เซมิกรุปการแปลงนัยทั่วไปที่ให้โครงสร้างไฮเปอร์ริง

นางสาวนกน้อย ร่มพฤกษ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2544

ISBN 974 - 03 - 0475 - 3

GENERALIZED TRANSFORMATION SEMIGROUPS ADMITTING

HYPERRING STRUCTURE

Miss Noknoi Rompurk

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Mathematics

Department of Mathematics
Faculty of Science
Chulalongkorn University

Academic Year 2001
ISBN 974 - 03 - 0475 - 3

Thesis Title	Generalized Transformation Semigroups Admitting Hyperring Structure
By	Miss. Noknoi Rompurk
Field of Study	Mathematics
Thesis Advisor	Associate Professor Yupaporn Kemprasit, Ph. D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

……………………………………………….. Dean of Faculty of Science
(Associate Professor Wanchai Phothiphichitr, Ph. D.)

THESIS COMMITTEE

……………………………………………….. Chairman
(Assistant Professor Amorn Wasanawichit, Ph. D.)

……………………………………….……. Thesis Advisor
(Associate Professor Yupaporn Kemprasit, Ph. D.)

…………………………………………….. Member
(Sajee Pianskool, Ph. D.)

นกน้อย ร่มพฤกษ์ : เซมิกรุปการแปลงนัยทั่วไปที่ให้โครงสร้างไฮเปอร์ริง
(GENERALIZED TRANSFORMATION SEMIGROUPS ADMITTING HYPERRING STRUCTURE)
อ. ที่ปรึกษา : รองศาสตราจารย์ ดร. ยุพาภรณ์ เข็มประสิทธิ์, 48 หน้า. ISBN 974 - 03 - 0475 - 3.

เรากล่าวว่าเซมิกรุป S *ให้โครงสร้างไฮเปอร์ริง* ถ้ามีไฮเปอร์โอเปอเรชัน + บน S^0 ที่ทำให้ $(S^0, +, \cdot)$ เป็น (คราสเนอร์) ไฮเปอร์ริง โดยที่ $\cdot$ เป็นโอเปอเรชันของ S^0 สำหรับเซมิกรุป S และ $\theta \in S^1$ ให้ (S, θ) เป็นเซมิกรุป S ภายใต้โอเปอเรชัน $*$ กำหนดโดย $x * y = x\theta y$ สำหรับทุก $x, y \in S$

เราให้ $T(X)$ แทนเซมิกรุปการแปลงเต็มบนเซต X ซึ่งเป็นเซตไม่ว่าง สำหรับปริภูมิเวกเตอร์ V บนริงการหาร ให้ $L(V)$ เป็นเซมิกรุปของการแปลงเชิงเส้น $\alpha : V \to V$ ทั้งหมดภายใต้การประกอบ

ในการวิจัยนี้เราให้ลักษณะที่จะบอกว่าเซมิกรุป (S, θ) โดย $\theta \in S^1$ ให้โครงสร้างไฮเปอร์ริงเมื่อใด โดยที่ S เป็นเซมิกรุปย่อยใดๆของ $T(X)$ และ $L(V)$ ต่อไปนี้

$T(X)$
$M(X) = \{ \alpha \in T(X) \mid \alpha$ หนึ่งต่อหนึ่ง$\}$
$E(X) = \{ \alpha \in T(X) \mid \mathrm{Im}\alpha = X \}$
$T_1(X) = \{ \alpha \in T(X) \mid \mathrm{Im}\alpha$ เป็นเซตอันตะ$\}$
$T_2(X) = \{ \alpha \in T(X) \mid X \setminus \mathrm{Im}\alpha$ เป็นเซตอันตะ$\}$
$T_3(X) = \{ \alpha \in T(X) \mid \mathrm{K}(\alpha)$ เป็นเซตอันตะ$\}$ เมื่อ $\mathrm{K}(\alpha) = \{ x \in X \mid \alpha$ ไม่หนึ่งต่อหนึ่งที่ $x\}$
$T_4(X) = \{ \alpha \in T(X) \mid \alpha$ หนึ่งต่อหนึ่ง และ $X \setminus \mathrm{Im}\alpha$ เป็นเซตอนันต์ $\}$ เมื่อ X เป็นเซตอนันต์
$T_5(X) = \{ \alpha \in T(X) \mid \mathrm{K}(\alpha)$ เป็นเซตอนันต์ และ $\mathrm{Im}\alpha = X \}$ เมื่อ X เป็นเซตอนันต์
$L(V)$
$M(V) = \{ \alpha \in L(V) \mid \alpha$ หนึ่งต่อหนึ่ง$\}$
$E(V) = \{ \alpha \in L(V) \mid \mathrm{Im}\alpha = V \}$
$L_1(V) = \{ \alpha \in L(V) \mid \dim \mathrm{Im}\alpha$ อันตะ$\}$
$L_2(V) = \{ \alpha \in L(V) \mid \dim (V / \mathrm{Im}\alpha)$ อันตะ$\}$
$L_3(V) = \{ \alpha \in L(V) \mid \dim \mathrm{Ker}\alpha$ อันตะ$\}$
$L_4(V) = \{ \alpha \in L(V) \mid \alpha$ หนึ่งต่อหนึ่ง และ $\dim (V / \mathrm{Im}\alpha)$ อนันต์$\}$ เมื่อ V เป็นปริภูมิเวกเตอร์ที่มีมิติอนันต์
$L_5(V) = \{ \alpha \in L(V) \mid \dim \mathrm{Ker}\alpha$ อนันต์ และ $\mathrm{Im}\alpha = V \}$ เมื่อ V เป็นปริภูมิเวกเตอร์ที่มีมิติอนันต์

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต...
สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา...
ปีการศึกษา **2544** ลายมือชื่ออาจารย์ที่ปรึกษาร่วม -

4272304023 : MAJOR MATHEMATICS
KEYWORD : HYPERRINGS / GENERALIZED TRANSFORMATION SEMIGROUPS
NOKNOI ROMPURK : GENERALIZED TRANSFORMATION SEMIGROUPS ADMITTING HYPERRING STRUCTURE. THESIS ADVISOR : ASSOC. PROF. YUPAPORN KEMPRASIT. Ph.D.
48 pp. ISBN 974 - 03 - 0475 - 3.

A semigroup S is said to *admit a hyperring structure* if there exists a hyperoperation + on S^0 such that $(S^0, +, \cdot)$ is a (Krasner) hyperring where $\cdot$ is the operation of S^0. For a semigroup S and $\theta \in S^1$, let (S, θ) be the semigroup S under the operation $*$ defined by $x * y = x\theta y$ for all $x, y \in S$.

The full transformation semigroup on a nonempty set X is denoted by $T(X)$. For a vector space V over a division ring, let $L(V)$ be the semigroup of all linear transformations $\alpha : V \to V$ under composition.

In this research, we give characterizations determining when the semigroup (S, θ) with $\theta \in S^1$ admits a hyperring structure where S is any of the following subsemigroups of $T(X)$ and of $L(V)$:

$T(X)$,
$M(X) = \{ \alpha \in T(X) \mid \alpha \text{ is } 1 - 1 \}$,
$E(X) = \{ \alpha \in T(X) \mid \text{Im}\alpha = X \}$,
$T_1(X) = \{ \alpha \in T(X) \mid \text{Im}\alpha \text{ is finite} \}$,
$T_2(X) = \{ \alpha \in T(X) \mid X \setminus \text{Im}\alpha \text{ is finite} \}$,
$T_3(X) = \{ \alpha \in T(X) \mid \text{K}(\alpha) \text{ is finite} \}$ where $\text{K}(\alpha) = \{ x \in X \mid \alpha \text{ is not } 1 - 1 \text{ at } x \}$,
$T_4(X) = \{ \alpha \in T(X) \mid \alpha \text{ is } 1 - 1 \text{ and } X \setminus \text{Im}\alpha \text{ is infinite} \}$ where X is infinite,
$T_5(X) = \{ \alpha \in T(X) \mid \text{K}(\alpha) \text{ infinite and } \text{Im}\alpha = X \}$ where X is infinite,
$L(V)$,
$M(V) = \{ \alpha \in L(V) \mid \alpha \text{ is } 1 - 1 \}$,
$E(V) = \{ \alpha \in L(V) \mid \text{Im}\alpha = V \}$,
$L_1(V) = \{ \alpha \in L(V) \mid \dim \text{Im}\alpha \text{ is finite} \}$,
$L_2(V) = \{ \alpha \in L(V) \mid \dim (V / \text{Im}\alpha) \text{ is finite} \}$,
$L_3(V) = \{ \alpha \in L(V) \mid \dim \text{Ker}\alpha \text{ is finite} \}$,
$L_4(V) = \{ \alpha \in L(V) \mid \alpha \text{ is } 1 - 1 \text{ and } \dim (V / \text{Im}\alpha) \text{ is infinite} \}$ where V is infinite dimensional,
$L_5(V) = \{ \alpha \in L(V) \mid \dim \text{Ker}\alpha \text{ is infinite and } \text{Im}\alpha = V \}$ where V is infinite dimensional.

Department **Mathematics** Student's signature................................

Field of Study **Mathematics** Advisor's signature..............................

Academic year **2001** Co-advisor's signature -

ACKNOWLEDGEMENT

I am greatly indebted to Assoc. prof. Dr. Yupaporn Kemprasit, my thesis advisor, for her untired offering me some thoughtful and helpful advice in preparing and writing my thesis. I am also grateful to Assist. Prof. Dr. Amorn Wasanawichit and Dr. Sajee Pianskool who served as the chairman and the member of my thesis committee, respectively. Moreover I would like to thank all of the lecturers for their previous valuable lectures while studying.

In particular, I would like to express my gratitude to my family and friends for their encouragement throughout my graduate study.

CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

For any set X, the cardinality of X will be denoted by $|X|$.

For a semigroup S, if S has an identity, let $S^1 = S$, and if S does not have an identity, let S^1 be the semigroup S with an identity 1 adjoined. The semigroup S^0 is defined to be S if S has a zero and $|S| > 1$; otherwise, let S^0 be the semigroup S with a zero 0 adjoined.

For an element a of a semigroup S, the *subsemigroup of S generated by a* is defined to be

$$\langle a \rangle = \{\ a^n \mid n \in \mathbb{N}\}$$

where $\mathbb{N}$ is the set of all positive integers. The *order* of a is defined to be $|\langle a \rangle|$. We have that $a^i \neq a^j$ for all distinct $i, j \in \mathbb{N}$ if and only if $\langle a \rangle$ is infinite.

Assume that $a \in S$ is such that $\langle a \rangle$ is finite. Then $a^i = a^j$ for some $i, j \in \mathbb{N}$ with $i < j$. Let

$$s = \min\{j \in \mathbb{N} \mid a^j = a^i \text{ for some } i < j\}.$$

Then there is a unique $r \in \mathbb{N}$ such that $r < s$ and $a^s = a^r$. Let $m = s - r$. Then $s = r + m$. It follows that

$$\langle a \rangle = \{a, a^2, ..., a^r, a^{r+1}, ..., a^{r+m-1}\},\ a^{r+m} = a^r,$$

$a, a^2, ..., a^{r+m-1}$ are all distinct and $\{a^r, a^{r+1}, ..., a^{r+m-1}\}$ is a cyclic subgroup of $\langle a \rangle$ of order m ([2], page 19 - 20). It then follows that $a^t \in \{a^r, a^{r+1}, ..., a^{r+m-1}\}$ for every positive integer $t \geq r$. Moreover, the numbers r and m are independent of a ([2], page 20), that is, if $\langle a\rangle = \langle b\rangle$, then

$$s = \min\{j \in \mathbb{N} \mid b^j = b^i \text{ for some } i < j\}$$

and $b^s = b^r$. We call r and m the *index* and the *period* of $\langle a\rangle$, respectively. Let index($\langle a\rangle$) and period($\langle a\rangle$) respectively denote the index and the period of $\langle a\rangle$. The following statements are clearly obtained.

(1) index($\langle a\rangle$) $= 1$ if and only if $\langle a\rangle$ is a cyclic group and

(2) period($\langle a\rangle$) $= 1$ if and only if a^r is the zero of $\langle a\rangle$ where $r =$ index($\langle a\rangle$).

A semigroup S is said to be *cyclic* if $S = \langle a\rangle$ for some $a \in S$ and a is called a *generator* of S. As was mentioned above, if S is a finite cyclic semigroup, index(S) and period(S) are independent of generators of S.

If S is a semigroup, $\theta \in S^1$ and define $*$ on S by

$$x * y = x\theta y \text{ for all } x, y \in S,$$

then $(S, *)$ is a semigroup which is called a *generalized semigroup* of S and we denote it by (S, θ). If $|S| = 1$ or (S, θ) has no zero, it is clear that $(S, \theta)^0 = (S \cup \{0\}, *)$ where 0 is a symbol not representing any element of S and

$$x * y = \begin{cases} x\theta y & \text{if } x, y \in S, \\ x\theta y = 0 & \text{otherwise.} \end{cases}$$

From now on, for this case, $(S, \theta)^0$ will be denoted by $(S \cup \{0\}, \theta)$. Hence the following proposition is directly obtained.

Proposition 1.1. *Let S be a semigroup and $\theta \in S^1$. If $|S| = 1$ or (S, θ) has no zero, then for all $x, y \in S \cup \{0\}, x\theta y = 0$ implies $x = 0$ or $y = 0$.*

If S has an identity and θ is a unit (an invertible element) of S, the map $x \mapsto x\theta$ is clearly an isomorphism of (S, θ) onto S, so $(S, \theta) \cong S$. In this case, θ^{-1} is the identity of (S, θ).

For a set A, let $P(A)$ denote the power set of A and let $P^*(A) = P(A) \setminus \{\phi\}$.

A *hyperoperation* $\circ$ on a nonempty set H is a mapping of $H \times H$ into $P^*(H)$.

A *hypergroupoid* is a system $(H, \circ)$ consisting of a nonempty set H together with a hyperoperation $\circ$ on H. We shall usually write H instead of $(H, \circ)$ when there is no danger of ambiguity.

Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets A, B of H, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} (a \circ b)$$

and let $A \circ x = A \circ \{x\}$ and $x \circ A = \{x\} \circ A$ for all $x \in H$. An element e of H is called an *identity* of $(H, \circ)$ if $x \in (x \circ e) \cap (e \circ x)$ for all $x \in H$. An element e of H is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$ for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is the unique identity of $(H, \circ)$.

The hyperoperation $\circ$ of a hypergroupoid $(H, \circ)$ is said to be *associative* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$.

A hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$.

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that the hyperoperation $\circ$ is associative. A semihypergroup $(H, \circ)$ is called a *hypergroup* if $H \circ x = H = x \circ H$ for all $x \in H$.

An element x of a semihypergroup $(H, \circ)$ is said to be an *inverse* of an element y in $(H, \circ)$ if there exists an identity e of $(H, \circ)$ such that $e \in (x \circ y) \cap (y \circ x)$, that is, $(x \circ y) \cap (y \circ x)$ contains at least one identity of $(H, \circ)$. Then every identity of a semihypergroup $(H, \circ)$ is an inverse of itself since $e \in e \circ e$ for every identity e of $(H, \circ)$.

A hypergroup H is said to be *regular* if every element of H has at least one inverse in H.

A regular hypergroup $(H,\circ)$ is said to be *reversible* if for $x,y,z\in H$, $x\in y\circ z$ implies $z\in u\circ x$ and $y\in x\circ v$ for some inverse u of y and inverse v of z in $(H,\circ)$.

A *canonical hypergroup* is a commutative reversible hypergroup H such that H has a scalar identity and every element of H has a unique inverse in H. Hence a hypergroup $(H,\circ)$ is a canonical hypergroup if and only if

1. $(H,\circ)$ is commutative,
2. $(H,\circ)$ has a scalar identity,
3. every element of H has a unique inverse in $(H,\circ)$ and
4. for $a,x,y\in H, y\in a\circ x$ implies $x\in a^{'}\circ y$ where $a^{'}$ denotes the unique inverse of a in $(H,\circ)$.

A (Krasner) *hyperring* is a system $(A,+,\cdot)$ such that

1. $(A,+)$ is a canonical hypergroup,
2. $(A,\cdot)$ is a semigroup with zero 0 where 0 is the scalar identity of $(A,+)$ and
3. $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(y+z)\cdot x=y\cdot x+z\cdot x$ for all $x,y,z\in A$.

The operations $+$ and $\cdot$ of a hyperring $(A,+,\cdot)$ are called the *addition* and the *multiplication* of A, respectively. We shall usually write A instead of $(A,+,\cdot)$ when there is no danger of ambiguity. Hence every ring is a hyperring.

Let $(A,+,\cdot)$ be a hyperring. The scalar identity of the canonical hypergroup $(A,+)$ which is the zero of the semigroup $(A,\cdot)$ is called the *zero* of the hyperring $(A,+,\cdot)$ and it is usually denoted by 0. For $x,y\in A$ and n a positive integer, let $-x$ denote the unique inverse of x in the canonical hypergroup $(A,+)$ which is called the *additive inverse* of x in $(A,+,\cdot)$, xy denote $x\cdot y$ and x^n denote $xx\cdots x$ (n times). Then the following statements hold.

1. $-0=0$,
2. $-(-x)=x$ for all $x\in A$,

3. $(-x)y = -(xy) = x(-y)$ for all $x, y \in A$ and
4. $(-x)(-y) = xy$ for all $x, y \in A$

([3], page 167). We give some examples of hyperrings as follows:

Example 1 ([12], page 16)**.** Define the hyperoperation $\oplus$ on $\mathbb{Z}_3$ as follows:

$\oplus$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{1\}$	$\mathbb{Z}_3$
2	$\{2\}$	$\mathbb{Z}_3$	$\{2\}$

Then $(\mathbb{Z}_3, \oplus, \cdot)$ is a hyperring where $\cdot$ is the usual multiplication in $\mathbb{Z}_3$. Observe that 0 is its zero and 1 is the additive inverse of 2 in this hyperring.

Example 2. For all $x, y \in [0, 1]$, define

$$x \oplus y = \begin{cases} \{\max\{x, y\}\} & \text{if } x \neq y, \\ [0, x] & \text{if } x = y. \end{cases}$$

From [3], page 95 - 96, $([0, 1], \oplus)$ is a canonical hypergroup. It was shown by Y. Punkla [12] that $([0, 1], \oplus, \cdot)$ is a hyperring where $\cdot$ is the usual multiplication on $[0, 1]$. In this hyperring, 0 is the zero and the additive inverse of $x \in [0, 1]$ is x itself.

Example 3. For all $x, y \in [-1, 1]$, define

$$x \oplus y = y \oplus x = \{x\} \text{ if } |y| < |x|,$$
$$x \oplus x = \{x\},$$
$$x \oplus (-x) = [-|x|, |x|\ ].$$

From [3], page 182 - 183, $([-1, 1], \oplus)$ is a canonical hypergroup. In fact, it is shown by Y. Kemprasit [8] that $([-1, 1], \oplus, \cdot)$ is a hyperring where $\cdot$ is the usual multiplication on $[-1,1]$. Note that 0 is its zero and the additive inverse of $x \in [-1,1]$ is $-x$.

Example 4. Let G be a group and define a hyperoperation $+$ on G^0 by

$$\begin{aligned} x+0 = 0+x &= \{x\} && \text{for all } x \in G^0, \\ x+x &= G^0 \setminus \{x\} && \text{for all } x \in G^0 \setminus \{0\} \text{ and} \\ x+y &= \{x,y\} && \text{for all } x,y \in G^0 \setminus \{0\} \text{ with } x \neq y. \end{aligned}$$

It is given in [3], page 170 that if G is an abelian group, then $(G^0,+,\cdot)$ is a hyperfield where $\cdot$ is the operation on G^0. A *hyperfield* is defined naturally to be a hyperring $(A,+,\cdot)$ such that $(A \setminus \{0\},\cdot)$ is an abelian group. In fact, it was proved by Y. Punkla [12] that $(G^0,+,\cdot)$ is a hyperring without assuming the commutativity of the group G. In this hyperring, 0 is the zero and the additive inverse of $x \in G^0$ is x itself.

A semigroup S is said to *admit a hyperring* [*ring*] *structure* if there exists a hyperoperation [operation] $+$ on S^0 such that $(S^0,+,\cdot)$ is a hyperring [ring]. Let $\boldsymbol{SR}$ and $\boldsymbol{SHR}$ denote the class of all semigroups admitting ring structure and the class of all semigroups admitting hyperring structure, respectively. Then $\boldsymbol{SHR}$ contains $\boldsymbol{SR}$ as a subclass. Note that for a semigroup S with $|S| = 1$, then $S^0 \cong (\mathbb{Z}_2,\cdot)$, so $S \in \boldsymbol{SR} \subseteq \boldsymbol{SHR}$. The following proposition follows from Example 4.

Proposition 1.2. *Every group belongs to* $\boldsymbol{SHR}$.

Since every finite division ring is a field (Wedderburn's Theorem for finite division rings), we deduce that every finite nonabelian group is in $\boldsymbol{SHR}$ but not in $\boldsymbol{SR}$. Consequently, $\boldsymbol{SR}$ is a proper subclass of $\boldsymbol{SHR}$.

Semigroups belonging to the class $\boldsymbol{SR}$ have long been studied. For examples, see [11], [13], [14], [1] and [15]. The purpose of this research is to study when certain semigroups belong to the class $\boldsymbol{SHR}$. However, characterizations of some semigroups in this class have been studied in [8], [9] and [12].

It was obtained from [7] by J. R. Isbell that every infinite cyclic semigroup is not in $\boldsymbol{SR}$. It was given in [11] that a finite cyclic semigroup S is in $\boldsymbol{SR}$ if and only if $|S| \leq 2$. In Chapter II, we characterize when any cyclic semigroup belongs to $\boldsymbol{SHR}$. It is shown that every infinite cyclic semigroup is in $\boldsymbol{SHR}$ and a finite cyclic semigroup S belongs to $\boldsymbol{SHR}$ if and only if index$(S) = 1$ or period$(S) = 1$.

Let X be a nonempty set. By a *transformation* of X we mean a mapping of X into itself. Let $T(X)$ denote the set of all transformations of X. Then under composition, $T(X)$ is a semigroup having 1_X as its identity where 1_X is the identity map on X and it is called the *full transformation semigroup* on X. For $\alpha \in T(X)$, let Imα denote the image of α. Then for $\alpha \in T(X), \alpha^2 = \alpha$ if and only if $x\alpha = x$ for all $x \in$ Imα. For $\alpha \in T(X)$ and $x \in X,\ \alpha$ is said to be 1 - 1 *at* x if $|(x\alpha)\alpha^{-1}| = 1$. The symmetric group on X is denoted by $G(X)$. Then

$$G(X) = \{\alpha \in T(X) \mid \alpha \text{ is 1 - 1 and } \mathrm{Im}\alpha = X\}.$$

The following two subsets of $T(X)$ are clearly subsemigroups of $T(X)$ containing $G(X)$:

$$M(X) = \{\alpha \in T(X) \mid \alpha \text{ is 1 - 1}\}$$

and

$$E(X) = \{\alpha \in T(X) \mid \mathrm{Im}\alpha = X\}.$$

Then $M(X)[E(X)] = G(X)$ if and only if X is finite. Since Im$\alpha\beta \subseteq$ Imβ for all $\alpha, \beta \in T(X)$, we have that

$$T_1(X) = \{\alpha \in T(X) \mid \mathrm{Im}\alpha \text{ is finite}\}$$

is a subsemigroup of $T(X)$ containing every constant map of X into X. Note that if X is infinite, $T_1(X)$ is an infinite
semigroup all of whose elements have finite order ([4], page 12). The subset

$$T_2(X) = \{\alpha \in T(X) \mid X \setminus \mathrm{Im}\alpha \text{ is finite}\}$$

of $T(X)$ is also considered. If $\alpha, \beta \in T(X)$, then $\mathrm{Im}\alpha\beta \subseteq \mathrm{Im}\beta \subseteq X$, so

$$\begin{aligned} X \setminus \mathrm{Im}\alpha\beta &= (X \setminus \mathrm{Im}\beta) \cup (\mathrm{Im}\beta \setminus \mathrm{Im}\alpha\beta) \\ &= (X \setminus \mathrm{Im}\beta) \cup (X\beta \setminus (X\alpha)\beta) \\ &\subseteq (X \setminus \mathrm{Im}\beta) \cup (X \setminus X\alpha)\beta. \end{aligned}$$

Consequently, $T_2(X)$ is a subsemigroup of $T(X)$ containing $E(X)$. $T_2(X)$ can be considered as the semigroup of all " almost onto transformations " of X. Then the set of all " almost 1 - 1 transformations " of X should be given as follows:

$$T_3(X) = \{\alpha \in T(X) \mid \mathrm{K}(\alpha) \text{ is finite}\}$$

where $\mathrm{K}(\alpha) = \{x \in X \mid \alpha \text{ is not 1 - 1 at } x\}$. Clearly, $\mathrm{K}(\alpha) \subseteq \mathrm{K}(\alpha\beta)$ for all $\alpha, \beta \in T(X)$. From [10], we have

$$\mathrm{K}(\alpha\beta) \subseteq \mathrm{K}(\alpha) \cup (\mathrm{K}(\beta))\alpha^{-1}$$

for all $\alpha, \beta \in T(X)$. It follows that for $\alpha, \beta \in T(X)$, if $\mathrm{K}(\alpha)$ and $\mathrm{K}(\beta)$ are finite, then $\mathrm{K}(\alpha\beta)$ is finite. Then $T_3(X)$ is a subsemigroup of $T(X)$ containing $M(X)$. Next, let

$$T_4(X) = \{\alpha \in T(X) \mid \alpha \text{ is 1 - 1 and } X \setminus \mathrm{Im}\alpha \text{ is infinite}\}$$

where X is infinite. Since X is infinite, there are subsets X_1, X_2 of X such that

$$X = X_1 \cup X_2,\ X_1 \cap X_2 = \varnothing \text{ and } |X_1| = |X| = |X_2|.$$

Then there exists a bijection $\lambda : X \to X_1$. Since $\lambda \in T(X), \lambda$ is 1 - 1 and $X \setminus \mathrm{Im}\lambda = X \setminus X_1 = X_2$ which is infinite, we have $\lambda \in T_4(X)$. This shows that $T_4(X) \neq \varnothing$. Since for $\alpha, \beta \in T(X)$, $\mathrm{Im}\alpha\beta \subseteq \mathrm{Im}\beta$, it follows that $\alpha\beta \in T_4(X)$ for all $\alpha, \beta \in T_4(X)$. Then if X is infinite, $T_4(X)$ is a subsemigroup of $T(X)$ contained in $M(X)$. If X is countably infinite, $T_4(X)$ is called the *Baer-Levi semigroup* on X and $\alpha T_4(X) = T_4(X)$ for all $\alpha \in T_4(X)$ ([4], page 14). The semigroup $T_4(X)$ motivates us to consider the set

$$T_5(X) = \{\alpha \in T(X) \mid \mathrm{K}(\alpha) \text{ is infinite and } \mathrm{Im}\alpha = X\}$$

which is defined in the opposite way. Let $X_1, X_2 \subseteq X$ be as above. Since $|X_1|$ $= |X|$, there is a bijection $\varphi : X_1 \to X$. Let $a \in X$ be fixed and define $\eta : X \to X$ by

$$x\eta = \begin{cases} x\varphi & \text{if } \ x \in X_1, \\ a & \text{if } \ x \in X_2. \end{cases}$$

Since $X_1\varphi = X$, $\mathrm{Im}\eta = X$. We can see that for every $x \in X_2, (x\eta)\eta^{-1} = a\eta^{-1}$ $= X_2$. Then $\mathrm{K}(\eta) = X_2 \cup \{a\varphi^{-1}\}$ which is infinite. Hence $\eta \in T_5(X)$, so $T_5(X)$ $\neq \varnothing$. Since $\mathrm{K}(\alpha) \subseteq \mathrm{K}(\alpha\beta)$ for all $\alpha, \beta \in T(X)$, $T_5(X)$ is a subsemigroup of $T(X)$ contained in $E(X)$.

The relations under inclusion of the transformation semigroups introduced above are as follows:

(1) If X is finite, then

$$G(X) = M(X) = E(X) \subseteq T_1(X) = T_2(X) = T_3(X) = T(X).$$

(2) If X is infinite, then

$$G(X) \subseteq E(X) \subseteq T_2(X) \subseteq T(X),\ G(X) \subseteq M(X) \subseteq T_3(X) \subseteq T(X),$$
$$T_4(X) \subseteq M(X) \subseteq T(X) \text{ and } T_5(X) \subseteq E(X) \subseteq T(X).$$

We note that if X is infinite, all the inclusions in (2) are proper. To see this, let λ and η be defined as above. Fix $b \in X$. Then $|X| = |X \setminus \{b\}|$. Let $\nu : X \to X \setminus \{b\}$ be a bijection and define $\xi : X \to X$ by

$$x\xi = \begin{cases} x\nu^{-1} & \text{if } \ x \in X \setminus \{b\}, \\ \\ b & \text{if } \ x = b. \end{cases}$$

Then $\mathrm{K}(\xi) = \{b\nu, b\}$, so $\xi \in E(X) \cap T_3(X)$. Hence we have

$$\eta \in E(X) \setminus G(X),\ \nu \in T_2(X) \setminus E(X),\ \lambda \in T(X) \setminus T_2(X),$$
$$\nu \in M(X) \setminus G(X),\ \xi \in T_3(X) \setminus M(X),\ \eta \in T(X) \setminus T_3(X),$$
$$1_X,\ \nu \in M(X) \setminus T_4(X),\ \xi \in T(X) \setminus M(X),$$
$$1_X,\ \xi \in E(X) \setminus T_5(X),\ \nu \in T(X) \setminus E(X).$$

The transformation semigroups $T(X), M(X), E(X), T_2(X)$ and $T_3(X)$ have been charactered in [8] and [12] when they belong to $\boldsymbol{SHR}$ as follows:

Proposition 1.3 ([8]). *For a nonempty set X,*

(1) $T(X) \in \boldsymbol{SHR}$ *if and only* $|X| = 1$,

(2) $M(X) \in \boldsymbol{SHR}$ *if and only if X is finite and*

(3) $E(X) \in \boldsymbol{SHR}$ *if and only if X is finite.*

Proposition 1.4 ([12]). *For a nonempty set X,*

(1) $T_2(X) \in \boldsymbol{SHR}$ *if and only if* $|X| = 1$,

(2) $T_3(X) \in \boldsymbol{SHR}$ *if and only if* $|X| = 1$.

The first main purpose of this research is the results in Chapter III. We give in Chapter III characterizations of determining when generalized semigroups of the transformation semigroups $T(X), G(X), M(X), E(X)$ and $T_1(X) - T_5(X)$ belong to $\boldsymbol{SHR}$. Proposition 1.3 and 1.4 are respectively lemmas to characterize generalized semigroups of $M(X)$ and $E(X)$ and of $T_2(X)$ and $T_3(X)$ belonging to $\boldsymbol{SHR}$. The following proposition will be useful for the characterizations in this chapter.

Proposition 1.5. *Let X be a nonempty set.*

(1) *If $S(X)$ is any of $T(X), M(X), E(X)$ and $T_1(X) - T_3(X)$ and $\theta \in S^1(X)$, then $|S(X)| = 1$ or $(S(X), \theta)$ has no zero.*

(2) *If X is infinite, $S(X)$ is $T_4(X)$ or $T_5(X)$ and $\theta \in S^1(X)$, then $(S(X), \theta)$ has no zero.*

Proof. First, recall that for $\alpha \in T(X)$, if $\alpha^2 = \alpha$, then $x\alpha = x$ for all $x \in \mathrm{Im}\alpha$.

Suppose that η is a zero of $(S(X), \theta)$. Then

$$\eta\theta\alpha = \eta = \alpha\theta\eta \text{ for all } \alpha \in S(X). \tag{1.5.1}$$

These imply that $(\eta\theta)^2 = \eta\theta$ and for every $\alpha \in S(X)$, $\mathrm{Im}\eta = \mathrm{Im}(\eta\theta\alpha) \subseteq \mathrm{Im}\alpha$. Hence we have

$$x(\eta\theta) = x \text{ for all } x \in \mathrm{Im}\eta\theta \tag{1.5.2}$$

and

$$\mathrm{Im}\eta \subseteq \mathrm{Im}\alpha \text{ for all } \alpha \in S(X). \tag{1.5.3}$$

Case 1: $S(X) = T(X)$ or $T_1(X)$ and $|S(X)| > 1$. Then $|X| > 1$. Let $a, b \in X$ be distinct. Then $X_a, X_b \in S(X)$. By (1.5.3), $\mathrm{Im}\eta \subseteq \mathrm{Im}X_a \cap \mathrm{Im}X_b = \{a\} \cap \{b\} = \varnothing$, a contradiction.

Case 2: $S(X) = M(X), E(X), T_4(X)$ or $T_5(X)$. Then $\eta\theta$ is 1 - 1 or $\mathrm{Im}\eta\theta = X$. By (1.5.2), $\eta\theta = 1_X$.

Subcase 2.1: $S(X) = M(X)$ or $E(X)$ and $|S(X)| > 1$. Let $\alpha \in S(X) \setminus \{\eta\}$. Then $\eta\theta\alpha = 1_X\alpha = \alpha$. By (1.5.1), $\eta\theta\alpha = \eta$, so $\alpha = \eta$, a contradiction.

Subcase 2.2: $S(X) = T_4(X)$ or $T_5(X)$ where X is infinite. From the proofs of those $T_4(X) \neq \varnothing$ and $T_5(X) \neq \varnothing$ in Chapter I, page 8 - 9, we can see that $|S(X)| > 1$ by interchanging X_1 and X_2. As the proof of Subcase 2.1, we also get a contradiction.

Case 3: $S(X) = T_2(X)$ or $T_3(X)$ and $|S(X)| > 1$. Then $|X| > 1$. For each $a \in X$, choose $a' \in X \setminus \{a\}$ and define $\alpha_a : X \to X$ by

$$x\alpha_a = \begin{cases} a' & \text{if } \ x = a, \\ x & \text{otherwise.} \end{cases}$$

Then for each $a \in X$, $\mathrm{Im}\alpha_a = X \setminus \{a\}$ and $\mathrm{K}(\alpha_a) = \{a, a'\}$. Hence $\alpha_a \in T_2(X) \cap T_3(X)$ for all $a \in X$. We have from (1.5.3) that

$$\mathrm{Im}\eta \subseteq \bigcap_{a \in X} \mathrm{Im}\alpha_a = \bigcap_{a \in X} (X \setminus \{a\}) = \varnothing,$$

a contradiction.

Therefore the proposition is completely proved. □

For a vector space V over a division ring, let $L(V)$ denote the set of all linear transformations from V into V. Then under composition, $L(V)$ is a semigroup having 1_V as its identity where 1_V is the identity map on V. The following three propositions are provided in this chapter. They are simple facts of vector spaces and linear transformations which will be used. The proofs are routine and elementary and they will be omitted.

Proposition 1.6. *Let $\alpha \in L(V)$ and B a basis of V. If $\alpha_{|B}$ is $1-1$ and $B\alpha$ is linearly independent, then α is $1-1$.*

Proposition 1.7. *Let B be a basis of V and $A \subseteq B$. If $\alpha \in L(V)$ is defined by*

$$v\alpha = \begin{cases} 0 & \text{if } \ v \in A, \\ v & \text{if } \ v \in B \setminus A, \end{cases}$$

then $Ker\alpha = \langle A \rangle$ *and* $Im\alpha = \langle B \setminus A \rangle$.

Proposition 1.8. *Let* B *be a basis of* V *and* $A \subseteq B$. *Then*

(1) $\{v + \langle A \rangle \mid v \in B \setminus A\}$ *is a basis of* $V \,/\, \langle A \rangle$ *and*

(2) $dim\ (V \,/\, \langle A \rangle) = |B \setminus A|$.

Let $G(V)$ denote the group of units of $L(V)$. Then

$$G(V) = \{\alpha \in L(V) \mid \alpha \text{ is } 1\text{ - }1 \text{ and } \mathrm{Im}\alpha = V\}.$$

The following two subsets of $L(V)$ are clearly subsemigroups of $L(V)$:

$$M(V) = \{\alpha \in L(V) \mid \alpha \text{ is } 1\text{ - }1\}$$

and

$$E(V) = \{\alpha \in L(V) \mid \mathrm{Im}\alpha = V\}.$$

Then $M(V)$ and $E(V)$ contain $G(V)$ as a subsemigroup and $M(V)[E(V)] = G(V)$ if and only if $\dim V$ is finite. Since $\mathrm{Im}\alpha\beta \subseteq \mathrm{Im}\beta$ for all $\alpha, \beta \in L(V)$, we have that

$$L_1(V) = \{\alpha \in L(V) \mid \dim \mathrm{Im}\alpha \text{ is finite}\}$$

is a subsemigroup of $L(V)$ containing 0.

Following $T_2(X)$ of $T(X)$, the subset

$$L_2(V) = \{\alpha \in L(V) \mid \dim\ (V \,/\, \mathrm{Im}\alpha) \text{ is finite}\}$$

of $L(V)$ is also considered. Then $E(V) \subseteq L_2(V)$. We will show that $L_2(V)$ is a subsemigroup of $L(V)$. Let V be a vector space over a division ring D. Let $\alpha, \beta \in L_2(V)$. Then $\dim\ (V \,/\, \mathrm{Im}\alpha)$ and $\dim\ (V \,/\, \mathrm{Im}\beta)$ are finite. Let $\dim\ (V \,/\, \mathrm{Im}\alpha) = n$, $\dim\ (V \,/\, \mathrm{Im}\beta) = m$ and $\{v_1 + \mathrm{Im}\alpha, ..., v_n + \mathrm{Im}\alpha\}$ and $\{w_1 + \mathrm{Im}\beta, ..., w_m + \mathrm{Im}\beta\}$ are bases of $V \,/\, \mathrm{Im}\alpha$ and $V \,/\, \mathrm{Im}\beta$, respectively. We claim that

$$\langle\{w_1+\mathrm{Im}\alpha\beta,...,w_m+\mathrm{Im}\alpha\beta,v_1\beta+\mathrm{Im}\alpha\beta,...,v_n\beta+\mathrm{Im}\alpha\beta\}\rangle \;=\; V\,/\,\mathrm{Im}\alpha\beta.$$

Step 1: We shall show that for every $v\in\mathrm{Im}\beta,\ v+\mathrm{Im}\alpha\beta\in\langle v_1\beta+\ \mathrm{Im}\alpha\beta,...,v_n\beta$ $+\ \mathrm{Im}\alpha\beta\rangle$. Let $v\in\ \mathrm{Im}\beta$. Then there exists $u\in V$ such that $v=u\beta$. Since $\{v_1+\mathrm{Im}\alpha,...,v_n+\mathrm{Im}\alpha\}$ is a basis of $V\,/\,\mathrm{Im}\alpha$, it follows that

$$u+\mathrm{Im}\alpha=\sum_{i=1}^{n}a_i(v_i+\mathrm{Im}\alpha)=\sum_{i=1}^{n}a_iv_i+\mathrm{Im}\alpha$$

for some elements $a_1,...,a_n$ of D. Then $u-\sum_{i=1}^{n}a_iv_i\in\mathrm{Im}\alpha$ and so

$$v-\sum_{i=1}^{n}a_i(v_i\beta)=(u-\sum_{i=1}^{n}a_iv_i)\beta\in(\mathrm{Im}\alpha)\beta=\mathrm{Im}\alpha\beta$$

which implies that $v+\mathrm{Im}\alpha\beta=\sum_{i=1}^{n}a_i(v_i\beta)+\mathrm{Im}\alpha\beta=\sum_{i=1}^{n}a_i(v_i\beta+\mathrm{Im}\alpha\beta)$.

Step 2: Let $v\in V$. Then $v+\mathrm{Im}\beta=\sum_{j=1}^{m}a_j(w_j+\mathrm{Im}\beta)$ for some elements $a_1,...,a_m$ of D and so $v+\mathrm{Im}\beta=\sum_{j=1}^{m}a_jw_j+\mathrm{Im}\beta$. It follows that $v-\sum_{j=1}^{m}a_jw_j\in\mathrm{Im}\beta$. By Step 1, we have that

$$(v-\sum_{j=1}^{m}a_jw_j)+\mathrm{Im}\alpha\beta=\sum_{i=1}^{n}c_i(v_i\beta+\mathrm{Im}\alpha\beta)$$

for some $c_1,...,c_n\in D$ which implies that

$$v+\mathrm{Im}\alpha\beta=\sum_{j=1}^{m}a_j(w_j+\mathrm{Im}\alpha\beta)+\sum_{i=1}^{n}c_i(v_i\beta+\mathrm{Im}\alpha\beta).$$

Hence we have the claim. It follows that dim $(V\,/\,\mathrm{Im}\alpha\beta)$ is finite. Therefore $L_2(V)$ is a subsemigroup of $L(V)$.

The subsemigroup $T_3(X)$ of $T(X)$ motivates us to consider

$$L_3(V)=\{\alpha\in L(V)\mid \dim\ \mathrm{Ker}\alpha \text{ is finite}\}.$$

Then $M(V)\subseteq L_3(V)$. To show that $L_3(V)$ is a subsemigroup of $L(V)$, let α,β $\in L_3(V)$. We claim that $\alpha_{|\mathrm{Ker}\alpha\beta}:\mathrm{Ker}\alpha\beta\longrightarrow\mathrm{Im}\alpha\cap\mathrm{Ker}\beta$ is an epimorphism and

$\text{Ker}(\alpha_{|\text{Ker}\alpha\beta}) = \text{Ker}\alpha$. It is clearly seen that $v\alpha \in \text{Im}\alpha \cap \text{Ker}\beta$ for all $v \in \text{Ker}\alpha\beta$. Let $v \in \text{Im}\alpha \cap \text{Ker}\beta$. Then $v\beta = 0$ and there exists $u \in V$ such that $u\alpha = v$. Since $u\alpha\beta = (u\alpha)\beta = v\beta = 0$, we have $u \in \text{Ker}\alpha\beta$. This shows that $\alpha_{|\text{Ker}\alpha\beta}$ is a map from $\text{Ker}\alpha\beta$ onto $\text{Im}\alpha \cap \text{Ker}\beta$. Thus $\alpha_{|\text{Ker}\alpha\beta} : \text{Ker}\alpha\beta \longrightarrow \text{Im}\alpha \cap \text{Ker}\beta$ is an epimorphism. Next, we will show that $\text{Ker}(\alpha_{|\text{Ker}\alpha\beta}) = \text{Ker}\alpha$. Trivially, $\text{Ker}(\alpha_{|\text{Ker}\alpha\beta}) \subseteq \text{Ker}\alpha$. Let $v \in \text{Ker}\alpha$. Then $v\alpha = 0$ which implies that $v\alpha\beta = 0\beta = 0$. It follows that $v \in \text{Ker}\alpha\beta$ and $v\alpha_{|\text{Ker}\alpha\beta} = v\alpha = 0$. Thus we get that $\text{Ker}(\alpha_{|\text{Ker}\alpha\beta}) = \text{Ker}\alpha$. Consequently,

$$\dim\ \text{Ker}\alpha\beta = \dim\ (\text{Im}\alpha \cap \text{Ker}\beta) + \dim\ \text{Ker}\alpha.$$

Since $\dim\ \text{Ker}\alpha$ and $\dim\ \text{Ker}\beta$ are finite, it follows that $\dim\ \text{Ker}\alpha\beta$ is finite. Therefore we have that $L_3(V)$ is a subsemigroup of $L(V)$, as required.

Next, let us consider

$$L_4(V) = \{\alpha \in L(V) \mid \alpha \text{ is 1 - 1 and } \dim\ (V \,/\, \text{Im}\alpha) \text{ is infinite}\}$$

which is motivated by $T_4(X)$ of $T(X)$ where V is infinite dimensional. Because we can define a linear transformation of V on its given basis, by the same idea of the proof of that $T_4(X) \neq \varnothing$ and the facts of Proposition 1.6 and 1.8(2), we have $L_4(V) \neq \varnothing$ where $\dim V$ is infinite. We have that $\text{Im}\alpha\beta \subseteq \text{Im}\beta$ and

$$V \,/\, \text{Im}\beta \cong (V \,/\, \text{Im}\alpha\beta) \,/\, (\text{Im}\beta \,/\, \text{Im}\alpha\beta)$$

for all $\alpha, \beta \in L_4(V)$. Since $\dim\ (V \,/\, \text{Im}\beta)$ is infinite, $\dim\ (V \,/\, \text{Im}\alpha\beta)$ is also infinite. Thus $L_4(V)$ is a subsemigroup of $L(V)$ contained in $M(V)$.

Finally, following $T_5(X)$ of $T(X)$, we put

$$L_5(V) = \{\alpha \in L(V) \mid \dim\ \text{Ker}\alpha \text{ is infinite and } \text{Im}\alpha = V\}$$

where V is infinite dimensional. The proof of that $L_5(V) \neq \varnothing$ can be given similarly to the proof of that $T_5(X) \neq \varnothing$ by defining a linear transformation of V

on its given basis and using a to be 0. Since $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}\alpha\beta$ for all $\alpha, \beta \in L(V)$, $L_5(V)$ is a subsemigroup of $L(V)$ contained in $E(V)$.

The following relations are also obtained similarly.

(1) If dim V is finite, then

$$G(V) = M(V) = E(V) \subseteq L_1(V) = L_2(V) = L(V).$$

(2) If dim V is infinite, then

$$G(V) \subsetneq E(V) \subsetneq L_2(V) \subsetneq L(V),\ G(V) \subsetneq M(V) \subsetneq L_3(V) \subsetneq L(V),$$
$$L_4(V) \subsetneq M(V) \subsetneq L(V) \text{ and } L_5(V) \subsetneq E(V) \subsetneq L(V).$$

Note that the proofs of the proper inclusions in (2) can be done similarly by defining linear transformations on bases and using Proposition 1.6, 1.7 and 1.8.

The second main purpose is the results in Chapter IV. We give in Chapter IV characterizations of determining when generalized semigroups of linear transformation semigroups $L(V), G(V), M(V), E(V)$ and $L_1(V) - L_5(V)$ belong to ***SHR***. The following Proposition will be used for the characterizations of this chapter.

Proposition 1.9. *Let V be a vector space over a division ring D.*

(1) If $S(V)$ is $M(V)$ or $E(V)$ and $\theta \in S(V)$, then $|S(V)| = 1$ or $(S(V), \theta)$ has no zero.

(2) If dim V is infinite, $S(V)$ is one of $L_2(V) - L_5(V)$ and $\theta \in S^1(V)$, then $(S(V), \theta)$ has no zero.

Proof. Assume that $(S(V), \theta)$ has a zero, say η. Then

$$\eta\theta\alpha = \eta = \alpha\theta\eta \text{ for all } \alpha \in S(V). \tag{1.9.1}$$

Consequently, $(\eta\theta)^2 = \eta\theta$ and $\mathrm{Im}\eta = \mathrm{Im}(\eta\theta\alpha) \subseteq \mathrm{Im}\alpha$ for all $\alpha \in S(V)$. Thus

$$v(\eta\theta) = v \text{ for all } v \in \ \mathrm{Im}(\eta\theta) \tag{1.9.2}$$

and

$$\mathrm{Im}\eta \subseteq \ \mathrm{Im}\alpha \text{ for all } \alpha \in S(V). \tag{1.9.3}$$

Case 1: $S(V) = M(V), E(V), L_4(V)$ or $L_5(V)$. Then $\eta\theta$ is 1 - 1 or $\mathrm{Im}\eta\theta = V$. By (1.9.2), $\eta\theta = 1_V$.

Subcase 1.1: $S(V) = M(V)$ or $E(V)$ and $|S(V)| > 1$. Let $\alpha \in S(V) \setminus \{n\}$. Then $\eta\theta\alpha = 1_V\alpha = \alpha$. By (1.9.1), $\eta\theta\alpha = \eta$. Then $\alpha = \eta$, a contradiction.

Subcase 1.2: $S(V) = L_4(V)$ or $L_5(V)$ where dim V is infinite. By the descriptions how to prove that $L_4(V)$ and $L_5(V)$ are not empty in Chapter I, page 15 - 16 and $|T_4(X)| > 1$ and $|T_5(X)| > 1$ in the proof of Proposition 1.5, one can see that $|S(V)| > 1$. From the proof of Subcase 1.1, we get a contradiction similarly.

Case 2: $S(V) = L_2(V)$ or $L_3(V)$ where dim V is infinite. Let B be a basis of V. Then B is infinite. For each $u \in B$, define $\alpha_u \in L(V)$ by

$$v\alpha_u = \begin{cases} 0 & \text{if } \ v = u, \\ v & \text{if } \ v \in B \setminus \{u\}. \end{cases}$$

By proposition 1.8, $\alpha_u \in L_2(V)$ for all $u \in B$ and by Proposition 1.7, $\alpha_u \in L_3(V)$ for all $u \in B$. From (1.9.3), we have

$$\mathrm{Im}\eta \subseteq \bigcap_{u\in B} \mathrm{Im}\alpha_u = \bigcap_{u\in B} \langle B \setminus \{u\}\rangle.$$

Let $v \in V \setminus \{0\}$. Then $v = a_1u_1 + ... + a_nu_n$ for some $u_1, ..., u_n \in B$ and nonzero $a_1, ..., a_n \in D$. If $v \in \langle B \setminus \{u_1\}\rangle$, then $a_1u_1 + ... + a_nu_n = b_1w_1 + ... + b_mw_m$ for some $w_1, ..., w_m \in B \setminus \{u_1\}, b_1, ..., b_m \in D$. Since B is linearly independent, we have $a_1 = 0$, a contradiction. Thus $v \notin \bigcap_{u\in B} \langle B \setminus \{u\}\rangle$. This proves that

$\bigcap_{u\in B}\langle B\setminus\{u\}\rangle = \langle 0\rangle$. Hence $\mathrm{Im}\eta = \{0\}$, so $\eta = 0$. Since $\dim V$ is infinite, $0 \notin L_2(V)$ and $0 \notin L_3(V)$, so we have a contradiction. $\square$

CHAPTER II

CYCLIC SEMIGROUPS

In this chapter, it will be shown that every infinite cyclic semigroup is in ***SHR***. Moreover, we shall show that for a finite cyclic semigroup S, the condition that index$(S) = 1$ or period$(S) = 1$ is necessary and sufficient for S to belong to ***SHR***.

Theorem 2.1. *Every infinite cyclic semigroup is in* ***SHR****.*

Proof. Let S be an infinite cyclic semigroup. Then there exists an element $a \in S$ such that

$$S = \{a^n \mid n \in \mathbb{N}\}.$$

Then $a^i \neq a^j$ if $i \neq j$ and so S has no zero. Define a hyperoperation $+$ on S^0 by

$$0 + 0 = \{0\},\ \ a^n + 0 = \{a^n\} = 0 + a^n,$$

$$a^n + a^m = \begin{cases} \{a^{min\{n,m\}}\} & \text{if } \ n \neq m, \\ \{a^n, a^{n+1}, ...\} \cup \{0\} & \text{if } \ n = m. \end{cases}$$

Then $(S^0, +)$ is a commutative hypergroupoid. It is clearly seen that for $x, y, z \in S^0$, if at least one of them is 0, then $(x + y) + z = x + (y + z)$. Let $n, m \in \mathbb{N}$. If $n \leq m$, then

$$(a^n + a^n) + a^m = (\{a^n, a^{n+1}, ...\} \cup \{0\}) + a^m = \{a^n, a^{n+1}, ...\} \cup \{0\}$$

and

$$a^n + (a^n + a^m) = \begin{cases} a^n + \{a^n\} & \text{if } \ n < m, \\ a^n + (\{a^n, a^{n+1}, ...\} \cup \{0\}) & \text{if } \ n = m \end{cases}$$

$$= \{a^n, a^{n+1}, ...\} \cup \{0\}.$$

If $n > m$, then

$$(a^n + a^n) + a^m = (\{a^n, a^{n+1}, ...\} \cup \{0\}) + a^m = \{a^m\}$$

and

$$a^n + (a^n + a^m) = a^n + \{a^m\} = a^n + a^m = \{a^m\}.$$

These imply that

$$(a^n + a^n) + a^m = a^n + (a^n + a^m) \tag{2.1.1}$$

for all $n, m \in \mathbb{N}$. It then follows from (2.1.1) and the commutativity of $+$ on S^0 that for $n, m \in \mathbb{N}$,

$$(a^m + a^n) + a^n = a^n + (a^m + a^n) = a^n + (a^n + a^m) = (a^n + a^n) + a^m = a^m + (a^n + a^n)$$

and

$$(a^n + a^m) + a^n = a^n + (a^n + a^m) = a^n + (a^m + a^n).$$

By the definition of $+$ on S^0, we have that for distinct elements n, m and k in $\mathbb{N}$,

$$(a^n + a^m) + a^k = \{a^{min\{n,m,k\}}\} = a^n + (a^m + a^k).$$

Then we prove that

$$(x + y) + z = x + (y + z) \text{ for all } x, y, z \in S^0$$

It is clear that

$$S^0 + x = S^0 \text{ for all } x \in S^0.$$

Hence $(S^0, +)$ is a hypergroup.

Since $0 + 0 = \{0\}$ and $0 + a^n = \{a^n\} = a^n + 0$ for all $n \in \mathbb{N}$, we have that 0 is a scalar identity of the hypergroup $(S^0, +)$. Since $0 \in \{a^n, a^{n+1}, ...\} \cup \{0\} = a^n + a^n$

for all $n \in \mathbb{N}$, it follows that for $n \in \mathbb{N}$, a^n is an inverse of a^n in $(S^0, +)$. Since 0 is the scalar identity of $(S^0, +)$, 0 is the unique inverse of 0 in $(S^0, +)$. For $n \in \mathbb{N}$, a^n is the unique inverse of a^n in $(S^0, +)$ since for every $m \in \mathbb{N} \setminus \{n\}$, $0 \notin a^n + a^m (= \{a^{min\{n,m\}}\})$.

To show that $(S^0, +)$ is reversible, it is clear that if $x, y, z \in S^0$ be such that $x \in y + z$ and at least one of them is 0, then $z \in x + y$. Next, let $n, m, k \in \mathbb{N}$ be such that $a^n \in a^m + a^k$. Then

$$a^n \in a^m + a^k = \begin{cases} \{a^m, a^{m+1}, ...\} \cup \{0\} & \text{if } \ m = k, \\ \{a^{min\{m,k\}}\} & \text{if } \ m \neq k. \end{cases}$$

We have that $n = m < k$, $n = k < m$, $n > m = k$ or $n = m = k$. Each case gives $a^k \in a^n + a^m$ as follows:

$$\begin{aligned} n = m < k \ &\Rightarrow \ a^k \in \{a^m, a^{m+1}, ...\} \cup \{0\} = a^m + a^m = a^n + a^m, \\ n = k < m \ &\Rightarrow \ a^k \in \{a^n\} = a^n + a^m, \\ n > m = k \ &\Rightarrow \ a^k \in \{a^m\} = a^n + a^m, \\ n = m = k \ &\Rightarrow \ a^k \in \{a^m, a^{m+1}, ...\} \cup \{0\} = a^m + a^m = a^n + a^m. \end{aligned}$$

This proves that $(S^0, +)$ is a canonical hypergroup.

Next, we shall show that $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in S^0$ where $\cdot$ is the operation of S^0. If $x, y, z \in S^0$ and at least one of them is 0, it is clear that $x \cdot (y + z) = x \cdot y + x \cdot z$. Let $n, m, k \in \mathbb{N}$. Then

$$a^n \cdot (a^m + a^k) = \begin{cases} a^n \cdot \{a^{min\{m,k\}}\} & \text{if } \ m \neq k, \\ a^n \cdot (\{a^m, a^{m+1}, ...\} \cup \{0\}) & \text{if } \ m = k \end{cases}$$

$$= \begin{cases} \{a^{n+min\{m,k\}}\} & \text{if } \ m \neq k, \\ \{a^{n+m}, a^{n+m+1}, ...\} \cup \{0\} & \text{if } \ m = k \end{cases}$$

$$= \begin{cases} \{a^{min\{n+m,n+k\}}\} & \text{if } \ m \neq k, \\ \{a^{n+m}, a^{n+m+1}, ...\} \cup \{0\} & \text{if } \ m = k \end{cases}$$

and

$$a^n \cdot a^m + a^n \cdot a^k = a^{n+m} + a^{n+k} = \begin{cases} \{a^{min\{n+m,n+k\}}\} & \text{if } \ m \neq k, \\ \{a^{n+m}, a^{n+m+1}, ...\} \cup \{0\} & \text{if } \ m = k. \end{cases}$$

Thus $a^n \cdot (a^m + a^k) = a^n \cdot a^m + a^n \cdot a^k$ for all $n, m, k \in \mathbb{N}$. Hence $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in S^0$.

Therefore $(S^0, +, \cdot)$ is a hyperring and so $S \in \boldsymbol{SHR}$. $\square$

Theorem 2.2. *Let S be a finite cyclic semigroup. Then $S \in \boldsymbol{SHR}$ if and only if index$(S) = 1$ or period$(S) = 1$.*

Proof. If index$(S) = 1$, then S is a finite cyclic group (Chapter I, page 2), so by Proposition 1.2, $S \in \boldsymbol{SHR}$.

Assume that period$(S) = 1$. Let $S = \langle a \rangle$ and index$(S) = r$. Then

$$S = \{a, a^2, ..., a^r\}, \ a^{r+1} = a^r,$$

$a, a^2, ..., a^r$ are all distinct and a^r which is the zero of S (Chapter I, page 1 - 2). Hence we have

$$a^t = a^r \text{ for all } t \in \{r, r+1, r+2, ...\}. \tag{2.2.1}$$

If $r = 1$, then $|S| = 1$, so $S \in \boldsymbol{SHR}$ (Chapter I, page 6). Assume that $r > 1$. Define a hyperoperation $+$ on S as follows: for $n, m \in \{1, 2, ..., r\}$,

$$a^n + a^m = \begin{cases} \{a^{min\{n,m\}}\} & \text{if } \ n \neq m, \\ \{a^n, a^{n+1}, ..., a^r\} & \text{if } \ n = m. \end{cases} \tag{2.2.2}$$

Then $(S,+)$ is a commutative hypergroupoid. From the definition of $+$ on S, we have that

$$a^n + a^r = \{a^n\} = a^r + a^n \text{ for all } n \in \{1,2,...,r\}. \tag{2.2.3}$$

Next, we claim that for all $n,m \in \mathbb{N}$,

$$a^n + a^m = \begin{cases} \{a^{min\{n,m\}}\} & \text{if } \ n \neq m, \\ \{a^n, a^{n+1}, ..., a^{n+r}\} & \text{if } \ n = m. \end{cases} \tag{2.2.4}$$

To prove (2.2.4), let $n,m \in \mathbb{N}$.

Case 1: $n \neq m$. If $n,m \leq r$, by (2.2.2), $a^n + a^m = a^{min\{n,m\}}$. If $n \leq r \leq m$, by (2.2.1) and (2.2.3), $a^n + a^m = a^n + a^r = \{a^n\} = \{a^{min\{n,m\}}\}$. Similarly, $m \leq r \leq n$ implies that $a^n + a^m = \{a^{min\{n,m\}}\}$. If $n,m \geq r$, then $\min\{n,m\} \geq r$, so by (2.2.1) and (2.2.3), $a^n + a^m = a^r + a^r = \{a^r\} = \{a^{min\{n,m\}}\}$.

Case 2: $n = m$. If $n = m \leq r$, then by (2.2.2), $a^n + a^m = \{a^n, a^{n+1}, ..., a^r\}$ and by (2.2.1), $a^r = a^{r+1} = ... = a^{n+r}$, so we have $a^n + a^m = \{a^n, a^{n+1}, ..., a^{n+r}\}$. By (2.2.1) and (2.2.3), $n = m \geq r$ implies $a^n + a^m = a^r + a^r = \{a^r\}$ and $a^r = a^n = a^{n+1} = ... = a^{n+r}$, so $a^n + a^m = \{a^r\} = \{a^n, a^{n+1}, ..., a^{n+r}\}$.

Hence (2.2.4) holds.

Let $n,m \in \{1,2,...,r\}$. Then

$$(a^n + a^n) + a^m = \{a^n, a^{n+1}, ..., a^r\} + a^m = \begin{cases} \{a^n, a^{n+1}, ..., a^r\} & \text{if } \ n \leq m, \\ \{a^m\} & \text{if } \ n > m \end{cases}$$

and

$$a^n + (a^n + a^m) = \begin{cases} a^n + \{a^n\} & \text{if } \ n < m, \\ a^n + \{a^n, a^{n+1}, ..., a^r\} & \text{if } \ n = m, \\ a^n + \{a^m\} & \text{if } \ n > m \end{cases}$$

$$= \begin{cases} \{a^n, a^{n+1}, ..., a^r\} & \text{if } n < m, \\ \{a^n, a^{n+1}, ..., a^r\} & \text{if } n = m, \\ \{a^m\} & \text{if } n > m \end{cases}$$

which implies that

$$(a^n + a^n) + a^m = a^n + (a^n + a^m) \tag{2.2.5}$$

for all $n, m \in \{1, 2, ..., r\}$. We have from (2.2.5) and the commutativity of $+$ on S that for $n, m \in \{1, 2, ..., r\}$,

$$(a^m + a^n) + a^n = a^n + (a^m + a^n) = a^n + (a^n + a^m) = (a^n + a^n) + a^m = a^m + (a^n + a^n)$$

and

$$(a^n + a^m) + a^n = a^n + (a^n + a^m) = a^n + (a^m + a^n).$$

By the definition of $+$ on S, it follows that for distinct elements n, m and k in $\{1, 2, ..., r\}$,

$$(a^n + a^m) + a^k = \{a^{min\{n,m,k\}}\} = a^n + (a^m + a^k).$$

It is clearly seen from the definition of $+$ on S that

$$S + x = S \text{ for all } x \in S.$$

Hence $(S, +)$ is a hypergroup. By (2.2.3), a^r is a scalar identity of the hypergroup $(S, +)$. Since $a^r \in \{a^n, a^{n+1}, ..., a^r\} = a^n + a^n$ for all $n \in \{1, 2, ..., r\}$, we have that for $n \in \{1, 2, ..., r\}$, a^n is an inverse of a^n in $(S, +)$. Moreover, for $n \in \{1, 2, ..., r\}$, a^n is the unique inverse of a^n in $(S, +)$ since for every $m \in \{1, 2, ..., r\} \setminus \{n\}, a^r \notin a^n + a^m (= \{a^{min\{n,m\}}\})$.

To show that $(S, +)$ is reversible, let $n, m, k \in \{1, 2, ..., r\}$ be such that $a^n \in$

$a^m + a^k$. Since

$$a^m + a^k = \begin{cases} \{a^m, a^{m+1}, ..., a^r\} & \text{if } \ m = k, \\ \{a^{min\{m,k\}}\} & \text{if } \ m \neq k, \end{cases}$$

we have that $n = m < k,\ n = k < m,\ n > m = k$ or $n = m = k$. Each case gives $a^k \in a^n + a^m$ as follows:

$$\begin{aligned} n = m < k \ &\Rightarrow\ a^k \in \{a^m, a^{m+1}, ..., a^r\} = a^m + a^m = a^n + a^m, \\ n = k < m \ &\Rightarrow\ a^k \in \{a^n\} = a^n + a^m, \\ n > m = k \ &\Rightarrow\ a^k \in \{a^m\} = a^n + a^m, \\ n = m = k \ &\Rightarrow\ a^k \in \{a^m, a^{m+1}, ..., a^r\} = a^m + a^m = a^n + a^m. \end{aligned}$$

This proves that $(S, +)$ is a canonical hypergroup.

Next, we shall show that $a^n \cdot (a^m + a^k) = a^n \cdot a^m + a^n \cdot a^k$ for all $n, m, k \in \{1, 2, ..., r\}$ where $\cdot$ is the operation of S. Let $n, m, k \in \{1, 2, ..., r\}$. Then by (2.2.4)

$$\begin{aligned} a^n \cdot (a^m + a^k) \ &= \begin{cases} a^n \cdot \{a^{min\{m,k\}}\} & \text{if } \ m \neq k, \\ a^n \cdot \{a^m, a^{m+1}, ..., a^{m+r}\} & \text{if } \ m = k \end{cases} \\ &= \begin{cases} \{a^{min\{n+m,n+k\}}\} & \text{if } \ m \neq k, \\ \{a^{n+m}, a^{n+m+1}, ..., a^{n+m+r}\} & \text{if } \ m = k \end{cases} \end{aligned}$$

and

$$a^n \cdot a^m + a^n \cdot a^k = a^{n+m} + a^{n+k} = \begin{cases} \{a^{min\{n+m,n+k\}}\} & \text{if } \ n+m \neq n+k, \\ \{a^{n+m}, a^{n+m+1}, ..., a^{n+m+r}\} & \text{if } \ n+m = n+k \end{cases}$$

$$= \begin{cases} \{a^{min\{n+m,n+k\}}\} & \text{if } \ m \neq k, \\ \{a^{n+m}, a^{n+m+1}, ..., a^{n+m+r}\} & \text{if } \ m = k. \end{cases}$$

Thus $a^n \cdot (a^m + a^k) = a^n \cdot a^m + a^n \cdot a^k$ for all $n, m, k \in \{1, 2, ..., r\}$.

Hence $(S, +, \cdot)$ is a hyperring. Therefore $S \in \boldsymbol{SHR}$.

For the converse, assume that $S = \langle a \rangle$, index$(S) = r > 1$ and period$(S) = m > 1$. Then

$$S = \{a, a^2, ..., a^r, a^{r+1}, ..., a^{r+m-1}\},\ a^{r+m} = a^r, r > 1, m > 1$$

and $a, a^2, ..., a^{r+m-1}$ are all distinct. Since period$(S) > 1, S$ has no zero (Chapter I, page 2). Consequently, for $x, y \in S^0, xy = 0$ implies $x = 0$ or $y = 0$. To show that $S \notin \boldsymbol{SHR}$, suppose on the contrary that $S \in \boldsymbol{SHR}$. Then there exists a hyperoperation $+$ on S^0 such that $(S^0, +, \cdot)$ is a hyperring where $\cdot$ is the operation on S^0. Then $0 \in a + a^k$ for some $k \in \{1, 2, ..., r + m - 1\}$.

Case 1: $k = 1$. Then $0 \in a + a$. Consequently,

$$0 \in a^{r-1}(a + a) = a^r + a^r = a^r + a^{r+m} = a^{r-1}(a + a^{m+1}).$$

This implies that $0 \in a + a^{m+1}$. Then $a^{m+1} = a$. But $1 < m + 1 \leq r + m - 1$ (since $r > 1$), so we have $a \neq a^{m+1}$, a contradiction.

Case 2: $k > 1$. Then $2k - 1 > 1$ and

$$0 \in a^{k-1}(a + a^k) = a^k + a^{2k-1}.$$

This implies that $a = a^{2k-1}$.

Subcase 2.1: $2k - 1 < r$. Then $1 < 2k - 1 < r + m - 1$, so $a^{2k-1} \neq a$, a contradiction.

Subcase 2.2: $2k - 1 \geq r$. Then $a^{2k-1} \in \{a^r, a^{r+1}, ..., a^{r+m-1}\}$ (Chapter I, page 1) which implies that $a \in \{a^r, a^{r+1}, ..., a^{r+m-1}\}$. This is a contradiction since $r > 1$. $\square$

CHAPTER III

GENERALIZED SEMIGROUPS OF SOME SEMIGROUPS OF TRANSFORMATIONS OF A SET

The purpose of this chapter is to characterize when generalized semigroups of the following transformation semigroups admit a hyperring structure where X is a nonempty set.

$T(X)$ = the full transformation semigroup on X,

$G(X) = \{\alpha \in T(X) \mid \alpha \text{ is } 1 - 1 \text{ and } \mathrm{Im}\alpha = X\}$,

$M(X) = \{\alpha \in T(X) \mid \alpha \text{ is } 1 - 1\}$,

$E(X) = \{\alpha \in T(X) \mid \mathrm{Im}\alpha = X\}$,

$T_1(X) = \{\alpha \in T(X) \mid \mathrm{Im}\alpha \text{ is finite}\}$,

$T_2(X) = \{\alpha \in T(X) \mid X \setminus \mathrm{Im}\alpha \text{ is finite}\}$,

$T_3(X) = \{\alpha \in T(X) \mid \mathrm{K}(\alpha) \text{ is finite}\}$

where $\mathrm{K}(\alpha) = \{x \in X \mid \alpha \text{ is not } 1 - 1 \text{ at } x\}$,

$T_4(X) = \{\alpha \in T(X) \mid \alpha \text{ is } 1 - 1 \text{ and } X \setminus \mathrm{Im}\alpha \text{ is infinite}\}$ where X is infinite and

$T_5(X) = \{\alpha \in T(X) \mid \mathrm{K}(\alpha) \text{ is infinite and } \mathrm{Im}\alpha = X\}$ where X is infinite.

We recall from Chapter I that $G(X) \in \boldsymbol{SHR}$ (Proposition 1.2), $M(X) \in \boldsymbol{SHR}$ if and only if X is finite (Proposition 1.3(2)) and $E(X) \in \boldsymbol{SHR}$ if and only if X is finite (Proposition 1.3(3)). Moreover, the condition that $|X| = 1$ is necessary and sufficient for $T_2(X)$ and for $T_3(X)$ to belong to $\boldsymbol{SHR}$ (Proposition 1.4).

Throughout this chapter, let X denote a nonempty set. For convenience, the following notation will be used. For $a \in X$, let X_a denote the constant transformation of X with image $\{a\}$ and for $a, b \in X$, let (a, b) be the element of $G(X)$ defined by

$$x(a,b) = \begin{cases} b & \text{if } x = a, \\ a & \text{if } x = b, \\ x & \text{if } x \in X \setminus \{a,b\}. \end{cases}$$

If $\theta \in G(X)$, we have $(G(X), \theta) \cong G(X)$ (Chapter I, page 2). Then the following theorem is obtained from Proposition 1.2.

Theorem 3.1. *For any $\theta \in G(X), (G(X), \theta) \in$ **SHR**.*

We recall that if $S(X)$ is any of the semigroups $T(X), M(X), E(X)$ and $T_1(X) - T_5(X)$ and $\theta \in S^1(X)$, by Proposition 1.5, $|S(X)| = 1$ or $(S(X), \theta)$ has no zero. Hence $(S(X), \theta)^0 = (S(X) \cup \{0\}, \theta)$ (Chapter I, page 2).

Theorem 3.2. *Let $S(X)$ be $T(X)$ or $T_1(X)$. For $\theta \in S^1(X), (S(X), \theta) \in$ **SHR** if and only if $|X| = 1$.*

Proof. Assume that $(S(X), \theta) \in$ ***SHR***. Then there exists a hyperoperation $+$ on $S(X) \cup \{0\}$ such that $(S(X) \cup \{0\}, +, \cdot)$ is a hyperring where $\cdot$ is the operation on $(S(X) \cup \{0\}, \theta)$. To show that $|X| = 1$, suppose on the contrary that $|X| \geq 2$. Let a and b be two distinct elements in X. Then $X_a, X_b \in S(X)$ and it is easily seen that $X_a \theta X_a = X_a$ and $X_b \theta X_a = X_a$. Thus we have

$$0 \in X_a - X_a = X_a \theta X_a - X_b \theta X_a = (X_a - X_b) \theta X_a$$

which implies by Proposition 1.1 and 1.5 that $0 \in X_a - X_b$. Hence $X_a = X_b$ which is a contradiction since $a \neq b$. Hence $|X| = 1$.

Conversely, if $|X| = 1$, then $|S(X)| = 1$, so $(S(X), \theta) \in$ ***SHR*** (Chapter I, page 6). $\square$

Hence Proposition 1.3(1) becomes a corollary of Theorem 3.2.

Corollary 3.3. *If $S(X) = T(X)$ or $T_1(X)$, then $S(X) \in \boldsymbol{SHR}$ if and only if $|X| = 1$.*

Theorem 3.4. *For $\theta \in M(X), (M(X), \theta) \in \boldsymbol{SHR}$ if and only if X is finite.*

Proof. If X is finite, then $M(X) = G(X)$, so by Theorem 3.1, $(M(X), \theta) \in \boldsymbol{SHR}$.

For the converse, assume that $(M(X), \theta) \in \boldsymbol{SHR}$. Then there exists a hyperoperation $+$ on $M(X) \cup \{0\}$ such that $(M(X) \cup \{0\}, +, \cdot)$ is a hyperring where $\cdot$ is the operation on $(M(X) \cup \{0\}, \theta)$. To show that $\mathrm{Im}\theta = X$, suppose that $\mathrm{Im}\theta \subsetneq X$. Since θ is 1 - 1, we have that $(\mathrm{Im}\theta)\theta \subsetneq X\theta$. Thus $\mathrm{Im}\theta^2 \subsetneq \mathrm{Im}\theta \subsetneq X$. This implies that $|X \setminus \mathrm{Im}\theta^2| \geq 2$. Let $a, b \in X \setminus \mathrm{Im}\theta^2$ be distinct. Consequently, $\theta^2(a, b) = \theta^2$. Since

$$0 \in \theta^2 - \theta^2 = \theta^2 - \theta^2(a, b) = \theta\theta(1_X - (a, b)),$$

by Proposition 1.1 and 1.5, $0 \in 1_X - (a, b)$. This implies that $(a, b) = 1_X$ which is a contradiction. Hence $\theta \in G(X)$. Then $(M(X), \theta) \cong M(X)$ (Chapter I, page 2). Since $(M(X), \theta) \in \boldsymbol{SHR}$, $M(X) \in \boldsymbol{SHR}$. By Proposition 1.3(2), X is finite. □

Theorem 3.5. *For $\theta \in E(X), (E(X), \theta) \in \boldsymbol{SHR}$ if and only if X is finite.*

Proof. If X is finite, then $E(X) = G(X)$ and hence $(E(X), \theta) \in \boldsymbol{SHR}$ by Theorem 3.1.

For the converse, assume that $(E(X), \theta) \in \boldsymbol{SHR}$. Then there exists a hyperoperation $+$ on $E(X) \cup \{0\}$ such that $(E(X) \cup \{0\}, +, \cdot)$ is a hyperring where $\cdot$ is the operation on $(E(X) \cup \{0\}, \theta)$. To show that $\theta \in G(X)$, suppose not. Because $\mathrm{Im}\theta = X$, θ is not 1 - 1. Then there exist distinct elements a and b in X such that $a\theta = b\theta$. Consequently, $(a, b)\theta = \theta$. Then

$$0 \in \theta - \theta = \theta - (a, b)\theta = (1_X - (a, b))\theta 1_X$$

which implies by Proposition 1.1 and 1.5 that $0 \in 1_X - (a,b)$. Hence $(a,b) = 1_X$, a contradiction. Therefore $\theta \in G(X)$. It follows that $(E(X),\theta) \cong E(X)$(Chapter I, page 2). But $(E(X),\theta) \in \boldsymbol{SHR}$, so $E(X) \in \boldsymbol{SHR}$. Hence X is finite by Proposition 1.3(3). $\square$

Theorem 3.6. *Let $S(X)$ be $T_2(X)$ or $T_3(X)$. For $\theta \in S(X), (S(X),\theta) \in \boldsymbol{SHR}$ if and only if $|X| = 1$.*

Proof. Assume that $(S(X),\theta) \in \boldsymbol{SHR}$ and $+$ is a hyperoperation on $S(X) \cup \{0\}$ such that $(S(X) \cup \{0\}, +, \cdot)$ is a hyperring where $\cdot$ is the operation on $(S(X) \cup \{0\}, \theta)$. First, we will prove that θ is 1 - 1. Suppose not. Then there exist distinct elements a and b in X such that $a\theta = b\theta$. Therefore $(a,b) \in S$ and $(a,b)\theta = \theta$. Then we have

$$0 \in \theta - \theta = \theta - (a,b)\theta = (1_X - (a,b))\theta 1_X$$

which implies by Proposition 1.1 and 1.5 that $0 \in 1_X - (a,b)$, so $(a,b) = 1_X$, a contradiction. Hence θ is $1 - 1$.

Next, we will prove that $\mathrm{Im}\theta = X$. Suppose that $\mathrm{Im}\theta \subsetneq X$. Since θ is 1 - 1, $(\mathrm{Im}\theta)\theta \subsetneq X\theta$. Then $\mathrm{Im}\theta^2 \subsetneq \mathrm{Im}\theta \subsetneq X$. Let $a, b \in X \setminus \mathrm{Im}\theta^2$ be distinct. Then $\theta^2(a,b) = \theta^2$. Since

$$0 \in \theta^2 - \theta^2 = \theta^2 - \theta^2(a,b) = \theta\theta(1_X - (a,b)),$$

we have $0 \in 1_X - (a,b)$. Hence $(a,b) = 1_X$, a contradiction. This proves that $\theta \in G(X)$. Consequently, $(S(X),\theta) \cong S(X)$ and hence $S(X) \in \boldsymbol{SHR}$. By Proposition 1.4, $|X| = 1$.

For the converse, assume that $|X| = 1$. Then $|S(X)| = 1$ and so $(S(X),\theta) \in \boldsymbol{SHR}$ (Chapter I, page 6). $\square$

The following lemma is required to prove that $(T_4(X),\theta) \notin \boldsymbol{SHR}$ if X is infinite and $\theta \in T_4^1(X)$.

Lemma 3.7. *$M(X)T_4(X) \subseteq T_4(X)$ where X is infinite.*

Proof. Let $\alpha \in M(X)$ and $\beta \in T_4(X)$. Then $\alpha\beta$ is 1 - 1 and Im$\alpha\beta \subseteq$ Imβ. Since $X \setminus$ Imβ is infinite, $X \setminus$ Im$\alpha\beta$ is infinite. Hence $\alpha\beta \in T_4(X)$. □

Theorem 3.8. *For any $\theta \in T_4^1(X)$, $(T_4(X), \theta) \notin$ **SHR** where X is infinite.*

Proof. Assume that $(T_4(X), \theta) \in$ ***SHR*** and let $+$ be a hyperoperation on $T_4(X) \cup \{0\}$ such that $(T_4(X) \cup \{0\}, +, \cdot)$ is a hyperring where $\cdot$ is the operation on $(T_4(X) \cup \{0\}, \theta)$. Let $\alpha \in T_4(X)$. Then $\alpha\theta \in T_4(X)$, so $X \setminus$ Im$\alpha\theta$ is infinite. Let a and b be distinct elements in $X \setminus$ Im$\alpha\theta$. Then $\alpha\theta(a, b) = \alpha\theta$, and so $\alpha\theta(a, b)\alpha = \alpha\theta\alpha$. By Lemma 3.7, we have that $(a, b)\alpha \in T_4(X)$. But

$$0 \in \alpha\theta\alpha - \alpha\theta\alpha = \alpha\theta\alpha - \alpha\theta(a, b)\alpha = \alpha\theta(\alpha - (a, b)\alpha),$$

so $0 \in \alpha - (a, b)\alpha$ by Proposition 1.1 and 1.5. This implies that $(a, b)\alpha = \alpha$. Hence

$$a\alpha = a(a, b)\alpha = b\alpha$$

which is a contradiction since $a \neq b$ and α is 1 - 1. □

Hence the following corollary is obtained.

Corollary 3.9. *If X is infinite, then $T_4(X) \notin$ **SHR**.*

The following lemma is given to prove that $(T_5(X), \theta) \notin$ ***SHR*** where X is infinite and $\theta \in T_5^1(X)$.

Lemma 3.10. *If X is infinite, then $T_5(X)E(X) \subseteq T_5(X)$.*

Proof. Let $\alpha \in T_5(X)$ and $\beta \in E(X)$. Then Im$\alpha\beta = X$ since Im$\alpha = X =$Imβ. If α is not 1 - 1 at $x \in X$, then $\alpha\beta$ is not 1 - 1 at x. Consequently,

K$(\alpha) = \{x \in X \mid \alpha$ is not 1 - 1 at $x\} \subseteq \{x \in X \mid \alpha\beta$ is not 1 - 1 at $x\} =$ K$(\alpha\beta)$.

Since K(α) is infinite, K$(\alpha\beta)$ is infinite. Hence $\alpha\beta \in T_5(X)$. □

Theorem 3.11. *For any $\theta \in T_5^1(X)$, $(T_5(X), \theta) \notin$ **SHR** where X is infinite.*

Proof. Assume that $(T_5(X), \theta) \in$ **SHR**. Then there exists a hyperoperation $+$ on $T_5(X) \cup \{0\}$ such that $(T_5(X) \cup \{0\}, +, \cdot)$ is a hyperring where $\cdot$ is the operation on $(T_5(X) \cup \{0\}, \theta)$.

Case 1: $\theta = 1_X$. Since X is infinite, there exist $X_1, X_2 \subseteq X$ such that

$$X_1 \cup X_2 = X,\ X_1 \cap X_2 = \varnothing,\ |X_1| = |X_2| = |X|.$$

Then there is a bijection $\varphi : X_1 \to X$. Let $a \in X_1$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} x\varphi & \text{if } x \in X_1, \\ a & \text{if } x \in X_2. \end{cases}$$

Thus Im$\alpha = X$ and α is not 1 - 1 at every $x \in X_2$. Therefore $\alpha \in T_5(X)$. Let $s, t \in X_2$ be such that $s \neq t$. Thus there exist unique $p, q \in X_1$ such that $p\alpha = s$ and $q\alpha = t$. Moreover, the following equalities hold.

$$\begin{aligned} p\alpha(s,t)\alpha &= a = p\alpha\alpha, \\ q\alpha(s,t)\alpha &= a = q\alpha\alpha, \\ X_2\alpha(s,t)\alpha &= \{a\alpha\} = X_2\alpha\alpha. \end{aligned} \tag{2.10.1}$$

Since $\varphi : X_1 \to X$ is 1 - 1, for $x \in X_1 \setminus \{p, q\}$, $x\alpha \notin \{s, t\}$. Then

$$\text{for every } x \in X_1 \setminus \{p,q\},\ x\alpha(s,t)\alpha = x\alpha\alpha. \tag{2.10.2}$$

From (2.10.1) and (2.10.2), we have $\alpha(s,t)\alpha = \alpha^2$. By Lemma 3.10, $\alpha(s,t) \in T_5(X)$. But

$$0 \in \alpha\alpha - \alpha\alpha = \alpha\alpha - \alpha(s,t)\alpha = (\alpha - \alpha(s,t))\alpha,$$

so by Proposition 1.1 and 1.5, $0 \in \alpha - \alpha(s,t)$. Hence $\alpha(s,t) = \alpha$. It then follows that

$$s = p\alpha = p\alpha(s,t) = s(s,t) = t,$$

a contradiction.

Case 2: $\theta \in T_5(X)$. Then $s\theta = t\theta$ for some distinct $s, t \in X$. Since $X \setminus \{s,t\}$ is infinite, there exist $X_1, X_2' \subseteq X \setminus \{s,t\}$ such

$$X_1 \cup X_2' = X \setminus \{s,t\},\ X_1 \cap X_2' = \varnothing,\ |X_1| = |X_2'| = |X \setminus \{s,t\}|.$$

Let $X_2 = X_2' \cup \{s,t\}$. Then

$$X_1 \cup X_2 = X,\ X_1 \cap X_2 = \varnothing,\ |X_1| = |X_2| = |X|.$$

Let $\varphi : X_1 \to X$ be a bijection. Let $a \in X_1$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} x\varphi & \text{if } x \in X_1, \\ a & \text{if } x \in X_2. \end{cases}$$

Then $\mathrm{Im}\alpha = X$ and α is not 1 - 1 at every $x \in X_2$, so $\alpha \in T_5(X)$. By Lemma 3.10, $\alpha(s,t) \in T_5(X)$. Let $p, q \in X_1$ be such that $p\alpha = s$ and $q\alpha = t$. Hence

$$\begin{aligned} p\alpha(s,t)\theta\alpha &= t\theta\alpha = s\theta\alpha = p\alpha\theta\alpha, \\ q\alpha(s,t)\theta\alpha &= s\theta\alpha = t\theta\alpha = q\alpha\theta\alpha, \\ X_2\alpha(s,t)\theta\alpha &= \{a\theta\alpha\} = X_2\alpha\theta\alpha. \end{aligned} \tag{2.10.3}$$

Since $\varphi : X_1 \to X$ is 1 - 1, for $x \in X_1 \setminus \{p,q\},\ x\alpha \notin \{s,t\}$. We deduce that

$$\text{for every } x \in X_1 \setminus \{p,q\},\ x\alpha(s,t)\theta\alpha = x\alpha\theta\alpha. \tag{2.10.4}$$

From (2.10.3) and (2.10.4), $\alpha(s,t)\theta\alpha = \alpha\theta\alpha$. Then

$$0 \in \alpha\theta\alpha - \alpha\theta\alpha = \alpha\theta\alpha - \alpha(s,t)\theta\alpha = (\alpha - \alpha(s,t))\theta\alpha,$$

so $0 \in \alpha - \alpha(s,t)$ by Proposition 1.1 and 1.5. Thus $\alpha(s,t) = \alpha$. This is a contradiction since

$$s = p\alpha = p\alpha(s,t) = s(s,t) = t. \qquad \square$$

In particular, we have

Corollary 3.12. *$T_5(X) \notin$ **SHR** where X is infinite.*

CHAPTER IV

GENERALIZED SEMIGROUPS OF SOME SEMIGROUPS OF LINEAR TRANSFORMATIONS OF A VECTOR SPACE

In this chapter, we characterize when generalized semigroups of the following semigroups of linear transformations under composition belong to the class ***SHR*** where V is a vector space over a division ring D.

$L(V) = \{\alpha : V \to V \mid \alpha \text{ is a linear transformation}\}$,

$G(V) = \{\alpha \in L(V) \mid \alpha \text{ is 1 - 1 and } \mathrm{Im}\alpha = V\}$,

$M(V) = \{\alpha \in L(V) \mid \alpha \text{ is 1 - 1}\}$,

$E(V) = \{\alpha \in L(V) \mid \mathrm{Im}\alpha = V\}$,

$L_1(V) = \{\alpha \in L(V) \mid \dim \mathrm{Im}\alpha \text{ is finite}\}$,

$L_2(V) = \{\alpha \in L(V) \mid \dim (V / \mathrm{Im}\alpha) \text{ is finite}\}$,

$L_3(V) = \{\alpha \in L(V) \mid \dim \mathrm{Ker}\alpha \text{ is finite}\}$,

$L_4(V) = \{\alpha \in L(V) \mid \alpha \text{ is 1 - 1 and } \dim (V / \mathrm{Im}\alpha) \text{ is infinite}\}$ if V is infinite dimensional and

$L_5(V) = \{\alpha \in L(V) \mid \dim \mathrm{Ker}\alpha \text{ is infinite and } \mathrm{Im}\alpha = V\}$ if V is infinite dimensional.

Throughout this chapter, let V be a vector space over a division ring D. The following notation will be used. If B is a basis of V and $u, w \in B$, let

$(u,w)_B \in L(V)$ be defined by

$$v(u,w)_B = \begin{cases} u & \text{if } v = w, \\ w & \text{if } v = u, \\ v & \text{if } v \in B \setminus \{u,w\}. \end{cases}$$

If the notation $(A, \oplus, \cdot)$ is used to denote a hyperring, then for $x, y \in A$, $\ominus x$ and $x \ominus y$ will denote the inverse of x in $(A, \oplus)$ and $x \oplus (\ominus y)$, respectively.

By Proposition 1.2, $G(V) \in \boldsymbol{SHR}$. It follows from the fact in Chapter I, page 2 that $(G(V), \theta) \cong G(V)$ for all $\theta \in G(V)$. Therefore we have

Theorem 4.1. *For $\theta \in G(V)$, $(G(V), \theta) \in \boldsymbol{SHR}$.*

We know that $L(V)$ is a ring under usual addition and composition. Moreover, $L_1(V)$ is an ideal of this ring ([6], page 424). Thus $L(V), L_1(V) \in \boldsymbol{SR}$ and so $L(V), L_1(V) \in \boldsymbol{SHR}$. Since for $\alpha, \beta, \gamma, \theta \in L(V)$,

$$\alpha\theta(\beta + \gamma) = \alpha\theta\beta + \alpha\theta\gamma$$

and

$$(\beta + \gamma)\theta\alpha = \beta\theta\alpha + \gamma\theta\alpha.$$

Hence $(L(V), \theta) \in \boldsymbol{SR}$ for every $\theta \in L(V)$ and $(L_1(V), \theta) \in \boldsymbol{SR}$ for every $\theta \in L_1^1(V)$. Consequently, we have

Theorem 4.2. *If $S(V)$ is $L(V)$ or $L_1(V)$, then for every $\theta \in S^1(V), (S(V), \theta) \in \boldsymbol{SHR}$.*

We recall the facts from Proposition 1.9 that if $S(V)$ is any of

(1) $M(V)$ and $E(V)$ and

(2) $L_2(V) - L_5(V)$ where $\dim V$ is infinite

and $\theta \in S^1(V)$, then $|S(V)| = 1$ or $(S(V), \theta)$ has no zero, and hence $(S(V), \theta)^0 = (S(V) \cup \{0\}, \theta)$ (Chapter I, page 2).

To characterize when $(M(V),\theta)$ with $\theta \in M(V)$ belongs to $\boldsymbol{SHR}$, we require the following lemma.

Lemma 4.3. *$M(V) \in \boldsymbol{SHR}$ if and only if dim V is finite.*

Proof. If dim V is finite, then $M(V) = G(V)$, so by Proposition 1.2, $M(V) \in \boldsymbol{SHR}$.

For the converse, assume that $M(V) \in \boldsymbol{SHR}$. Then there exists a hyperoperation $\oplus$ on $M(V) \cup \{0\}$ such that $(M(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(M(V) \cup \{0\}, 1_V)$. To show that dim V is finite, suppose on the contrary that dim V is infinite. Let B be a basis of V and $u, w \in B$ such that $u \neq w$. Since B is infinite, it follows that $|B| = |B \setminus \{u, w\}|$. Then there exists a 1 - 1 map φ from B onto $B \setminus \{u, w\}$. Let $\alpha \in L(V)$ be defined by $v\alpha = v\varphi$ for all $v \in B$. By Proposition 1.6, α is 1 - 1 and so $\alpha \in M(V)$. Since $B\alpha = B \setminus \{u, w\}$, we have $v\alpha(u, w)_B = v\alpha$ for all $v \in B$. It follows that $\alpha(u, w)_B = \alpha$. Therefore

$$0 \in \alpha \ominus \alpha = \alpha 1_V \ominus \alpha(u, w)_B = \alpha(1_V \ominus (u, w)_B).$$

But $\alpha \neq 0$, so by Proposition 1.1 and 1.9, $0 \in 1_V \ominus (u, w)_B$. Hence $(u, w)_B = 1_V$, a contradiction. Therefore dim V is finite. $\square$

Theorem 4.4. *For $\theta \in M(V)$, $(M(V), \theta) \in \boldsymbol{SHR}$ if and only if dim V is finite.*

Proof. First, we recall that if A is a linearly independent subset of V and $u \in V \setminus \langle A \rangle$, then $A \cup \{u\}$ is linearly independent.

If dim V is finite, then $M(V) = G(V)$ and thus $(M(V), \theta) \in \boldsymbol{SHR}$ by Theorem 4.1.

For the converse, assume that $(M(V), \theta) \in \boldsymbol{SHR}$. Let $\oplus$ be a hyperoperation on $M(V) \cup \{0\}$ such that $(M(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(M(V) \cup \{0\}, \theta)$. To show that $\theta \in E(V)$, suppose on the contrary that

$\theta \notin E(V)$. Then $\mathrm{Im}\alpha \subsetneq V$. Since θ is 1 - 1, we have $(\mathrm{Im}\theta)\theta \subsetneq V\theta$. Hence $\mathrm{Im}\theta^2 \subsetneq \mathrm{Im}\theta \subsetneq V$.

Next, let $u \in V \setminus \mathrm{Im}\theta, w \in \mathrm{Im}\theta \setminus \mathrm{Im}\theta^2$ and B_1 a basis of $\mathrm{Im}\theta^2$. Then $w \in V \setminus \langle B_1 \rangle$. It follows that $B_1 \cup \{w\}$ is linearly independent. But $\langle B_1 \cup \{w\} \rangle \subseteq \mathrm{Im}\theta$, so $u \in V \setminus \langle B_1 \cup \{w\} \rangle$. It follows that $B_1 \cup \{u, w\}$ is linearly independent. Let B be a basis of V containing $B_1 \cup \{u, w\}$. Since $u, w \notin \langle B_1 \rangle$ and for $v \in V, v\theta^2 \in \langle B_1 \rangle$, we deduce that $v\theta^2(u, w)_B = v\theta^2$ for all $v \in V$. Hence $\theta^2(u, w)_B = \theta^2$. But

$$0 \in \theta^2 \ominus \theta^2 = \theta^2 \ominus \theta^2(u, w)_B = \theta\theta(1_V \ominus (u, w)_B),$$

so $0 \in 1_V \ominus (u, w)_B$ by Proposition 1.1 and 1.9. Consequently, $(u, w)_B = 1_V$, a contradiction. Now, we have $\theta \in G(V)$. It follows that $(M(V), \theta) \cong M(V)$ (Chapter I, page 2). Therefore $M(V) \in \boldsymbol{SHR}$. By Lemma 4.3, dim V is finite. $\square$

Next, we shall prove that $(E(V), \theta)$ with $\theta \in E(V)$ belongs to $\boldsymbol{SHR}$ if and only if dim V is finite. The following lemma is required.

Lemma 4.5. $E(V) \in \boldsymbol{SHR}$ *if and only if dim* V *is finite.*

Proof. If dim V is finite, then $E(V) = G(V)$, so by Proposition 1.2, $E(V) \in \boldsymbol{SHR}$.

For the converse, assume that $E(V) \in \boldsymbol{SHR}$ and let $\oplus$ be a hyperoperation on $E(V) \cup \{0\}$ such that $(E(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(E(V) \cup \{0\}, 1_V)$. To show that dim V is finite, suppose on the contrary that dim V is infinite. Let B be a basis of V and let $u, w \in B$ be such that $u \neq w$. Since B is infinite, it follows that $|B| = |B \setminus \{u, w\}|$. Then there exists a 1 - 1 map φ from $B \setminus \{u, w\}$ onto B. Let $\alpha \in L(V)$ be defined by

$$v\alpha = \begin{cases} v\varphi & \text{if } \ v \in B \setminus \{u, w\}, \\ u & \text{if } \ v = u \text{ or } v = w. \end{cases}$$

Then $\mathrm{Im}\alpha = \langle B \rangle = V$, so $\alpha \in E(V)$. Moreover,

$$\begin{aligned} v(u,w)_B\alpha &= v\alpha \text{ for all } v \in B \setminus \{u,w\}, \\ u(u,w)_B\alpha &= w\alpha = u = u\alpha \text{ and} \\ w(u,w)_B\alpha &= u\alpha = u = w\alpha. \end{aligned}$$

It follows that $(u,w)_B\alpha = \alpha$. Hence

$$0 \in \alpha \ominus \alpha = \alpha \ominus (u,w)_B\alpha = (1_V \ominus (u,w)_B)\alpha$$

which implies by Proposition 1.1 and 1.9 that $0 \in 1_V \ominus (u,w)_B$. Therefore $(u,w)_B = 1_V$, a contradiction. This proves that dim V is finite. □

Theorem 4.6. *For $\theta \in E(V), (E(V),\theta) \in$ **SHR** if and only if dim V is finite.*

Proof. If dim V is finite, then $E(V) = G(V)$, so by Theorem 4.1, $(E(V),\theta) \in$ ***SHR***.

For the converse, assume that $(E(V),\theta) \in$ ***SHR***. Let $\oplus$ be a hyperoperation on $E(V) \cup \{0\}$ such that $(E(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(E(V) \cup \{0\}, \theta)$. To show that $\theta \in M(V)$, suppose that $\theta \notin M(V)$. Then $\mathrm{Ker}\theta \neq \{0\}$. Let B_1 be a basis of $\mathrm{Ker}\theta$ and B a basis of V such that $B_1 \subseteq B$. Since $\mathrm{Ker}\theta \neq \{0\}$, it follows that $B_1 \neq \varnothing$. Let $\alpha \in L(V)$ be defined by

$$v\alpha = \begin{cases} v\theta & \text{if } v \in B \setminus B_1, \\ v & \text{if } v \in B_1. \end{cases}$$

Then $\alpha \neq \theta$. Since

$$\begin{aligned} \mathrm{Im}\alpha = V\alpha = \langle B\alpha \rangle & \\ &= \langle (B \setminus B_1)\alpha \cup B_1\alpha \rangle \\ &\supseteq \langle (B \setminus B_1)\alpha \rangle \\ &= \langle (B \setminus B_1)\theta \rangle \end{aligned}$$

$$= \langle (B \setminus B_1)\theta \cup B_1\theta \rangle \text{ since } B_1\theta = \{0\}$$
$$= \langle B\theta \rangle$$
$$= \langle B \rangle \theta = V\theta = V \text{ since } \theta \in E(V),$$

we have $\alpha \in E(V)$. The following proof shows that $\alpha\theta = \theta^2$.

$$v \in B \setminus B_1 \Rightarrow v\alpha\theta = v\theta\theta = v\theta^2 \text{ and}$$
$$v \in B_1 \Rightarrow v\alpha\theta = v\theta = 0 = 0\theta = (v\theta)\theta = v\theta^2.$$

Then

$$0 \in \theta^2 \ominus \theta^2 = \theta^2 \ominus \alpha\theta = (\theta \ominus \alpha)\theta 1_V.$$

This implies from Proposition 1.1 and 1.9 that $0 \in \theta \ominus \alpha$. Hence $\theta = \alpha$, a contradiction. This proves that $\theta \in M(V)$. Thus $\theta \in G(V)$. Consequently, $(E(V), \theta) \cong E(V)$. Therefore $E(V) \in \boldsymbol{SHR}$. By Lemma 4.5, dim V is finite. $\square$

We show in the next theorem that finiteness of dim V is necessary and sufficient for $(L_2(V), \theta)$ with $\theta \in L_2(V)$ and $(L_3(V), \theta)$ with $\theta \in L_3(V)$ to belong to $\boldsymbol{SHR}$. The following two lemmas will be used.

Lemma 4.7. $L_2(V) \in \boldsymbol{SHR}$ *if and only if dim* V *is finite.*

Proof. If dim V is finite, then $L_2(V) = L(V) \in \boldsymbol{SHR}$.

Conversely, assume that dim V is infinite. Suppose that there exists a hyperoperation $\oplus$ on $L_2(V) \cup \{0\}$ such that $(L_2(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(L_2(V) \cup \{0\}, 1_V)$. Let B be a basis of V and u, w distinct elements of B. Then $|B| = |B \setminus \{u, w\}|$, so there exists a bijection $\varphi : B \to B \setminus \{u, w\}$. Define $\alpha \in L(V)$ by $v\alpha = v\varphi$ for all $v \in B$. Then $\mathrm{Im}\alpha = V\alpha = \langle B \rangle \alpha = \langle B\alpha \rangle = \langle B\varphi \rangle = \langle B \setminus \{u, w\} \rangle$. By Proposition 1.8(2), dim $(V / \mathrm{Im}\alpha) = $ dim $(V / \langle B \setminus \{u, w\} \rangle) = 2$. Therefore $\alpha \in L_2(V)$. Since $B\alpha = B \setminus \{u, w\}$, we have $v\alpha(u, w)_B = v\alpha$ for all

$v \in B$. Consequently, $\alpha(u,w)_B = \alpha$. Therefore

$$0 \in \alpha \ominus \alpha = \alpha \ominus \alpha(u,w)_B = \alpha(1_V \ominus (u,w)_B).$$

We then have from Proposition 1.1 and 1.9 that $0 \in 1_V \ominus (u,w)_B$. Hence $(u,w)_B = 1_V$, a contradiction. This proves that if dim V is infinite, then $(L_2(V),\theta) \notin$ ***SHR***. $\square$

Lemma 4.8. *$L_3(V) \in$ **SHR** if and only if dim V is finite.*

Proof. If dim V is finite, then $L_3(V) = L(V) \in$ ***SHR***.

Conversely, assume that dim V is infinite and suppose that $L_3(V) \in$ ***SHR***. Let $\oplus$ be a hyperoperation on $L_3(V) \cup \{0\}$ such that $(L_3(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(L_3(V) \cup \{0\}, 1_V)$. Let B be a basis of V and $u, w \in B$ be distinct. Define α as in the proof of Lemma 4.7. Then $\alpha(u,w)_B = \alpha$. By Proposition 1.6, $\alpha \in M(V) \subseteq L_3(V)$. Thus

$$0 \in \alpha \ominus \alpha = \alpha(1_V \ominus (u,w)_B).$$

It follows from Proposition 1.1 and 1.9 that $0 \in 1_V \ominus (u,w)_B$ which is a contradiction since $(u,w)_B \neq 1_V$. Hence if dim V is infinite, then $L_3(V) \notin$ ***SHR***. $\square$

Theorem 4.9. *Let $S(V)$ be $L_2(V)$ or $L_3(V)$ and $\theta \in S(V)$. Then $(S(V),\theta) \in$ **SHR** if and only if dim V is finite.*

Proof. If dim V is finite, then $S(V) = L(V)$, and so $(S(V),\theta) \in$ ***SHR*** by Theorem 4.2.

For the converse, assume that $(S(V),\theta) \in$ ***SHR***, and suppose that dim V is infinite. Let $\oplus$ be a hyperoperation on $S(V) \cup \{0\}$ such that $(S(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(S(V) \cup \{0\}, \theta)$.

Case 1. θ is not 1 - 1. Then $\mathrm{Ker}\theta \neq \{0\}$. Let $u \in \mathrm{Ker}\theta \setminus \{0\}$ and B a basis of V containing u. Define $\alpha \in L(V)$ by

$$v\alpha = \begin{cases} 0 & \text{if } \; v = u, \\ v & \text{if } \; v \in B \setminus \{u\}. \end{cases}$$

By Proposition 1.7, $\mathrm{Ker}\alpha = \langle u \rangle$ and $\mathrm{Im}\alpha = \langle B \setminus \{u\} \rangle$. By Proposition 1.8(2), $\dim\,(V\,/\,\mathrm{Im}\alpha) = \dim\,(V\,/\,\langle B \setminus \{u\}\rangle) = 1$. Then $\alpha \in S(V)$. From the fact that

$$\begin{aligned} &u\alpha\theta = 0 = u\theta, \\ &v\alpha\theta = v\theta \text{ for all } v \in B \setminus \{u\}, \end{aligned}$$

we have $\alpha\theta = \theta$. Consequently,

$$0 \in \theta \ominus \theta = \theta \ominus \alpha\theta = (1_V \ominus \alpha)\theta 1_V.$$

This implies by Proposition 1.1 and 1.9 that $0 \in 1_V \ominus \alpha$ and so $\alpha = 1_V$, a contradiction.

Case 2: θ is 1 - 1 and onto. Then $S(V) \cong (S(V), \theta) \in \boldsymbol{SHR}$. By Lemma 4.7 and 4.8, dim V is finite, a contradiction.

Case 3: θ is 1 - 1 but not onto. Then $\mathrm{Im}\theta \subsetneq V$ and $(\mathrm{Im}\theta)\theta \subsetneq V\theta$. Consequently, $\mathrm{Im}\theta^2 \subsetneq \mathrm{Im}\theta \subsetneq V$. Let $u \in V \setminus \mathrm{Im}\theta$ and $w \in \mathrm{Im}\theta \setminus \mathrm{Im}\theta^2$. Let B_1 be a basis of $\mathrm{Im}\theta^2$. Then

$$w \notin \mathrm{Im}\theta^2 = \langle B_1 \rangle \text{ and } u \notin \mathrm{Im}\theta \supseteq \langle B_1 \cup \{w\} \rangle$$

which imply that $B_1 \cup \{u, w\}$ is linearly independent. Let B be a basis of V containing $B_1 \cup \{u, w\}$. Since for every $v \in B, v\theta^2 \in \langle B_1 \rangle$ and $u, w \notin \langle B_1 \rangle$, it follows that $v\theta^2(u, w)_B = v\theta^2$ for all $v \in B$. Therefore $\theta^2(u, w)_B = \theta^2$. We then have

$$0 \in \theta^2 \ominus \theta^2 = \theta^2 \ominus \theta^2(u, w)_B = \theta\theta(1_V \ominus (u, w)_B).$$

By Proposition 1.1 and 1.9, $0 \in 1_V \ominus (u, w)_B$, so $(u, w)_B = 1_V$, a contradiction.

This proves that $(S(V), \theta) \in \boldsymbol{SHR}$ implies that dim V is finite. $\square$

Next, to show that $(L_4(V), \theta) \notin \boldsymbol{SHR}$ for any infinite dimension of V, we require following lemma.

Lemma 4.10. *$M(V)L_4(V) \subseteq L_4(V)$ where dim V is infinite.*

Proof. Let $\alpha \in M(V)$ and $\beta \in L_4(V)$. Since α and β are 1 - 1, $\alpha\beta$ is 1 - 1. We have that

$$V \,/\, \mathrm{Im}\beta \;\cong\; (V \,/\, \mathrm{Im}\alpha\beta) \,/\, (\mathrm{Im}\beta \,/\, \mathrm{Im}\alpha\beta).$$

Since dim $(V \,/\, \mathrm{Im}\beta)$ is infinite, dim $(V \,/\, \mathrm{Im}\alpha\beta)$ is also infinite. Hence $\alpha\beta \in L_4(V)$. $\square$

Theorem 4.11. *For $\theta \in L_4^1(V)$, $(L_4(V), \theta) \notin \boldsymbol{SHR}$ where dim V is infinite.*

Proof. Assume that there exists a hyperoperation $\oplus$ on $L_4(V) \cup \{0\}$ such that $(L_4(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(L_4(V) \cup \{0\}, \theta)$. Let $\alpha \in L_4(V)$. Then $\alpha\theta \in L_4(V)$. Let B_1 be a basis of $\mathrm{Im}\alpha\theta$ and B a basis of V containing B_1. Since $\langle B_1 \rangle = \mathrm{Im}\alpha\theta$, by Proposition 1.8(2), dim $(V \,/\, \mathrm{Im}\alpha\theta) = |B \setminus B_1|$ which is infinite. Let $u, w \in B \setminus B_1$ be distinct. Then $u, w \notin \langle B_1 \rangle = \mathrm{Im}\alpha\theta$ and also $u\alpha \neq w\alpha$ because α is 1 - 1. We have that for every $v \in B, v\alpha\theta(u, w)_B = v\alpha\theta$. Hence $\alpha\theta(u, w)_B = \alpha\theta$ and so $\alpha\theta(u, w)_B\alpha = \alpha\theta\alpha$. By Lemma 4.10, $(u, w)_B\alpha \in L_4(V)$. Thus

$$0 \in \alpha\theta\alpha \ominus \alpha\theta\alpha = \alpha\theta\alpha \ominus \alpha\theta(u, w)_B\alpha = \alpha\theta(\alpha \ominus (u, w)_B\alpha).$$

From Proposition 1.1 and 1.9, we have $0 \in \alpha \ominus (u, w)_B\alpha$. Therefore $(u, w)_B\alpha = \alpha$ and so $u(u, w)_B\alpha = u\alpha$. But $u(u, w)_B\alpha = w\alpha$, so $w\alpha = u\alpha$. This is a contradiction.

This proves that $(L_4(V), \theta) \notin \boldsymbol{SHR}$, as required. $\square$

The following corollary is an immediate consequence of Theorem 4.11.

Corollary 4.12. *$L_4(V) \notin \boldsymbol{SHR}$ where dim V is infinite.*

Finally, we shall show that for $\theta \in L_5^1(V), (L_5(V), \theta) \notin \boldsymbol{SHR}$ for any infinite dimension of V. The following lemma will be used.

Lemma 4.13. *$L_5(V)E(V) \subseteq L_5(V)$ where dim V is infinite.*

Proof. Let $\alpha \in L_5(V)$ and $\beta \in E(V)$. Since Im$\alpha = V =$ Imβ, we have Im$\alpha\beta = V$. Since Ker$\alpha\beta \supseteq$ Kerα and dim Kerα is infinite, it follows that dim Ker$\alpha\beta$ is infinite. Hence $\alpha\beta \in L_5(V)$. $\square$

Theorem 4.14. *For $\theta \in L_5^1(V)$, $(L_5(V), \theta) \notin \boldsymbol{SHR}$ where dim V is infinite.*

Proof. Suppose that there exists a hyperoperation $\oplus$ on $L_5(V) \cup \{0\}$ such that $(L_5(V) \cup \{0\}, \oplus, \cdot)$ is a hyperring where $\cdot$ is the operation on $(L_5(V) \cup \{0\}, \theta)$. Let $\alpha \in L_5(V)$. Then $\theta\alpha \in L_5(V)$, so dim Ker$\theta\alpha$ is infinite. Let $u, w \in$ Ker$\theta\alpha$ be linearly independent. Then $u\theta\alpha = 0 = w\theta\alpha$. Let B be a basis of V containing u and w. Since $B \setminus \{u, w\}$ is infinite, there are two subsets B_1 and B_2' of $B \setminus \{u, w\}$ such that

$$B \setminus \{u, w\} = B_1 \cup B_2',\ B_1 \cap B_2' = \varnothing \text{ and } |B_1| = |B_2'| = |B \setminus \{u, w\}|.$$

Let $B_2 = B_2' \cup \{u, w\}$. Then

$$B = B_1 \cup B_2,\ B_1 \cap B_2 = \varnothing \text{ and } |B_1| = |B_2| = |B|.$$

Let $\varphi : B_1 \to B$ be a bijection. Define $\beta \in L(V)$ by

$$v\beta = \begin{cases} v\varphi & \text{if } v \in B_1, \\ 0 & \text{if } v \in B_2. \end{cases}$$

Then $\mathrm{Im}\beta = \langle B\beta \rangle = \langle B_1\beta \rangle = \langle B_1\varphi \rangle = \langle B \rangle = V$ and $B_2 \subseteq \mathrm{Ker}\beta$. Thus $\dim \mathrm{Ker}\beta \geq |B_2|$, so $\dim \mathrm{Ker}\beta$ is infinite. Hence $\beta \in L_5(V)$. Let $u^{'}, w^{'} \in B_1$ be such that $u^{'}\varphi = u$ and $w^{'}\varphi = w$. Thus $u^{'}\beta = u$ and $w^{'}\beta = w$. Since $\beta_{|_{B_1}} = \varphi : B_1 \to B$ is a bijection, for all $v \in B_1 \setminus \{u^{'}, w^{'}\}, v\beta \in B \setminus \{u, w\}$, and so $v\beta(u,w)_B = v\beta$ for all $v \in B \setminus \{u, w\}$. The following equalities yield $\beta(u,w)_B\theta\alpha\beta = \beta\theta\alpha\beta$.

$u^{'}\beta(u,w)_B\theta\alpha\beta = u(u,w)_B\theta\alpha\beta = w\theta\alpha\beta = (w\theta\alpha)\beta = 0\beta = 0,$

$u^{'}\beta\theta\alpha\beta = u\theta\alpha\beta = (u\theta\alpha)\beta = 0\beta = 0,$

$w^{'}\beta(u,w)_B\theta\alpha\beta = w(u,w)_B\theta\alpha\beta = u\theta\alpha\beta = (u\theta\alpha)\beta = 0\beta = 0,$

$w^{'}\beta\theta\alpha\beta = w\theta\alpha\beta = (w\theta\alpha)\beta = 0\beta = 0,$

$v\beta(u,w)_B\theta\alpha\beta = 0 = v\beta\theta\alpha\beta$ for all $v \in B_2$ and

for $v \in B_1 \setminus \{u^{'}, w^{'}\}, v\beta(u,w)_B\theta\alpha\beta = (v\beta(u,w)_B)\theta\alpha\beta = v\beta\theta\alpha\beta.$

By Lemma 4.13, $\beta(u,w)_B \in L_5(V)$. Then

$$0 \in \beta\theta\alpha\beta \ominus \beta\theta\alpha\beta = \beta\theta\alpha\beta \ominus \beta(u,w)_B\theta\alpha\beta = (\beta \ominus \beta(u,w)_B)\theta\alpha\beta.$$

This implies that $0 \in \beta \ominus \beta(u,w)_B$ by Proposition 1.1 and 1.9. Thus $\beta(u,w)_B = \beta$. But $u^{'}\beta(u,w)_B = u(u,w)_B = w, u^{'}\beta = u$ and $u \neq w$, so we have a contradiction.

This proves that $(L_5(V), \theta) \notin$ ***SHR***, as required. $\square$

The following corollary is a particular case of Theorem 4.14.

Corollary 4.15. *$L_5(V) \notin$ **SHR** where $\dim V$ is infinite.*

REFERENCES

[1]. Chu, D. D., and Shyr, H. J. Monoids of languages admitting ring structure. **Semigroup Forum** 19(1980) : 127 - 132.

[2]. Clifford, A. H., and Preston, G. B. **The algebraic theory of semigroups**

(Vol. I). Mathematical Surveys, American Mathematical Society, Providence, R. I., 1961.

[3]. Corsini, P. **Prolegomena of hypergroup theory**. Aviani Editore, 1992.

[4]. Higgins, P. M. **Techniques of semigroup theory**. New York : Oxford University Press, 1992.

[5]. Howie, J. M. **An introduction to semigroup theory**. London : Academic Press, 1976.

[6]. Hungerford, T. W. **Algebra**. New York : Springer - Verlag, 1974.

[7]. Isbell, J. R. On the multiplicative semigroup of a commutative ring. **Proceedings American Mathematical Society** 10(1959) : 908 - 909.

[8]. Kemprasit, Y., and Punkla, Y. Transformation semigroups admitting hyperring structure. **Italian Journal of Pure and Applied Mathematics**, 12(to appear).

[9]. Kemprasit, Y. Multiplicative interval semigroups on $\mathbb{R}$ admitting hyperring structure. **Italian Journal of Pure and Applied Mathematics**, 13(to appear).

[10]. Limbupasiriporn, P. **Absolutely closed transformation semigroups**. Master's Thesis, Department of Mathematics, Graduate School, Chulalongkorn University, 1989.

[11]. Peinado, R. E. On semigroups admitting ring structure. **Semigroup Forum** 1(1970) : 189 - 208.

[12]. Punkla, Y. **Hyperrings and transformation semigroups admitting hyperring structure**. Master's Thesis, Department of Mathematics, Graduate School, Chulalongkorn University, 1991.

[13]. Satyanarayana, M. On semigroups admitting ring structure. **Semigroup Forum** 3(1971) : 43 - 50.

[14]. Satyanarayana, M. On semigroups admitting ring structure II. **Semigroup Forum** 6(1973) : 189 - 197.

[15]. Satyanarayana, M. On 0 - simple semigroups admitting ring structure. **Semigroup Forum** 19(1980) : 307 - 312.

VITA

Miss Noknoi Rompurk was born on December 7, 1975 in Kalasin, Thailand. She graduated with a Bachelor Degree of Science in Mathematics from Khonkaen University in 1997. In 1998, she was admitted into the Master Degree program in Mathematics, Department of Mathematics, Faculty of Science, Chulalongkorn University. During the study for her Master's degree, she received a financial support from University Development Commission (UDC) for two years.