_i^{T_i}) = \mathbb{FV}(t_i^{T_i}) \setminus \{v^T\} \cup \mathbb{FV}(t^T)$. By the previous lemma, for all $i \notin I$, $[t^T/v^T]t_i^{T_i} = t_i^{T_i}$, so $\mathbb{FV}([t^T/v^T]t_i^{T_i}) = \mathbb{FV}(t_i^{T_i}) = \mathbb{FV}(t_i^{T_i}) \setminus \{v^T\}$. Hence

$$\begin{aligned}
\mathbb{FV}([t^T/v^T]s^S) &= \bigcup_{i=1}^{m} \mathbb{FV}([t^T/v^T]t_i^{T_i}) \\
&= \bigcup_{i\in I} \mathbb{FV}([t^T/v^T]t_i^{T_i}) \cup \bigcup_{i\notin I} \mathbb{FV}([t^T/v^T]t_i^{T_i}) \\
&= \bigcup_{i\in I} (\mathbb{FV}(t_i^{T_i}) \setminus \{v^T\} \cup \mathbb{FV}(t^T)) \\
&\quad \cup \bigcup_{i\notin I} \mathbb{FV}(t_i^{T_i}) \setminus \{v^T\} \\
&= \bigcup_{i\in I} \mathbb{FV}(t_i^{T_i}) \setminus \{v^T\} \cup \bigcup_{i\notin I} \mathbb{FV}(t_i^{T_i}) \setminus \{v^T\} \\
&\quad \cup \mathbb{FV}(t^T))
\end{aligned}$$

$$= \bigcup_{i=1}^{m} \mathbb{FV}(t_i^{T_i}) \setminus \{v^T\} \cup \mathbb{FV}(t^T)$$
$$= \mathbb{FV}(s^S) \setminus \{v^T\} \cup \mathbb{FV}(t^T).$$

(ii) The statement that we need to prove is, "For all terms t^T, and all formulas φ, and all variables v^T, if $v^T \in \mathbb{FV}(\varphi)$, then $\mathbb{FV}([t^T/v^T]\varphi) = (\mathbb{FV}(\varphi)\setminus\{v^T\})\cup\mathbb{FV}(t^T)$." We will prove this by induction on the formula φ.

Let t^T be a term, φ a formula, and v^T a variable such that $v^T \in \mathbb{FV}(\varphi)$. If $rank(\varphi) = 0$, then there are two cases that we must consider.

Case(i) $\varphi \equiv (t_1^S =^S t_2^S)$ for some $t_1^S, t_2^S \in \mathbb{T}$. Since $v^T \in \mathbb{FV}(\varphi)$, $v^T \in \mathbb{FV}(t_1^S)$ or $v^T \in \mathbb{FV}(t_2^S)$. This case is similar to the case $s^S \equiv f^{U\to S}(u^U)$ for terms.

Case(ii) $\varphi \equiv r^{T_1*\cdots*T_m}(t_1^{T_1}, ..., t_m^{T_m})$ for some $r^{T_1*\cdots*T_m} \in \mathcal{R}$ and $t_1^{T_1}, ..., t_m^{T_m} \in \mathbb{T}$. Since $v^T \in \varphi$, $v^T \in \bigcup_{i=1}^{m} \mathbb{FV}(t_i^{T_i})$. This case is similar to the case $s^S \equiv (t_1^{T_1}, ..., t_m^{T_m})$ for terms.

Assume that $rank(\varphi) = k$, where $k > 0$, and that for all formulas ψ of rank less than k and all variables x^T, if $x^T \in \mathbb{FV}(\psi)$, then $\mathbb{FV}([t^T/x^T]\psi) = (\mathbb{FV}(\psi) \setminus \{x^T\}) \cup \mathbb{FV}(t^T)$.

Case $\varphi \equiv (\neg\psi)$ for some formula ψ of rank less than k. Since $\mathbb{FV}(\varphi) = \mathbb{FV}(\psi)$, $v^T \in \mathbb{FV}(\psi)$. By induction, $\mathbb{FV}([t^T/v^T]\psi) = \mathbb{FV}(\psi)\setminus\{v^T\}\cup \mathbb{FV}(t^T)$. Hence

$$\begin{aligned}
\mathbb{FV}([t^T/v^T]\varphi) &= \mathbb{FV}([t^T/v^T](\neg\psi)) \\
&= \mathbb{FV}(\neg[t^T/v^T]\psi) \\
&= \mathbb{FV}([t^T/v^T]\psi) \\
&= \mathbb{FV}(\psi) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(\neg\psi) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(\varphi) \setminus \{v^T\} \cup \mathbb{FV}(t^T).
\end{aligned}$$

Case $\varphi \equiv (\psi \wedge \chi)$ for some formulas ψ and χ of rank less than k. Since $v^T \in \mathbb{FV}(\varphi)$, $v^T \in \mathbb{FV}(\psi)$ or $v^T \in \mathbb{FV}(\chi)$. This case is similar to the case $f^{U\to S}(u^U)$ for terms.

Case $\varphi \equiv (\exists x^S\psi)$ for some $x^S \in \mathbb{V}$ and $\psi \in \mathbb{F}$. Since $v^T \in \mathbb{FV}(\varphi)$, $v^T \not\equiv x^S$ and $v^T \in \mathbb{FV}(\psi)$. There are two cases that we must consider.

Subcase $x^S \notin \mathbb{FV}(t^T)$: By definition, $[t^T/v^T]\varphi \equiv (\exists x^S[t^T/v^T]\psi)$. By induction,

$$\mathbb{FV}([t^T/v^T]\varphi) = \mathbb{FV}(\exists x^S[t^T/v^T]\psi)$$

$$
\begin{aligned}
&= \mathbb{FV}([t^T/v^T]\psi) \setminus \{x^S\} \\
&= (\mathbb{FV}(\psi) \setminus \{v^T\} \cup \mathbb{FV}(t^T)) \setminus \{x^S\} \\
&= (\mathbb{FV}(\psi) \setminus \{x^S\}) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(\exists x^S \psi) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(\varphi) \setminus \{v^T\} \cup \mathbb{FV}(t^T)
\end{aligned}
$$

Subcase $x^S \in \mathbb{FV}(t^T)$: Let y^S be the first variable of type S not occurring in either ψ or t^T. By definition, $[t^T/v^T]\varphi \equiv \exists y^S[t^T/v^T][y^S/x^S]\psi)$. Since $v^T \not\equiv x^S$, $v^T \in \mathbb{FV}([y^S/x^S]\psi)$. Hence

$$
\begin{aligned}
\mathbb{FV}([t^T/v^T]\varphi) &= \mathbb{FV}(\exists y^S[t^T/v^T][y^S/x^S]\psi) \\
&= \mathbb{FV}([t^T/v^T][y^S/x^S]\psi) \setminus \{y^S\} \\
&= (\mathbb{FV}([y^S/x^S]\psi) \setminus \{v^T\} \cup \mathbb{FV}(t^T)) \setminus \{y^S\} \\
&= ((\mathbb{FV}([y^S/x^S]\psi) \setminus \{y^S\}) \setminus \{v^T\}) \\
&\quad \cup (\mathbb{FV}(t^T) \setminus \{y^S\}) \\
&= ((\mathbb{FV}(\psi) \setminus \{x^S\} \cup \{y^S\}) \setminus \{y^S\} \setminus \{v^T\} \\
&\quad \cup (\mathbb{FV}(t^T) \setminus \{y^S\}) \\
&= (\mathbb{FV}(\psi) \setminus \{x^S\}) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(\exists x^S \psi) \setminus \{v^T\} \cup \mathbb{FV}(t^T) \\
&= \mathbb{FV}(\varphi) \setminus \{v^T\} \cup \mathbb{FV}(t^T)
\end{aligned}
$$

In all cases, we have $\mathbb{FV}([t^T/v^T]\varphi) = \mathbb{FV}(\varphi) \setminus \{v^T\} \cup \mathbb{FV}(t^T)$.

2.3 The semantics of predicate logic with types

The formulas we have defined are merely strings of symbols; they have no meaning yet. It is the job of the semantics to give meaning to the formulas. This will be done in two steps: The first step is to choose meanings for all of the symbols in the language by choosing a structure for the language. The second step is to use the structure we have chosen to detemine the meanings of all of the terms and formulas that can be constructed from the language.

To save ourselves some writing in this section, let us fix a language $\mathcal{L} = \mathcal{C} \cup \mathcal{R}$ over a type system $\langle P_1, ..., P_n\rangle$ to use in all of our discussions.

2.3.1 The structures

It should not be surprising that the structures for this logic are more complicated than the traditional ones. The purpose of a structure is to give meaning to all of the symbols in $\mathcal{L}$. In order to do that in a consistent way, we must give meaning to all of the types in the type system first. And since types are interpreted as sets, our very first step must be to choose a set from which to construct the interpretations of the types.

Definition 2.3.1 *A* **structure** *for the language* $\mathcal{L}$ *over* $\mathbb{T}$ *is a fourtuple of the form*

$$\mathfrak{A} = \langle A, \mathcal{P}^{\mathfrak{A}}, \mathcal{R}^{\mathfrak{A}}, \mathcal{C}^{\mathfrak{A}} \rangle$$

where

A *is a non-empty set,*
$\mathcal{P}^{\mathfrak{A}}$ *is a* **type assignment,**
$\mathcal{R}^{\mathfrak{A}}$ *is an* **interpretation of** $\mathcal{R}$*, and*
$\mathcal{C}^{\mathfrak{A}}$ *is an* **interpretation of** $\mathcal{C}$.

Type assignments and the interpretations of relation and constant symbols are defined as follows.

Definition 2.3.2 *A* **type assignment** $\mathcal{P}^{\mathfrak{A}}$ *is a mapping from* $\{P_1, ..., P_n\}$ *to* $\mathfrak{P}(A)$. *i.e.*

$$\mathcal{P}^{\mathfrak{A}} : \{P_1, ..., P_n\} \rightarrow \mathfrak{P}(A).$$

We denote $\mathcal{P}^{\mathfrak{A}}(P_\imath)$ *by* $P_\imath^{\mathfrak{A}}$ *for all* $\imath \in \{1, ..., n\}$.

Before we can define interpretations of relation and constant symbols, we need to define the universe of a structure and the interpretation of a type system. We use "the universe constructed from A" which is defined in Chapter 1 (Definition 1.2.1) to define the structure universe.

Definition 2.3.3 The universe of the structure $\mathfrak{A}$, $\mathcal{U}_{\mathfrak{A}}$, *is defined to be the universe of the set* A, $\mathcal{U}_A$. *i.e,*

$$\mathcal{U}_{\mathfrak{A}} = \mathcal{U}_A$$

Now our types and type systems can be interpreted as follows.

Definition 2.3.4 *An* **interpretation of a type system** *under a structure* $\mathfrak{A}$ *, denoted by* $\mathbb{T}^{\mathfrak{A}}$*, is a mapping from* $\mathbb{T}$ *to* $\mathcal{U}_{\mathfrak{A}}$. *It is defined by induction as follows. Let* $T \in \mathbb{T}$*, so* $T \in \mathbb{T}_k$ *for some* $k \in \mathbb{N}$. *If* $k = 0$*, then* $T \equiv P_\imath$ *for some* $\imath \in \{1, ..., n\}$*, and we define*

$$\mathbb{T}^{\mathfrak{A}}(T) = \mathcal{P}^{\mathfrak{A}}(P_\imath).$$

Assume that $k > 1$*, and that for all* $\ell < k$ *and all* $S \in \mathbb{T}(\ell)$*,* $\mathbb{T}^{\mathfrak{A}}(S)$ *is defined. Since* $k > 1$*,* $T \equiv (T_1 \rightarrow T_2)$ *for some types* T_1 *and* $T_2 \in \mathbb{T}(\ell)$ *where is* $\ell < k$*, or* $T \equiv (T_1 * \cdots * T_m)$ *for some types* $T_1, ..., T_m \in \mathbb{T}(\ell)$ *where is* $\ell < k$.

Case $T \equiv (T_1 \to T_2)$: *Define*

$$\mathbb{T}^{\mathfrak{A}}(T) = \{f : \mathbb{T}^{\mathfrak{A}}(T_1) \to \mathbb{T}^{\mathfrak{A}}(T_2)\}.$$

Case $T \equiv (T_1 * \cdots * T_m)$: *Define*

$$\mathbb{T}^{\mathfrak{A}}(T) = \mathbb{T}^{\mathfrak{A}}(T_1) \times \cdots \times \mathbb{T}^{\mathfrak{A}}(T_m).$$

In order to simplify the writing, we will introduce the following abbreviations. For $T, T_1, T_2 \in \mathbb{T}$,

- $T^{\mathfrak{A}}$ will abbreviate $\mathbb{T}^{\mathfrak{A}}(T)$,
- $T_1^{\mathfrak{A}} \to T_2^{\mathfrak{A}}$ will abbreviate $\{f : \mathbb{T}^{\mathfrak{A}}(T_1) \to \mathbb{T}^{\mathfrak{A}}(T_2)\}$.
(Thus $(T_1 \to T_2)^{\mathfrak{A}} = T_1^{\mathfrak{A}} \to T_2^{\mathfrak{A}}$.)

Note that it follows from corollary 1.2.3 and proposition 1.2.4 that $\mathbb{T}^{\mathfrak{A}}(T)$ for all types T.

Now, we are ready to interpret relation and constant symbols.

Definition 2.3.5 *An* **interpretation of the relation symbols under a structure** $\mathfrak{A}$, $\mathcal{R}^{\mathfrak{A}}$, *is a mapping from* $\mathcal{R}$ *to* $\mathcal{U}_{\mathfrak{A}}$ *satisfying the condition that for all* $r^T \in \mathcal{R}$,

$$\mathcal{R}^{\mathfrak{A}}(r^T) \subseteq T^{\mathfrak{A}}.$$

Definition 2.3.6 *An* **interpretation of the constant symbols under a structure** $\mathfrak{A}$, $\mathcal{C}^{\mathfrak{A}}$, *is a mapping from* $\mathcal{C}$ *to* $\mathcal{U}_{\mathfrak{A}}$ *satisfying the condition that for all* $c^T \in \mathcal{C}$,

$$\mathcal{C}^{\mathfrak{A}}(c^T) \in T^{\mathfrak{A}}.$$

2.3.2 Interpretations of terms and formulas

Once we have chosen a structure $\mathfrak{A}$, we can use it to help determine the meanings of all of the terms and formulas. A moment's thought shows that a structure is not enough, because it does not determine the meaning of one of the simplest terms we can have: a single variable. So, in addition to a structure, we must have a variable assignment in order to determine the meanings of terms and formulas. A **variable assignment** is just a function $\alpha : \mathbf{V} \to \mathcal{U}_{\mathfrak{A}}$ such that for every $x^T \in \mathbf{V}$, $\alpha(x^T) \in T^{\mathfrak{A}}$. Given a structure $\mathfrak{A}$ and a variable assignment α, we can determine the meanings of terms and formulas, which are called their **interpretations**.

Definition 2.3.7 *The* **interpretation of a term** t^T *in a structure* $\mathfrak{A}$ *under a variable assignment* α, *denoted by* $I^{\mathfrak{A}}(t^T, \alpha)$, *is defined by induction on* t^T *as follows. If* $rank(t^T) = 0$, *then either* $t^T \in \mathbf{V}$ *or* $t^T \in \mathcal{C}$.

Case $t^T \in \mathbf{V}$: *Define* $I^{\mathfrak{A}}(t^T, \alpha) = \alpha(t^T)$.

Case $t^T \in \mathcal{C}$: *Define* $I^{\mathfrak{A}}(t^T, \alpha) = \mathcal{C}^{\mathfrak{A}}(t^T)$.

Assume that $k > 0$, *and that for all* $\ell < k$ *and all* $s^S \in \mathbb{T}(\ell)$, $I^{\mathfrak{A}}(s^S, \alpha)$ *is defined. Since* $k > 0$, *there are two cases that we must consider.*

Case $t^T \equiv f^{S \to T}(s^S)$ *where* $f^{S \to T}, s^S \in \mathbb{T}_{k-1}$. *Define*

$$I^{\mathfrak{A}}(t^T, \alpha) = I^{\mathfrak{A}}(f^{S \to T}, \alpha)(I^{\mathfrak{A}}(s^S, \alpha)).$$

Case $t^T \equiv (t_1^{T_1}, ..., t_m^{T_m})$ *where* $t_1^{T_1}, \ldots, t_m^{T_m} \in \mathbb{T}_{k-1}$. *Define*

$$I^{\mathfrak{A}}(t^T, \alpha) = (I^{\mathfrak{A}}(t_1^{T_1}, \alpha), \ldots, I^{\mathfrak{A}}(t_m^{T_m}, \alpha)).$$

Note that for any term t^T, $I^{\mathfrak{A}}(t^T, \alpha)$ *is an element of* $T^{\mathfrak{A}}$.

Definition 2.3.8 *Let* φ *be a formula.* **The interpretation of a formula** φ *in a structure* $\mathfrak{A}$ *under an assignment* α, *denoted by* $I^{\mathfrak{A}}(\varphi, \alpha)$, *is defined as follows. If* $rank(\varphi) = 0$, *then there are two cases that we must consider.*

Case $\varphi \equiv (t_1^T =^T t_2^T)$ *where* $t_1^T, t_2^T \in \mathbb{T}$. *Define*

$$I^{\mathfrak{A}}(\varphi, \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{A}}(t_1^T, \alpha) \text{ is the same as } I^{\mathfrak{A}}(t_2^T, \alpha) \\ F & \text{otherwise} \end{cases}$$

Case $\varphi \equiv r^{T_1 * \cdots * T_m}(t_1^{T_1}, ..., t_m^{T_m})$, *where* $r^{T_1 * \cdots * T_m} \in \mathcal{R}$ *and* $t_1^{T_1}, ..., t_m^{T_m} \in \mathbb{T}$. *Define*

$$I^{\mathfrak{A}}(\varphi, \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{A}}((t_1^{T_1}, ..., t_m^{T_m}), \alpha) \in \mathcal{R}^{\mathfrak{A}}(r^{T_1 * \cdots * T_m}) \\ F & \text{otherwise.} \end{cases}$$

Assume that $rank(\varphi) > 0$, *and that for all variable assignments* β *and all formulas* ψ *of rank less than* k, $I^{\mathfrak{A}}(\psi, \beta)$ *is defined. Since* $k > 0$, *there are three cases that we must consider.*

Case $\varphi \equiv (\neg\psi)$ *where* ψ *is a formula of rank less than* k. *Then*

$$I^{\mathfrak{A}}(\varphi, \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{A}}(\psi, \alpha) = F \\ F & \text{if } I^{\mathfrak{A}}(\psi, \alpha) = T \end{cases}$$

Case $\varphi \equiv (\psi \wedge \chi)$ *where* ψ *and* χ *are formulas of rank less than* k. *Then*

$$I^{\mathfrak{A}}(\varphi, \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{A}}(\psi, \alpha) = T \text{ and } I^{\mathfrak{A}}(\chi, \alpha) = T \\ F & \text{otherwise} \end{cases}$$

Case $\varphi \equiv (\exists x^T \psi)$, *where* $x^T \in \mathbf{V}$ *and* ψ *is a formula of rank less than* k. *Then*

$$I^{\mathfrak{A}}(\varphi, \alpha) = \begin{cases} T & \text{if there exists a variable assignment } \beta \\ & \text{with } \beta(v^S) = \alpha(v^S) \text{ for all variables } v^S \neq x^T \\ & \text{such that } I^{\mathfrak{A}}(\psi, \beta) = T \\ F & \text{otherwise} \end{cases}$$

The following lemma tells us how to interpret the abbreviations we introduced above.

Lemma 2.3.9 *Let* φ, ψ, *and* $\varphi_1, \varphi_2, ..., \varphi_n$ *be formulas, let* $\mathfrak{A}$ *be a structure, and let* α *be a variable assignment.*

(i) The interpretation of $(\varphi \vee \psi)$ *is given by*

$$I^{\mathfrak{A}}((\varphi \vee \psi), \alpha) = \begin{cases} F & \text{if } I^{\mathfrak{A}}(\varphi, \alpha) = F \text{ and } I^{\mathfrak{A}}(\psi, \alpha) = F \\ T & \text{otherwise.} \end{cases}$$

(ii) The interpretation of $(\varphi \Rightarrow \psi)$ *is given by*

$$I^{\mathfrak{A}}((\varphi \Rightarrow \psi), \alpha) = \begin{cases} F & \text{if } I^{\mathfrak{A}}(\varphi, \alpha) = T \text{ and } I^{\mathfrak{A}}(\psi, \alpha) = F \\ T & \text{otherwise.} \end{cases}$$

(iii) The interpretation of $(\varphi \Leftrightarrow \psi)$ *is given by*

$$I^{\mathfrak{A}}((\varphi \Leftrightarrow \psi), \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{A}}(\varphi, \alpha) = I^{\mathfrak{A}}(\psi, \alpha) \\ F & \text{if } I^{\mathfrak{A}}(\varphi, \alpha) \neq I^{\mathfrak{A}}(\psi, \alpha) \end{cases}$$

(iv) The interpretation of $(\forall x^T \varphi)$ *is given by*

$$I^{\mathfrak{A}}((\forall x^T \varphi), \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{A}}(\varphi, \beta) = T \text{ for all variable} \\ & \text{assignments } \beta \text{ with } \beta(v^S) = \alpha(v^S) \\ & \text{for all variables } v^S \neq x^T \\ F & \text{otherwise} \end{cases}$$

(v) The interpretation of $(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$ *is given by*

$$I^{\mathfrak{A}}((\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n), \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{A}}(\varphi_\imath, \alpha) = T \text{ for all} \\ & \imath \in \{1, ..., n\} \\ F & \text{otherwise.} \end{cases}$$

(vi) The interpretation of $(\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$ *is given by*

$$I^{\mathfrak{A}}((\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n), \alpha) = \begin{cases} T & \text{if } I^{\mathfrak{A}}(\varphi_\imath, \alpha) = T \text{ for at} \\ & \text{least one } \imath \in \{1, ..., n\} \\ F & \text{otherwise.} \end{cases}$$

Proof: Let φ, ψ, and $\varphi_1, \varphi_2, ..., \varphi_n$ be formulas, and let α be a variable assignment. Parts (i) and (ii) follow directly from the definitions.

(iii) Since $(\varphi \Leftrightarrow \psi)$ abbreviates $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$, we can argue as follows.

Case $I^{\mathfrak{A}}(\varphi, \alpha) = I^{\mathfrak{A}}(\psi, \alpha)$: There are two subcases that we must consider.

Subcase $I^{\mathfrak{A}}(\varphi, \alpha) = T$: So $I^{\mathfrak{A}}(\psi, \alpha) = T$. By definition, $I^{\mathfrak{A}}(\varphi \Rightarrow \psi, \alpha)$ and $I^{\mathfrak{A}}(\psi \Rightarrow \varphi, \alpha)$ are true. Hence $I^{\mathfrak{A}}(\varphi \Leftrightarrow \psi, \alpha) = T$.

Subcase $I^{\mathfrak{A}}(\varphi, \alpha) = F$: So $I^{\mathfrak{A}}(\psi, \alpha) = F$. By definition, $I^{\mathfrak{A}}(\varphi \Rightarrow \psi, \alpha)$ and $I^{\mathfrak{A}}(\psi \Rightarrow \varphi, \alpha)$ are true. Hence $I^{\mathfrak{A}}(\varphi \Leftrightarrow \psi, \alpha) = T$.

Case $I^{\mathfrak{A}}(\varphi, \alpha) \neq I^{\mathfrak{A}}(\psi, \alpha)$: Again, there are two subcases that we must consider.

Subcase $I^{\mathfrak{A}}(\varphi, \alpha) = T$: So $I^{\mathfrak{A}}(\psi, \alpha) = F$. Then $I^{\mathfrak{A}}(\varphi \Rightarrow \psi, \alpha) = F$. Hence $I^{\mathfrak{A}}(\varphi \Leftrightarrow \psi, \alpha) = F$.

Subcase $I^{\mathfrak{A}}(\varphi, \alpha) = F$: So $I^{\mathfrak{A}}(\psi, \alpha) = T$. Then $I^{\mathfrak{A}}(\psi \Rightarrow \varphi, \alpha) = F$. Hence $I^{\mathfrak{A}}(\varphi \Leftrightarrow \psi, \alpha) = F$.

(iv) Since $(\forall x^T \varphi)$ abbreviates $(\neg\exists x^T(\neg\varphi))$, we can give the following proof.

Suppose $I^{\mathfrak{A}}(\forall x^T \varphi, \alpha) = T$, so that $I^{\mathfrak{A}}(\exists x^T neg \varphi), \alpha) = F$. Let β be any variable assignment such that $\beta(v^S) = \alpha(v^S)$ for all variables $v^S \neq x^T$. Then $I^{\mathfrak{A}}(\neg\varphi, \beta) = F$. So $I^{\mathfrak{A}}(\varphi, \beta) = T$. Hence for all a variable assignment β with $\beta(v^S) = \alpha(v^S)$ for all variables $v^S \neq x^T$, $I^{\mathfrak{A}}(\varphi, \beta) = T$.

Conversely, assume that $I^{\mathfrak{A}}(\varphi, \beta) = T$ for all variable assignments β with $\beta(v^S) = \alpha(v^S)$ for all variables $v^S \neq x^T$. Then $I^{\mathfrak{A}}(\neg\varphi, \beta) = F$ for any variable assignment β s.t. $\beta(v^S) = \alpha(v^S)$ for all variables $v^S \neq x^T$. Hence $I^{\mathfrak{A}}(\exists x^T(\neg\varphi), \alpha) = F$. Thus $I^{\mathfrak{A}}(\neg\exists x^T(\neg\varphi), \alpha) = T$.

Parts (v) and (vi) can both be proved by induction on n.

The next proposition tells us that the interpretations of terms and formulas only depend on the values assigned to their free variables.

Proposition 2.3.10 *Let $\mathfrak{A}$ be a structure and let α and β be variable assignments.*

(i) Let t^S be a term and suppose that $\alpha(v^T) = \beta(v^T)$ for all $v^T \in \mathbb{FV}(t^S)$. Then $I^{\mathfrak{A}}(t^S, \alpha) = I^{\mathfrak{A}}(t^S, \beta)$.

(ii) Let φ be a formula and suppose that $\alpha(v^T) = \beta(v^T)$ for all $v^T \in \mathbb{FV}(\varphi)$. Then $I^{\mathfrak{A}}(\varphi, \alpha) = I^{\mathfrak{A}}(\varphi, \beta)$.

Proof: Let $\mathfrak{A}$ be a structure.

(i) Let t^S be a term and let α and β be variable assignments. We will prove this by induction on t^S. Let $k = rank(t^S)$. If $k = 0$, then either $t^S \in \mathbb{V}$ or $t^S \in \mathcal{C}$.

Case $t^S \in \mathbb{V}$: Then $\mathbb{FV}(t^S) = \{t^S\}$. Hence $\alpha(t^S) = \beta(t^S)$, so $I^{\mathfrak{A}}(t^S, \alpha) = \alpha(t^S) = \beta(t^S) = I^{\mathfrak{A}}(t^S, \beta)$.

Case $t^S \in \mathcal{C}$: We have $I^{\mathfrak{A}}(t^S, \alpha) = \mathcal{C}^{\mathfrak{A}}(t^S) = I^{\mathfrak{A}}(t^S, \beta)$.

Assume that $k > 0$ and that for all $\ell < k$ and all $u^U \in \mathbb{T}_\ell$, if $\alpha(v^T) = \beta(v^T)$ for all $v^T \in \mathbb{FV}(u^U)$, then $I^{\mathfrak{A}}(u^U, \alpha) = I^{\mathfrak{A}}(u^U, \beta)$. Since $k > 0$, there are two cases that we must consider.

Case $t^S \equiv f^{U \to S}(u^U)$ where $f^{U \to S}$ and u^U are terms of rank less than k. Since $\mathbb{FV}(f^{U \to S}) \subseteq \mathbb{FV}(t^S)$ and $\mathbb{FV}(u^U) \subseteq \mathbb{FV}(t^S)$, then for all $v^T \in \mathbb{FV}(f^{U \to S})$, $\alpha(v^T) = \beta(v^T)$, and for all $v^T \in \mathbb{FV}(u^U)$, $\alpha(v^T) = \beta(v^T)$. By induction,

$$\begin{aligned} I^{\mathfrak{A}}(t^S, \alpha) &= I^{\mathfrak{A}}(f^{U \to S}(u^U), \alpha) \\ &= I^{\mathfrak{A}}(f^{U \to S}, \alpha)(I^{\mathfrak{A}}(u^U, \alpha)) \\ &= I^{\mathfrak{A}}(f^{U \to S}, \beta)(I^{\mathfrak{A}}(u^U, \beta)) \\ &= I^{\mathfrak{A}}(f^{U \to S}(u^U), \beta) \\ &= I^{\mathfrak{A}}(t^S, \beta) \end{aligned}$$

Case $t^S \equiv (t_1^{T_1}, ..., t_m^{T_m})$ where $t_1^{T_1}, ..., t_m^{T_m}$ are terms of rank less than k. For all $i \in \{1, ..., m\}$ we have that $\alpha(v^T) = \beta(v^T)$ for all $v^T \in \mathbb{FV}(t_i^{T_i})$, since $\mathbb{FV}(t_i^{T_i}) \subseteq \mathbb{FV}(t^S)$. By induction,

$$\begin{aligned} I^{\mathfrak{A}}(t^S, \alpha) &= I^{\mathfrak{A}}((t_1^{T_1}, ..., t_m^{T_m}), \alpha) \\ &= (I^{\mathfrak{A}}(t_1^{T_1}, \alpha), ..., I^{\mathfrak{A}}(t_m^{T_m}, \alpha)) \\ &= (I^{\mathfrak{A}}(t_1^{T_1}, \beta), ..., I^{\mathfrak{A}}(t_m^{T_m}, \beta)) \\ &= I^{\mathfrak{A}}((t_1^{T_1}, ..., t_m^{T_m}), \beta) \\ &= I^{\mathfrak{A}}(t^S, \beta) \end{aligned}$$

(ii) As usual for formulas, we need to prove something that is stronger than what we state in the proposition. The statement that we need to show is true is "For any formula φ and any variable assignments α and β such that $\alpha(v^T) = \beta(v^T)$ for all variables $v^T \in \mathbb{FV}(\varphi)$, $I^{\mathfrak{A}}(\varphi, \alpha) = I^{\mathfrak{A}}(\varphi, \beta)$." We will prove this statement by induction on φ.

Let φ be a formula and α and β be variable assignments such that for all $v^T \in \mathbb{FV}(\varphi)$, $\alpha(v^T) = \beta(v^T)$. Let $k = rank(\varphi)$. If $k = 0$, then there are two cases that we must consider.

Case $\varphi \equiv (t_1^T =^T t_2^T)$ where t_1^T, t_2^T are terms. By part(i),

$$\begin{aligned} I^{\mathfrak{A}}(\varphi,\alpha) = T \ \text{ iff } & I^{\mathfrak{A}}((t_1^T =^T t_2^T),\alpha) = T \\ \text{iff } & I^{\mathfrak{A}}(t_1^T,\alpha) \text{ is the same element as } I^{\mathfrak{A}}(t_2^T,\alpha) \\ \text{iff } & I^{\mathfrak{A}}(t_1^T,\beta) \text{ is the same element as } I^{\mathfrak{A}}(t_2^T,\beta) \\ \text{iff } & I^{\mathfrak{A}}((t_1^T =^T t_2^T),\beta) = T \\ \text{iff } & I^{\mathfrak{A}}(\varphi,\beta) = T \end{aligned}$$

Case $\varphi \equiv r^{T_1*\cdots*T_m}(t_1^{T_1},...,t_m^{T_m})$ where $r^{T_1*\cdots*T_m} \in \mathcal{R}$, $t_1^{T_1},...,t_m^{T_m} \in \mathbb{T}$. By part(i),

$$\begin{aligned} I^{\mathfrak{A}}(\varphi,\alpha) = T \ \text{ iff } & I^{\mathfrak{A}}(r^{T_1*\cdots*T_m}(t_1^{T_1},...,t_m^{T_m}),\alpha) = T \\ \text{iff } & I^{\mathfrak{A}}((t_1^{T_1},...,t_m^{T_m}),\alpha) \in \mathcal{R}^{\mathfrak{A}}(r^{T_1*\cdots*T_m}) \\ \text{iff } & I^{\mathfrak{A}}((t_1^{T_1},...,t_m^{T_m}),\beta) \in \mathcal{R}^{\mathfrak{A}}(r^{T_1*\cdots*T_m}) \\ \text{iff } & I^{\mathfrak{A}}(r^{T_1*\cdots*T_m}(t_1^{T_1},...,t_m^{T_m}),\beta) = T \\ \text{iff } & I^{\mathfrak{A}}(\varphi,\beta) = T \end{aligned}$$

Assume that $k > 0$, and for all formulas ψ of rank less than k and all variable assignments α_0 and β_0 with $\alpha_0(v^T) = \beta_0(v^T)$ for all $v^T \in \mathbb{FV}(\psi)$, $I^{\mathfrak{A}}(\psi,\alpha_0) = I^{\mathfrak{A}}(\psi,\beta_0)$. Since $k > 0$, there are three cases that we must consider.

Case $\varphi \equiv (\neg\psi)$, where ψ is a formula of rank less than k. Since $\mathbb{FV}(\psi) = \mathbb{FV}(\varphi)$, for all $v^T \in \mathbb{FV}(\psi)$, $\alpha(v^T) = \beta(v^T)$. By induction,

$$\begin{aligned} I^{\mathfrak{A}}(\varphi,\alpha) = T \ \text{ iff } & I^{\mathfrak{A}}(\psi,\alpha) = F \\ \text{iff } & I^{\mathfrak{A}}(\psi,\beta) = F \\ \text{iff } & I^{\mathfrak{A}}(\varphi,\beta) = T \end{aligned}$$

Case $\varphi \equiv (\psi \wedge \chi)$, where ψ and χ are formulas of rank less than k. Since $\mathbb{FV}(\varphi) = \mathbb{FV}(\psi) \cup \mathbb{FV}(\chi)$, $\alpha(v^T) = \beta(v^T)$ for all $v^T \in \mathbb{FV}(\psi)$ and $\alpha(v^T) = \beta(v^T)$ for all $v^T \in \mathbb{FV}(\chi)$. By induction,

$$\begin{aligned} I^{\mathfrak{A}}(\varphi,\alpha) = T \ \text{ iff } & I^{\mathfrak{A}}((\psi \wedge \chi),\alpha) = T \\ \text{iff } & I^{\mathfrak{A}}(\psi,\alpha) = T \text{ and } I^{\mathfrak{A}}(\chi,\alpha) = T \\ \text{iff } & I^{\mathfrak{A}}(\psi,\beta) = T \text{ and } I^{\mathfrak{A}}(\chi,\beta) = T \\ \text{iff } & I^{\mathfrak{A}}((\psi \wedge \chi),\beta) = T \\ \text{iff } & I^{\mathfrak{A}}(\varphi,\beta) = T \end{aligned}$$

Case $\varphi \equiv (\exists x^S\psi)$, where $x^S \in \mathbf{V}$ and ψ is a formula of rank less than k.

Assume that $I^{\mathfrak{A}}((\exists x^S\psi),\alpha) = T$. Then there exists a variable assignment α_0 with $\alpha_0(v^T) = \alpha(v^T)$ for all variables $v^T \not\equiv x^S$ such that $I^{\mathfrak{A}}(\psi,\alpha) = T$. Define a variable assignment β_0 by $\beta_0(v^T) = \beta(v^T)$ for all $v^T \not\equiv x^S$ and $\beta_0(x^S) = \alpha_0(x^S)$. Let $v^T \in \mathbb{FV}(\psi)$. If $v^T \equiv x^S$, then $\alpha_0(v^T) = \beta_0(v^T)$ by the definition of β_0. If $v^T \not\equiv x^S$, then $v^T \in \mathbb{FV}(\varphi)$, so $\alpha_0(v^T) = \alpha(v^T) = \beta(v^T) = \beta_0(v^T)$. Hence for all $v^T \in \mathbb{FV}(\psi)$, $\alpha_0(v^T) = \beta_0(v^T)$. By induction $I^{\mathfrak{A}}(\psi,\beta_0) = I^{\mathfrak{A}}(\psi,\alpha_0) = T$. So there exists a variable assignment β_0 with $\beta_0(v^T) = \beta(v^T)$ for all $v^T \not\equiv x^S$ such that $I^{\mathfrak{A}}(\psi,\beta_0) = T$. Hence $I^{\mathfrak{A}}(\varphi,\beta) = T$. The converse is similar.

Finally, we have a proposition relating interpretations and substitution.

Proposition 2.3.11 *Let $\mathfrak{A}$ be a structure, t^T a term, x^T a variable, and α a variable assignment. Let β be the variable assignment defined by $\beta(v^S) = \alpha(v^S)$ for all variables $v^S \not\equiv x^T$ and $\beta(x^T) = I^{\mathfrak{A}}(t^T,\alpha)$.*

(i) If u^U is a term, then $I^{\mathfrak{A}}([t^T/x^T]u^U,\alpha) = I^{\mathfrak{A}}(u^U,\beta)$.

(ii) If φ is a formula, then $I^{\mathfrak{A}}([t^T/x^T]\varphi,\alpha) = I^{\mathfrak{A}}(\varphi,\beta)$.

Proof: Let $\mathfrak{A}$ be a structure, t^T a term, x^T a variable, and α a variable assignment. Let β be the variable assignment defined by $\beta(v^S) = \alpha(v^S)$ for all variables $v^S \not\equiv x^T$ and $\beta(x^T) = I^{\mathfrak{A}}(t^T,\alpha)$.

(i) Let u^U be a term. Let $n = rank(u^U)$. If $n = 0$, then u^U is a variable or a constant. If $u^U \in \mathcal{C}$, then $I^{\mathfrak{A}}([t^T/x^T]u^U,\alpha) = \mathcal{C}^{\mathfrak{A}}(u^U) = I^{\mathfrak{A}}(u^U,\beta)$. Assume that $u^U \in \mathbf{V}$. If $u^U \equiv x^T$, then $[t^T/x^T]u^U \equiv t^T$. Then $I^{\mathfrak{A}}([t^T/x^T]u^U,\alpha) = I^{\mathfrak{A}}(t^T,\alpha) = \beta(x^T) = I^{\mathfrak{A}}(u^U,\beta)$. If $u^U \not\equiv x^T$, then $[t^T/x^T]u^U \equiv u^U$. Since $\beta(u^U) = \alpha(u^U)$, $I^{\mathfrak{A}}([t^T/x^T]u^U,\alpha) = \alpha(u^U) = \beta(u^U) = I^{\mathfrak{A}}(u^U,\beta)$.

Assume that $n > 0$ and that for all terms $u_1^{U_1}$ of rank less than n, we have $I^{\mathfrak{A}}([t^T/x^T]u_1^{U_1},\alpha) = I^{\mathfrak{A}}(u_1^{U_1},\beta)$.

Case $u^U \equiv (t_1^{T_1},...,t_m^{T_m})$ where $t_1^{T_1},...,t_m^{T_m}$ are terms of rank less than n. Then by induction,

$$\begin{aligned}
I^{\mathfrak{A}}([t^T/x^T]u^U,\alpha) &= I^{\mathfrak{A}}([t^T/x^T](t_1^{T_1},...,t_m^{T_m}),\alpha) \\
&= (I^{\mathfrak{A}}([t^T/x^T]t_1^{T_1},\alpha),...,I^{\mathfrak{A}}([t^T/x^T]t_m^{T_m},\alpha)) \\
&= (I^{\mathfrak{A}}(t_1^{T_1},\beta),...,I^{\mathfrak{A}}(t_m^{T_m},\beta)) \\
&= I^{\mathfrak{A}}((t_1^{T_1},...,t_m^{T_m}),\beta) \\
&= I^{\mathfrak{A}}(u^U,\beta)
\end{aligned}$$

Case $u^U \equiv f^{U_1\to U}(u_1^{U_1})$ where $f^{U_1\to U}$ and $u_1^{U_1}$ are terms of rank less than n.

$$\begin{aligned} I^{\mathfrak{A}}([t^T/x^T]u^U,\alpha) &= I^{\mathfrak{A}}([t^T/x^T]f^{U_1\to U}(u_1^{U_1}),\alpha) \\ &= I^{\mathfrak{A}}([t^T/x^T]f^{U_1\to U},\alpha)(I^{\mathfrak{A}}([t^T/x^T]u_1^{U_1},\alpha)) \\ &= I^{\mathfrak{A}}(f^{U_1\to U},\beta)(I^{\mathfrak{A}}(u_1^{U_1},\beta)) \\ &= I^{\mathfrak{A}}(f^{U_1\to U}(u_1^{U_1}),\beta) \\ &= I^{\mathfrak{A}}(u^U,\beta) \end{aligned}$$

(ii) Let φ be a formula and let $n = rank(\varphi)$. If $n = 0$, then there are two cases that we must consider.

Case $\varphi \equiv r^{T_1*\cdots*T_m}(t_1^{T_1},...,t_m^{T_m})$ where $r^{T_1*\cdots*T_m} \in \mathcal{R}$ and $t_1^{T_1},...,t_m^{T_m} \in \mathbb{T}$. By part (i),

$I^{\mathfrak{A}}([t^T/x^T]r^{T_1*\cdots*T_m}(t_1^{T_1},...,t_m^{T_m}),\alpha) = T$

iff $I^{\mathfrak{A}}(r^{T_1*\cdots*T_m}([t^T/x^T](t_1^{T_1},...,t_m^{T_m})),\alpha) = T$
iff $I^{\mathfrak{A}}([t^T/x^T](t_1^{T_1},...,t_m^{T_m}),\alpha) \in \mathcal{R}^{\mathfrak{A}}(r^{T_1*\cdots*T_m})$
iff $I^{\mathfrak{A}}((t_1^{T_1},...,t_m^{T_m}),\beta) \in \mathcal{R}^{\mathfrak{A}}(r^{T_1*\cdots*T_m})$
iff $I^{\mathfrak{A}}(r^{T_1*\cdots*T_m}(t_1^{T_1},...,t_m^{T_m}),\beta) = T$.

Case $\varphi \equiv (t_1^S = t_2^S)$ where t_1^S and $t_2^S \in \mathbb{T}$. By part (i),

$I^{\mathfrak{A}}([t^T/x^T](t_1^S = t_2^S),\alpha) = T$

iff $I^{\mathfrak{A}}(([t^T/x^T]t_1^S = [t^T/x^T]t_2^S),\alpha) = T$
iff $I^{\mathfrak{A}}([t^T/x^T]t_1^S,\alpha)$ is the same element as $I^{\mathfrak{A}}([t^T/x^T]t_2^S,\alpha)$
iff $I^{\mathfrak{A}}(t_1^S,\beta)$ is the same element as $I^{\mathfrak{A}}(t_2^S,\beta)$
iff $I^{\mathfrak{A}}((t_1^S = t_2^S),\beta) = T$.

Now assume that $n > 0$ and that for any formula ψ of rank less than n and any variable assignment α_1 if β_1 is the variable assignment defined by $\beta_1(v^S) = \alpha_1(v^S)$ for all $v^S \not\equiv x^T$ and $\beta_1(x^T) = I^{\mathfrak{A}}(t^T,\alpha_1)$, then $I^{\mathfrak{A}}([t^T/x^T]\psi,\alpha_1) = I^{\mathfrak{A}}(\psi,\beta_1)$. Since $n > 0$, φ is in one of following forms: $\neg\psi$, $\psi\wedge\chi$, or $\exists y^S\psi$, where ψ and χ are formulas of rank less than n and y^S a variable.

Case $\varphi \equiv \neg\psi$: Then

$$\begin{aligned} I^{\mathfrak{A}}([t^T/x^T](\neg\psi),\alpha) = T \quad &\text{iff} \quad I^{\mathfrak{A}}([t^T/x^T]\psi,\alpha) = F \\ &\text{iff} \quad I^{\mathfrak{A}}(\psi,\beta) = F \\ &\text{iff} \quad I^{\mathfrak{A}}(\neg\psi,\beta) = T. \end{aligned}$$

Case $\varphi \equiv \psi \wedge \chi$: Then

$$I^{\mathfrak{A}}([t^T/x^T](\psi \wedge \chi), \alpha) = T$$

$$\begin{aligned} &\text{iff } I^{\mathfrak{A}}([t^T/x^T]\psi \wedge [t^T/x^T]\chi, \alpha) = T \\ &\text{iff } I^{\mathfrak{A}}([t^T/x^T]\psi, \alpha) = T \text{ and } I^{\mathfrak{A}}([t^T/x^T]\chi, \alpha) = T \\ &\text{iff } I^{\mathfrak{A}}(\psi, \beta) = T \text{ and } I^{\mathfrak{A}}(\chi, \beta) = T \\ &\text{iff } I^{\mathfrak{A}}(\psi \wedge \chi, \beta) = T. \end{aligned}$$

Case $\varphi \equiv \exists y^S \psi$: First, suppose that $x^T \notin \mathbb{FV}(\varphi)$. Then $[t^T/x^T]\varphi = \varphi$ and $\alpha(v^U) = \beta(v^U)$ for all variables $v^U \in \mathbb{FV}(\varphi)$. Hence $I^{\mathfrak{A}}([t^T/x^T]\varphi, \alpha) = I^{\mathfrak{A}}(\varphi, \alpha) = I^{\mathfrak{A}}(\varphi, \beta)$. Therefore, for the remainder of this case we way assume $x^T \in \mathbb{FV}(\varphi)$, which implies that $x^T \not\equiv y^S$ and $x^T \in \mathbb{FV}(\psi)$. There are two subcases to consider.

Subcase $y^S \notin \mathbb{FV}(t^T)$. Then $[t^T/x^T]\varphi \equiv \exists y^S[t^T/x^T]\psi$. Suppose $I^{\mathfrak{A}}(\exists y^S[t^T/x^T]\psi, \alpha) = T$, so that there is a variable assignment α_1 with $\alpha_1(v^U) = \alpha(v^U)$ for all $v^U \not\equiv y^S$ such that $I^{\mathfrak{A}}([t^T/x^T]\psi, \alpha_1) = T$. Let β_1 be the variable assignment defined by $\beta_1(v^U) = \alpha_1(v^U)$ for all $v^U \not\equiv x^T$ and $\beta_1(x^T) = I^{\mathfrak{A}}(t^T, \alpha_1)$. By induction, $I^{\mathfrak{A}}(\psi, \beta_1) = T$. Since $y^S \notin \mathbb{FV}(t^T)$, $\beta_1(x^T) = I^{\mathfrak{A}}(t^T, \alpha_1) = I^{\mathfrak{A}}(t^T, \alpha) = \beta(x^T)$. If $v^U \not\equiv y^S$ and $v^U \not\equiv x^T$, then $\beta_1(v^U) = \alpha_1(v^U) = \alpha(v^U) = \beta(v^U)$. Thus, $\beta_1(v^U) = \beta(v^U)$ for all variables $v^U \not\equiv y^S$, So $I^{\mathfrak{A}}(\exists y^S \psi, \beta) = T$.

Conversely, suppose $I^{\mathfrak{A}}(\exists y^S \psi, \beta) = T$, and let β_1 be a variable assignment such that $\beta_1(v^U) = \beta(v^U)$ for all variables $v^U \not\equiv y^S$ and $I^{\mathfrak{A}}(\psi, \beta_1) = T$. Define α_1 by $\alpha_1(v^U) = \beta_1(v^U)$ for $v^U \not\equiv x^T$ and $\alpha_1(x^T) = \alpha(x^T)$. Note that $\alpha_1(v^U) = \alpha(v^U)$ for all $v^U \in \mathbb{FV}(t^T)$. Indeed, $\alpha_1(x^T) = \alpha(x^T)$ by definition and for $v^U \not\equiv x^T$, we must have $v^U \not\equiv y^S$, so $\alpha_1(v^U) = \beta_1(v^U) = \beta(v^U) = \alpha(v^U)$. Thus, since $x^T \not\equiv y^S$, $\beta_1(x^T) = \beta(x^T) = I^{\mathfrak{A}}(t^T, \alpha) = I^{\mathfrak{A}}(t^T, \alpha_1)$. By induction, $I^{\mathfrak{A}}([t^T/x^T]\psi, \alpha_1) = I^{\mathfrak{A}}(\psi, \beta_1) = T$. Since we showed above that $\alpha_1(v^U) = \alpha(v^U)$ for all $v^U \not\equiv y^S$, this shows $I^{\mathfrak{A}}(\exists y^S[t^T/x^T]\psi, \alpha) = T$.

Subcase $y^S \in \mathbb{FV}(t^T)$. Then let z^S be the first variable of type S not occuring in either ψ or t^T. So $[t^T/x^T]\varphi \equiv \exists z^S[t^T/x^T][z^S/y^S]\psi$. Let $\chi \equiv [z^S/y^S]\psi$. By the Lemma 2.2.12, $rank(\chi) = rank(\psi)$. Since $z^S \notin \mathbb{FV}(t^T)$, by the previous subcase, $I^{\mathfrak{A}}([t^T/x^T]\varphi, \alpha) = I^{\mathfrak{A}}(\varphi, \beta)$.

2.3.3 Sentences, models, and semantic implication.

All of the definitions and results in this section are identical to those in standard first-order logic, so we will just state them briefly.

A formula σ such that $\mathbf{FV}(\sigma) = \emptyset$ is called a **sentence**. If σ is a sentence and $\mathfrak{A}$ is a structure, then $I^{\mathfrak{A}}(\sigma, \alpha)$ is the same for all variable assignments α. If $I^{\mathfrak{A}}(\sigma, \alpha) = T$ for all α, then we say σ is true in $\mathfrak{A}$, or that $\mathfrak{A}$ is a **model** of σ, and write $\mathfrak{A} \models \sigma$. If $I^{\mathfrak{A}}(\sigma, \alpha) = F$ for all α, we say that σ is false in $\mathfrak{A}$, or that σ is not a model of σ, and write $\mathfrak{A} \not\models \sigma$. Similarly, if Σ is a set of sentences and $\mathfrak{A} \models \sigma$ for all $\sigma \in \Sigma$, then we say that $\mathfrak{A}$ is a model of Σ and write $\mathfrak{A} \models \Sigma$, whereas if there is at least one $\sigma \in \Sigma$ such that $\mathfrak{A} \not\models \sigma$, then we say that $\mathfrak{A}$ is not a model of Σ and write $\mathfrak{A} \not\models \Sigma$.

The following lemma is a direct consequence of the definitions.

Lemma 2.3.12 *Let $\mathfrak{A}$ be a structure and let Σ and Φ be sets of sentences such that $\Sigma \subseteq \Phi$. If $\mathfrak{A} \models \Phi$, then $\mathfrak{A} \models \Sigma$ also.*

Let Σ be a set of sentences and σ a single sentence. If $\mathfrak{A} \models \sigma$ for all structures $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$, then we say that Σ **semantically implies** σ, and write $\Sigma \models \sigma$. We also say that σ is a **consequence** of Σ in this case. We call σ **valid** whenever $\emptyset \models \sigma$. Usually we just write $\models \sigma$ to indicate that σ is valid. Note that σ is valid iff $\mathfrak{A} \models \sigma$ for all structures $\mathfrak{A}$. Then we have the following lemma.

Lemma 2.3.13 *Let Σ and Φ be sets of sentences, and let γ be a sentence.*

(i) $\Sigma \models \sigma$ for all $\sigma \in \Sigma$.

(ii) If $\Sigma \models \varphi$ for all $\varphi \in \Phi$ and $\Phi \models \gamma$, then $\Sigma \models \gamma$.

2.4 The Formal Proof

The definitions and basic results on formal proofs are also quite similar to those for standard first-order logic, so we will often be brief in this section as well.

2.4.1 Deductive systems and formal proofs

A **deductive system** consists of two sets: a set of formulas, whose members are called **logical axioms**, and a set of objects called **inference rules**. An inference rule is a pair (P, φ), where P is a set of formulas and φ is a single formula. The set P is called the set of **premises** of the rule, and the formula φ is called the **conclusion** of the rule. If $R = (P, \varphi)$ is an inference rule, we say that φ follows from P by R. The logical axioms and inference rules are used when writing formal proofs.

Definition 2.4.1 *Let Σ be a set of formulas and σ a single formula. A* **formal proof** *of σ from Σ is a finite sequence $\varphi_1, ..., \varphi_n$ of formulas (called the* **steps** *of the proof) such that $\varphi_n \equiv \sigma$, and for each $i \in \{1, ..., n\}$ we have that φ_i satisfies one of the following conditions:*

(i) φ_i is a logical axiom; or

(ii) $\varphi_i \in \Sigma$; or

(iii) there are an inference rule R and formulas $\varphi_{j_1}, ..., \varphi_{j_m}$ such that $j_\ell < i$ for $\ell \in \{1, ..., m\}$ and φ_i follows from the formulas $\varphi_{j_1}, ..., \varphi_{j_m}$ by the rule R.

If there is a formal proof of σ from Σ, then we say Σ **proves** σ, and write $\Sigma \vdash \sigma$. If $\emptyset \vdash \sigma$, then we call σ **a theorem**; usually we write $\vdash \sigma$ to indicate that σ is a theorem.

Note that for semantic implication we required that σ and the elements of Σ be sentences, but for formal proofs there is no such requirement: σ and the elements of Σ can be arbitrary formulas.

2.4.2 Our rules of inference and axioms.

There are many choices for the sets of inference rules and axioms which are usable, and the particular choice made is often a matter of personal taste. The sets that we will use in this thesis are as follows.

Inference rules.

We will follow a common convention and write inference rules as fractions, with the premises above the bar and the conclusion below the bar, like this:

$$\frac{P_1, ..., P_n}{C}$$

Our inference rules are divided into three group: modus ponens rules, generalization rules, and specialization rules.

i. Modus ponens rules (MP). This group is the set of all rules of the form

$$\frac{\varphi \quad \varphi \Rightarrow \psi}{\psi}$$

where $\varphi, \psi \in \mathbb{F}$.

ii. Generalization rules (GN). This group is the set of all rules of the form

$$\frac{\varphi}{\forall x^T \varphi}$$

where $\varphi \in \mathbb{F}$ and $x^T \in \mathbf{V}$.

iii. Specialization rules (SP). This group is the set of all rules of the form

$$\frac{\forall x^T \varphi}{[t^T / x^T]\varphi}$$

where $\varphi \in \mathbb{F}$ and $x^T \in \mathbf{V}$, and $t^T \in \mathbb{T}$.

Logical Axioms

Our axioms are also divided into three group: propositional axioms, quantifier axioms, and identity axioms.

i. Propositional axioms (PA). This group is the set of all formulas which can be obtained from some tautology in propositional logic by substituting a formula for each propositional variable in that tautology. We can make this completely rigoruos as follows.

In order to define the substitution of formulas for the propositional variables in a propositional formula, we need to select a formula for each propositional variable that occurs in Φ. To simplify the definition, in fact we select formulas for all propositional variables by defining a **formula selector** to be a function from $\mathbb{V}^p$ to $\mathbb{F}$. The substitution of formulas for propositional variables in a propositional formula will be defined by induction as follows.

Definition 2.4.2 *Let Φ be a formula in propositional logic, $\eta : \mathbb{V}^p \to \mathbb{F}$ a formula selector.* **The subsititution of formulas for the propositional variables in a propositional formula Φ under a formula selector** *η, denoted by $\Phi\langle\eta\rangle$, is defined by induction on Φ. Since $\Phi \in \mathbb{F}^p$, there is a smallest $k \in \mathbb{N}$ such that $\Phi \in \mathbb{F}^p_k$. If $k = 0$, then define $\Phi\langle\eta\rangle = \eta(\Phi)$. Assume that $k > 0$, and that for all $\ell < k$ and all $\Psi \in \mathbb{F}^p_\ell$, $\Psi\langle\eta\rangle$ is defined. Since $k > 0$, there are two cases that we must consider.*

Case $\Phi \equiv \neg\Psi$ *for some* $\Psi \in \mathbb{F}^p_\ell, \ell < k$. *Define*

$$\Phi\langle\eta\rangle = (\neg\Psi\langle\eta\rangle)$$

Case $\Phi \equiv (\Psi \wedge \Theta)$ *for some* $\Psi, \Theta \in \mathbb{F}^p_\ell, \ell < k$. *Define*

$$\Phi\langle\eta\rangle = (\Psi\langle\eta\rangle \wedge \Theta\langle\eta\rangle)$$

Proposition 2.4.3 *Let Φ be a formula in propositional logic, $\eta : \mathbb{V}^p \to \mathbb{F}$ a formula selector. Then $\Phi\langle\eta\rangle$ is a formula.*

Proof: This is clear from the definitions.

This seems like an appropriate place to state the following proposition (which we will need in our proof of the Soundness theorem).

Proposition 2.4.4 *Let Φ be a formula in propositional logic, η a formula selector, $\mathfrak{A}$ a structure, and α a variable assignment. Then*

$$I^{\mathfrak{A}}(\Phi\langle\eta\rangle, \alpha) = I^{\mathfrak{P}}(\Phi, \mu),$$

where μ is the truth assignment defined by

$$\mu(P_i) = I^{\mathfrak{A}}(\eta(P_i), \alpha)$$

for all $P_i \in \mathbb{V}^p$.

Proof: This follows from the definitions by a straightforward induction on k, where k is the smallest element of $\mathbb{N}$ such that $\Phi \in \mathbb{V}_k^p$.

ii. Quantifier axioms. (QA) This group is the set of all formulas of the form

$$(\forall x^T(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow (\forall x^T \psi))$$

$$\forall x^T \psi \Rightarrow \psi$$

$$(\forall x^T(\neg\psi)) \Rightarrow (\neg\exists x^T \psi)$$

$$(\neg\exists x^T \psi) \Rightarrow (\forall x^T(\neg\psi)).$$

where $\varphi, \psi \in \mathbb{F}$ and x^T is a variable not in $\mathbb{FV}(\varphi)$.

iii. Equality axioms (EA). This group is the set of all formulas of the form

$$x^S = x^S$$

$$(x^S = y^S) \Rightarrow ([x^S/v^S]t^T = [y^S/v^S]t^T)$$

$$(x^S = y^S) \Rightarrow ([x^S/v^S]\varphi \Rightarrow [y^S/v^S]\varphi)$$

where $x^S, y^S, v^S \in \mathbb{V}$, $t^T \in \mathbb{T}$, and φ is an atomic formula.

Example 2.4.5 *This example will show how types can help us to write a homomophism and prove some statements about it. Let $\mathcal{L} = \{e^N, e^Z, +^{N*N\to N}, \oplus^{Z*Z\to Z}, h^{N\to Z}\}$ be a language over the type system $\mathbb{T} = \langle N, Z\rangle$. Let Σ be the set consisting of the following formulas.*

- $\forall x_1^N(+^{N*N\to N}(x_1^N, e^N) = x_1^N)$,
- $\forall x_1^Z(\oplus^{Z*Z\to Z}(x_1^Z, e^Z) = x_1^Z)$,
- $\forall x_1^N \forall x_2^N(h^{N\to Z}(+^{N*N\to N}(x_1^N, x_2^N)) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(x_2^N)))$,
- $\forall x_3^Z \forall x_4^Z \forall x_5^Z((\oplus^{Z*Z\to Z}(x_3^Z, x_4^Z) = \oplus^{Z*Z\to Z}(x_3^Z, x_5^Z)) \Rightarrow (x_4^Z = x_5^Z))$.

We will show that $\Sigma \vdash h^{N\to Z}(e^N) = e^Z$.

φ_1	$\equiv$	$x_1^N = x_2^N \Rightarrow h^{N\to Z}(x_1^N) = h^{N\to Z}(x_2^N)$	φ_1 *is an EA,* $t^T \equiv h^{N\to Z}(x_5^N)$.
φ_2	$\equiv$	$\forall x_2^N(x_1^N = x_2^N \Rightarrow h^{N\to Z}(x_1^N) = h^{N\to Z}(x_2^N))$	φ_1 *and GN.*
φ_3	$\equiv$	$\forall x_1^N \forall x_2^N(x_1^N = x_2^N \Rightarrow h^{N\to Z}(x_1^N) = h^{N\to Z}(x_2^N))$	φ_2 *and GN.*
φ_4	$\equiv$	$\forall x_2^N(+^{N\ast N\to N}(x_1^N, e^N) = x_2^N \Rightarrow$ $h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = h^{N\to Z}(x_2^N))$	φ_3 *and SP.*
φ_5	$\equiv$	$(+^{N\ast N\to N}(x_1^N, e^N) = x_1^N \Rightarrow$ $h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = h^{N\to Z}(x_1^N))$	φ_4 *and SP.*
φ_6	$\equiv$	$\forall x_1^N(+^{N\ast N\to N}(x_1^N, e^N) = x_1^N)$	$\varphi_6 \in \Sigma$
φ_7	$\equiv$	$(+^{N\ast N\to N}(x_1^N, e^N) = x_1^N)$	φ_6 *and SP.*
φ_8	$\equiv$	$h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = h^{N\to Z}(x_1^N))$	φ_5, φ_7 *and MP.*
φ_9	$\equiv$	$(x_4^Z = x_3^Z) \Rightarrow ((x_4^Z = x_5^Z) \Rightarrow (x_3^Z = x_5^Z))$	φ_9 *is an EA.*
φ_{10}	$\equiv$	$\forall x_5^Z(x_4^Z = x_3^Z) \Rightarrow ((x_4^Z = x_5^Z) \Rightarrow (x_3^Z = x_5^Z))$	φ_9 *and GN.*
φ_{11}	$\equiv$	$\forall x_4^Z \forall x_5^Z(x_4^Z = x_3^Z) \Rightarrow ((x_4^Z = x_5^Z) \Rightarrow (x_3^Z = x_5^Z))$	φ_{10} *and GN.*
φ_{12}	$\equiv$	$\forall x_3^Z \forall x_4^Z \forall x_5^Z(x_4^Z = x_3^Z) \Rightarrow ((x_4^Z = x_5^Z) \Rightarrow (x_3^Z = x_5^Z))$	φ_{11} *and GN.*
φ_{13}	$\equiv$	$\forall x_4^Z \forall x_5^Z(x_4^Z = h^{N\to Z}(x_1^N)) \Rightarrow ((x_4^Z = x_5^Z) \Rightarrow$ $(h^{N\to Z}(x_1^N) = x_5^Z))$	φ_{12} *and SP.*
φ_{14}	$\equiv$	$\forall x_5^Z(h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = h^{N\to Z}(x_1^N)) \Rightarrow$ $((h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = x_5^Z) \Rightarrow$ $(h^{N\to Z}(x_1^N) = x_5^Z))$	φ_{13} *and SP.*
φ_{15}	$\equiv$	$(h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = h^{N\to Z}(x_1^N)) \Rightarrow$ $((h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = \oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N),$ $h^{N\to Z}(e^N))) \Rightarrow (h^{N\to Z}(x_1^N) = \oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N),$ $h^{N\to Z}(e^N))))$	φ_{14} *and SP.*
φ_{16}	$\equiv$	$((h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = \oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N),$ $h^{N\to Z}(e^N))) \Rightarrow (h^{N\to Z}(x_1^N) = \oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N),$ $h^{N\to Z}(e^N))))$	φ_8, φ_{15} *and MP.*
φ_{17}	$\equiv$	$\forall x_1^N \forall x_2^N(h^{N\to Z}(+^{N\ast N\to N}(x_1^N, x_2^N)) =$ $\oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(x_2^N)))$	$\varphi_{17} \in \Sigma$
φ_{18}	$\equiv$	$\forall x_2^N(h^{N\to Z}(+^{N\ast N\to N}(x_1^N, x_2^N)) =$ $\oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(x_2^N)))$	φ_{17} *and SP.*
φ_{19}	$\equiv$	$(h^{N\to Z}(+^{N\ast N\to N}(x_1^N, e^N)) = \oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N),$ $h^{N\to Z}(e^N)))$	φ_{18} *and SP.*
φ_{20}	$\equiv$	$h^{N\to Z}(x_1^N) = \oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N))$	$\varphi_{16}, \varphi_{19}$ *and MP.*
φ_{21}	$\equiv$	$\forall x_4^Z \forall x_5^Z(x_4^Z = \oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N))) \Rightarrow$ $((x_4^Z = x_5^Z) \Rightarrow (\oplus^{Z\ast Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N))$ $= x_5^Z))$	φ_{12} *and SP.*

φ_{22}	$\equiv$	$\forall x_5^Z(h^{N\to Z}(x_1^N) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N))) \Rightarrow ((h^{N\to Z}(x_1^N) = x_5^Z) \Rightarrow (\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N)) = x_5^Z))$	φ_{21} *and SP.*
φ_{23}	$\equiv$	$(h^{N\to Z}(x_1^N) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N))) \Rightarrow ((h^{N\to Z}(x_1^N) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)) \Rightarrow (\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N)) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)))$	φ_{22} *and SP.*
φ_{24}	$\equiv$	$(h^{N\to Z}(x_1^N) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)) \Rightarrow (\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N)) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z))$	$\varphi_{20}, \varphi_{23}$ *and MP.*
φ_{25}	$\equiv$	$\forall x_1^Z(\oplus^{Z*Z\to Z}(x_1^Z, e^Z) = x_1^Z)$	$\varphi_{25} \in \Sigma$
φ_{26}	$\equiv$	$\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z) = h^{N\to Z}(x_1^N)$	φ_{25} *and SP.*
φ_{27}	$\equiv$	$(x_5^Z = x_3^Z) \Rightarrow ((x_5^Z = x_5^Z) \Rightarrow (x_3^Z = x_5^Z))$	φ_{27} *is an EA.*
φ_{28}	$\equiv$	$\forall x_5^Z((x_5^Z = x_3^Z) \Rightarrow ((x_5^Z = x_5^Z) \Rightarrow (x_3^Z = x_5^Z)))$	φ_{27} *and GN.*
φ_{29}	$\equiv$	$\forall x_3^Z \forall x_5^Z((x_5^Z = x_3^Z) \Rightarrow ((x_5^Z = x_5^Z) \Rightarrow (x_3^Z = x_5^Z)))$	φ_{28} *and GN.*
φ_{30}	$\equiv$	$\forall x_5^Z((x_5^Z = h^{N\to Z}(x_1^N)) \Rightarrow ((x_5^Z = x_5^Z) \Rightarrow (h^{N\to Z}(x_1^N) = x_5^Z)))$	φ_{29} *and SP.*
φ_{31}	$\equiv$	$(\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z) = h^{N\to Z}(x_1^N)) \Rightarrow ((\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)) \Rightarrow (h^{N\to Z}(x_1^N) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)))$	φ_{30} *and SP.*
φ_{32}	$\equiv$	$(\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)) \Rightarrow (h^{N\to Z}(x_1^N) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z))$	$\varphi_{26}, \varphi_{31}$ *and MP.*
φ_{33}	$\equiv$	$x_3^Z = x_3^Z$	φ_{33} *is an EA.*
φ_{34}	$\equiv$	$\forall x_3^Z(x_3^Z = x_3^Z)$	φ_{33} *and GN.*
φ_{35}	$\equiv$	$(\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z))$	φ_{34} *and SP.*
φ_{36}	$\equiv$	$h^{N\to Z}(x_1^N) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)$	$\varphi_{32}, \varphi_{35}$ *and MP.*
φ_{37}	$\equiv$	$\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N)) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)$	$\varphi_{24}, \varphi_{36}$ *and MP.*
φ_{38}	$\equiv$	$\forall x_3^Z \forall x_4^Z \forall x_5^Z((\oplus^{Z*Z\to Z}(x_3^Z, x_4^Z) = \oplus^{Z*Z\to Z}(x_3^Z, x_5^Z)) \Rightarrow (x_4^Z = x_5^Z))$	$\varphi_{38} \in \Sigma$
φ_{39}	$\equiv$	$\forall x_4^Z \forall x_5^Z((\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), x_4^Z) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), x_5^Z)) \Rightarrow (x_4^Z = x_5^Z))$	φ_{38} *and SP.*
φ_{40}	$\equiv$	$\forall x_5^Z((\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N)) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), x_5^Z)) \Rightarrow (h^{N\to Z}(e^N) = x_5^Z))$	φ_{39} *and SP.*
φ_{41}	$\equiv$	$(\oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), h^{N\to Z}(e^N)) = \oplus^{Z*Z\to Z}(h^{N\to Z}(x_1^N), e^Z)) \Rightarrow (h^{N\to Z}(e^N) = e^Z)$	φ_{40} *and SP.*
φ_{42}	$\equiv$	$h^{N\to Z}(e^N) = e^Z$	$\varphi_{37}, \varphi_{41}$ *and MP.*