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HENSTOCK-STIELTJES INTEGRALS OF L_2-VALUED FUNCTIONS

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ในงานวิจัยนี้ เราให้นิยามของการอินทิเกรตแบบ เฮนสต็อก-สตีลเซสของฟังก์ชันซึ่งมีค่าอยู่ในปริภูมิ L_2 และเราศึกษาสมบัติของการอินทิเกรตดังกล่าว งานวิจัยนี้มีแนวคิดจากการอินทิเกรตแบบเฮนสต็อกซึ่งคิดค้นโดย J.KURZWEIL และ R.HENSTOCK.

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The Henstock-Stieltjes integration for functions whose values are in L_2-space is defined and its properties are investigated. The work is based on the idea of Henstock integration introduced by J.Kurzweil and R.Henstock.

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CONTENTS

CHAPTER I

Introduction

In theory of Riemann integration, we consider a function f defined on a non-degenerate $(a < b)$ interval $[a, b]$ into $\mathbb{R}$. By a partition of $[a, b]$, we mean a finite collection

$$\mathcal{P} = \{I_i \mid i = 1, 2, ..., n\} = \{I_i\}_{i=1}^{n}$$

of nonoverlapping (except at end points) subintervals I_i such that $[a, b] = \cup_{i=1}^{n} I_i$. It is always possible to arrange the subintervals in an increasing order; that is $\max(I_i) = \min(I_{i+1})$ for $i = 1, 2, ..., n-1$.

Let $\mathcal{P} = \{I_i\}_{i=1}^{n}$ be a partition of $[a, b]$. The *mesh* of $\mathcal{P}$ is denoted by $\|\mathcal{P}\|$ and defined by

$$\|\mathcal{P}\| = \max\{l(I_i) \mid i = 1, 2, ..., n\}$$

where $l(I_i)$ is the length of I_i.

If $\mathcal{P} = \{I_i \mid i = 1, 2, .., n\}$ and $\mathcal{Q} = \{J_j \mid j = 1, 2, ..., m\}$ are partitions of $[a, b]$, we say that $\mathcal{Q}$ is *finer than* $\mathcal{P}$ if every end point of subintervals in $\mathcal{P}$ is an endpoint of some subinterval in $\mathcal{Q}$. It is obvious that if $\mathcal{Q}$ is finer than $\mathcal{P}$ then $\|\mathcal{Q}\| \leq \|\mathcal{P}\|$. A set $t = \{t_1, t_2, ..., t_n\}$ of points in $[a, b]$ is called a *set of intermediate points* of $\mathcal{P}$ if for each i, $t_i \in I_i$. For each $i = 1, 2, ..., n$, let $I_i = [x_{i-1}, x_i]$. The number

$$S(f, \mathcal{P}, t) = \sum_{i=1}^{n} f(t_i)[x_i - x_{i-1}].$$

is called the *Riemann sum* of f with respect to the partition $\mathcal{P} = \{I_i\}_{i=1}^{n}$ and the set $t = \{t_1, t_2, ..., t_n\}$ of intermediate points of $\mathcal{P}$.

We say that the Riemann sum of f approaches a real number A as $\|\mathcal{P}\|$ tends to zero if for every $\epsilon > 0$, there is a partition $\mathcal{P}_\epsilon$ of $[a, b]$ such that for any partition $\mathcal{P}$ finer than $\mathcal{P}_\epsilon$ and for any set t of intermediate points of $\mathcal{P}$,

$$|S(f, \mathcal{P}, t) - A| < \epsilon.$$

And if this is the case, we write

$$\lim_{\|\mathcal{P}\| \to 0} S(f, \mathcal{P}, t) = A.$$

It is noted that if such a number A exists, it is unique. We usually denote A by $\int_a^b f$, and we say that f is (Riemann) integrable on $[a, b]$ with the *integral* $\int_a^b f$.

By the above definition, it is seen that the idea of Riemann integration is intuitive and simple. However, there are many elementary functions which are not integrable in the sense of Riemann. For example, the Dirichlet function f which assigns the value 1 for rationals x in $[0, 1]$ and 0 elsewhere in $[0, 1]$, is not Riemann integrable on $[0, 1]$. However, it can be proved that if a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then it is bounded on $[a, b]$.

In 1904, Henri Lebesgue introduced a new approach of integration called Lebesgue integration that can be applied to a wider class of functions, and every Riemann integrable functions belong to this class. For many years, many scientists satisfied with this kind of integration, since the class of Lebesgue integrable functions is larger than that of Riemann type. However the class of Lebesgue integrable functions is still not *big* enough, and the computation is rather complicated. The new integration introduced by Henstock in the year 1957 fulfilled this requirement. That is it can be applied to a wider class of functions (than that of the Lebesgue integration) and most of all the computation is straightforward and intuitive.

Our work is inspired by the work of Jean Christophe Feauveau [1] and [2], in which an idea of a generalized Riemann integral for Banach-valued functions was introduced. We develop an integral called Henstock-Stieltjes integral or a generalized Riemann-Stieltjes integral for function whose values are in an L_2-space and investigate some properties of the integral.

CHAPTER II

Henstock Integrals of Real-Valued Functions

2.1 Preliminaries

In this section, we recall the concept of Henstock integration that was initiated by Jaroslav Kurzweil, in 1957, in his research work on differential equations. He gave an elementary definition of the integral and later in 1961, Ralph Henstock rediscovered Kurzweil's approach and developed his theory. The integration is then called $Kurzweil - Henstock\ integration$ or shortly $Henstock\ integration$. The integration is remarked as a generalized Riemann integration, because the tools of Riemann type are used in the definition.

Let $[a, b]$ be a nondegenerate closed interval in $\mathbb{R}$. By a *gauge* on $[a, b]$, we mean a positive function δ defined on $[a, b]$. We sometimes refer to a partition $\mathcal{P} = \{[x_{i-1}, x_i] \mid i = 1, 2, ..., n\}$ simply as $\mathcal{P} = \{x_0, x_1, ..., x_n\}$. For a partition $\mathcal{P} = \{[x_{i-1}, x_i] \mid i = 1, 2, ..., n\}$ of $[a, b]$ and a set $t = \{t_1, t_2, ..., t_n\}$ of intermediate points in $[a, b]$ with $t_i \leq t_{i+1}$ for $i = 1, 2, ..., n-1$, the set

$$D = \{([x_{i-1}, x_i], t_i) | i = 1, 2, ..., n\}$$

is called a tagged-partition if δ is a gauge on $[a, b]$, a tagged-partition $D = \{([x_{i-1}, x_i], t_i) | i = 1, 2, ..., n\}$ is said to be a δ-fine partition if for each $i = 1, 2, ..., n$

$$t_i - \delta(t_i) < x_{i-1} \leq t_i \leq x_i < t_i + \delta(t_i).$$

The points $t_1, t_2, ..., t_n$ are sometimes called tags of the tagged partition D.

In the above definition it is not immediately clear that for a given positive function δ there will be a δ-fine partition. It might look a little amazing that a δ-fine partition always exists no matter how the positive function δ behaves. The assertion is interesting. It is known as *Cousin lemma*. The proof is interesting and it is shown in the following lemma.

Lemma 2.1.1. *(Cousin's lemma)*[5] If δ is a gauage on a closed and bounded interval $[a, b]$ and $[c, d]$ is any closed subinterval of $[a, b]$, then there always exists a δ-fine partition of $[c, d]$.

Proof. Suppose on the contrary that there is no δ-fine partition on $[c, d]$. Then at least one of the two intervals $[c, \frac{c+d}{2}]$ or $[\frac{c+d}{2}, d]$ has no δ-fine partition. Let us denote the half of $[c, d]$ without a δ-fine partition by $[c_1, d_1]$. Now we continue this halving process indefinitely, we obtain a sequence $([c_n, d_n])$ of nested intervals with $d_n - c_n = \frac{d-c}{2^n}$, and hence, $d_n - c_n \to 0$ as $n \to \infty$.So there exists a point z which lies in all $[c_n, d_n]$. Since $\delta(z) > 0$, there exists a number $N \in \mathbb{N}$ such that for $n > N$, we have

$$d_n - c_n < \delta(z).$$

This inequality shows that if $D = ([c_n, d_n], z)$, then D is a δ-fine partition of $[c_n, d_n]$. Which contradicts to the assumption for $[c_n, d_n]$. $\square$

Let $f:[a,b]\to\mathbb{R}$, δ a gauge on $[a,b]$ and $D=\{([x_{i-1},x_i],t_i)\mid i=1,2,...,n\}$ a δ-fine partition of $[a,b]$. The Riemann sum of f on $[a,b]$ with respect to the gauge δ and the δ-fine partition D is denoted by $S(f,D)$. It is defined by

$$S(f,D)=\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1}).$$

The Kurzweil-Henstock integral (or just the Henstock integral) is defined as the limit of Riemann sums, the same way as the Riemann integrals, except that the δ-fineness of partition is measured by the gauge δ instead of a positive constant δ. More precisely, the definition is as follows:

Definition 2.1.2. Let $f:[a,b]\to\mathbb{R}$. We say that f is *Kurzweil-Henstock integrable,* or briefly *Henstock integrable on [a,b]*, if for each positive real number ϵ, there exists a gauge δ on $[a,b]$ such that for any δ-fine partitions

$$D_1=\{([x_{i-1},x_i],t_i)\mid i=1,2,...,n\}$$

and

$$D_2=\{([x'_{j-1},x'_j],t'_j)\mid j=1,2,...,m\}$$

of $[a,b]$, we have

$$|S(f,D_1)-S(f,D_2)|\le\epsilon.$$

Definition 2.1.3. Let $I=[a,b]$ be a nondegenerate interval. By a *subpartition* of I we mean a collection $\{J_j\mid j=1,2,...,s\}$ of nonoverlapping closed intervals in I. If δ is a gauge on I, and $\{J_j\mid j=1,2,...,s\}$ a subpartition of I, the collection $\{(J_j,t_j)\mid j=1,2,...,s\}$ is called a *δ-fine subpartition* of I if

$$t_j\in J_j\subset(t_j-\delta(t_j),t_j+\delta(t_j))$$

for $j=1,2,...,s$.

2.2 Properties of the Integral

By the definition of integrability given in the previous section, we list here some important properties which will be useful in our study.

Theorem 2.2.1. Let $f : [a,b] \to \mathbb{R}$. Then f is Henstock integrable on $[a,b]$ if and only if there is a real number A with the property that for every $\epsilon > 0$, there exists a positive function δ on $[a,b]$ such that for any δ-fine partition $D = \{([x_{i-1}, x_i], \xi_i) \mid i = 1, 2, ..., n\}$ of $[a,b]$ we have

$$|S(f, D) - A| < \epsilon.$$

The number A in the definition is obviously unique. It is so-called the *integral of f on* $[a,b]$ and is denoted by $\int_{[a,b]} f$ or $\int_a^b f$.

The next theorem is known as *Henstock's lemma*, it is fundamentally important in proving deeper properties of the generalized Riemann integral.

Theorem 2.2.2. Let f be Henstock integrable on $[a,b]$. Then for each $\epsilon > 0$, there exists a gauge δ on $[a,b]$ such that whenever
$D = \{([u_i, v_i], \xi_i) | i = 1, 2, ..., n\}$ is a δ-fine subpartition of $[a,b]$ we have

$$\sum_{i=1}^{n} \left| f(\xi_i)(v_i - u_i) - \int_{u_i}^{v_i} f(x)dx \right| \leq \epsilon.$$

The statement of theorem are not surprising and the proof is straightforward. The following theorems are basic properties of the integral.

Theorem 2.2.3. Let $\alpha, \beta \in \mathbb{R}$ and f_1, f_2 be Henstock integrable on $[a,b]$. Then $\alpha f_1 + \beta f_2$ are Henstock integrable on $[a,b]$ and

$$\int_a^b \alpha f_1 + \beta f_2 = \alpha \int_a^b f_1 + \beta \int_a^b f_2.$$

Theorem 2.2.4. Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. Then f is Henstock integrable on $[a, b]$ if and only if its restriction to $[a, c]$ and $[c, b]$ are both Henstock integrable. In this case, we have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 2.2.5. If f is Henstock integrable on $[a, b]$ and $[c, d] \subset [a, b]$, then it is Henstock integrable on $[c, d]$.

Theorem 2.2.6. If $f(x) = 0$ for almost all x in $[a, b]$, that is, for every x in $[a, b]$ except a set E of measure zero, then f is integrable on $[a, b]$ and $\int_a^b f = 0$.

Theorem 2.2.7. If f and g are Henstock integrable on $[a, b]$ and if $f(x) \leq g(x)$ for almost all x in $[a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

It is the fact that every function that is Lebesgue integrable on $[a, b]$ is also Henstock integrable. The proof of this assertion needs some of the properties of Lebesgue integration and requires some effort. However, it should be mentioned that E.J. McShane has given an equivalent definition of the Lebesgue integral that makes it clear that the Lebesgue integral is a special case of Henstock integral. By using a δ-fine division is used instead of δ-fine partition.

However, it is noticed that a function f is Lebesgue integrable if and only if both f and $|f|$ are Henstock integrable.

2.3 Examples

In this section, some examples are given so that we can clearly see the idea of Henstock integration. All of them are not Riemann integrable.

Example 2.3.1. Consider the discontinuous function introduced by Peter G.L. Dirichelet in 1829. The function is so called the Dirichelet function and defined on $[0,1]$ by

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational,}\\ 0 & \text{if } x \text{ is irrational.}\end{cases}$$

It is well know that f is not Riemann integrable on $[0,1]$. However, we will show that f is Henstock integrable on $[0,1]$.

Let $\{r_k | k\in\mathbb{N}\}$ be an enumeration of the rationals in $[0,1]$ and $\epsilon>0$ be given. We define a gauge δ on $[0,1]$ by

$$\delta(t)=\begin{cases}\frac{\epsilon}{2^{k+1}} & \text{if } t=r_k,\\ 1 & \text{if } t \text{ is irrational.}\end{cases}$$

Let $D=\{([x_{i-1},x_i],t_i) | i=1,2,...,n\}$ be a δ-fine partition on $[0,1]$. If t_i is irrational, then $f(t_i)=0$ and so

$$f(t_i)[x_i-x_{i-1}]=0.$$

If t_i is rational, then $f(t_1)=1$ and

$$f(t_i)[x_i-x_{i-1}]=x_i-x_{i-1}.$$

So if $t_i=r_k$, then $f(t_i)[x_i-x_{i-1}]=x_i-x_{i-1}<2\delta(r_k)=\frac{\epsilon}{2^k}$.

If r_k is the tag for two consecutive subintervals in D, the sum of the length of these two nonoverlapping subintervals does not exceed $\frac{\epsilon}{2^k}$. So we have

$$|S(f,D)|\le\sum_{k=1}^{\infty}\frac{\epsilon}{2^k}=\epsilon.$$

This implies that the Dirichlet functions is Henstock integrable with $\int_0^1 f=0$.

Example 2.3.2. Now consider a modification of the function known as the *Thomae's function* or the *ruler function*. The original one was introduced by Karl J. Thomae in 1875. It is the function g defined on $[0,1]$ by

$$g(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}\ , \ x \neq 0 \ (p,q) = 1 \\ 0 & \text{if } t \text{ is irrational or } x = 0. \end{cases}$$

The function g is Riemann integrable with $\int_0^1 g = 0$. But we slightly change the value of the function at nonzero rationals, namely define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} q & \text{if } x = \frac{p}{q}\ , \ x \neq 0 \ (p,q) = 1 \\ 0 & \text{if } t \text{ is irrational or } x = 0. \end{cases}$$

It is easy to see that f is not continuous at any point in $[0,1]$ and is unbounded on any nondegenerate subinterval of $[0,1]$. So f is not Riemann integrable on $[0,1]$. However, it is Henstock integrable on the interval. To show this, let $\epsilon > 0$ be given. Let $\{r_k | k \in \mathbb{N}\}$ be an enumeration or rationals in $[0,1]$.
Define $\delta : [0,1] \to \mathbb{R}$ by

$$\delta(\frac{p_r}{q_k}) = \frac{\epsilon}{q_k 2^{k+1}}$$

and $\delta(t) = 1$ if t is an irrational or $t = 0$. Then for any δ-fine partition D on $[0,1]$, we have (similar to Example 2.3.1) that

$$|S(f,D)| \leq \epsilon.$$

So, f is Henstock integrable on $[0,1]$ with $\int_0^1 f = 0$.

It is interesting to notice that the Henstock integral is not absolutely integrable. This is in contrast to the situation for the Riemann integral and the Lebesgue integral where the absolute value of an integrable function is also integrable. Then there are a Henstock integrable function which is not Lebesgue integrable and the class of Henstock integrable functions is bigger than that of Lebesgue.

Example 2.3.3. [6] Let $\sum_{n=1}^{\infty} a_n$ be any convergent series of real numbers, and A be its limit. Let $c_n = 1 - \frac{1}{2^n}$ for $n = 0, 1, ...,$ that is $c_0 = 0,\ c_1 = \frac{1}{2},\ c_2 = \frac{3}{4},$ $c_3 = \frac{7}{8}, ...$ We define a function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2^k a_k & \text{if } \ x \in [c_{k-1}, c_k), \\ 0 & \text{if } \ x = 1. \end{cases}$$

Then f is Henstock integrable on $[0, 1]$ with $\int_0^1 f = A$.

The next example is remarkable as it shows that the absolute value of Henstock integrable function need not be Henstock integrable. This implies that the class of Henstock integrable function is strictly *bigger* than that of Lebesgue.

Example 2.3.4. [6] Let $c_k = 1 - \frac{1}{2^k}$ for each $k \in \mathbb{N}$. Define a function $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} (-1)^{k+1} \frac{2^k}{k} & \text{if } \ x \in [c_{k-1}, c_k), \\ 0 & \text{if } \ x = 1. \end{cases}$$

Since the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ converges , then by Example 2.3.3 g is Henstock integrable on $[0, 1]$ and $\int_0^1 g = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.

Now consider $|g|$, we have

$$|g|(x) = \begin{cases} \frac{2^n}{n} & \text{if } \ x \in [c_{n-1}, c_n), \\ 0 & \text{if } \ x = 1. \end{cases}$$

We claim that $|g|$ is not Henstock integrable on $[0, 1]$. Indeed, for each $n \in \mathbb{N}$, let

$$h_n(x) = \begin{cases} |g|(x) & \text{if } \ x \in [0, c_n), \\ 0 & \text{if } \ x \in [c_n, 1]. \end{cases}$$

It follows from Example 2.3.3 that h_n is Henstock integrable on $[0,1]$ and

$$\int_0^1 h_n = \sum_{k=1}^{n} \frac{1}{k}.$$

But since $0 \le h_n(x) \le |g|(x)$ for all $x \in [0,1]$, it is clear that if $|g|$ is integrable on $[0,1]$, then we must have $\sum_{k=1}^{n} \frac{1}{k} = \int_0^1 h_n \le \int_0^1 |g|$ for every $n \in \mathbb{N}$, which is impossible. Then $|g|$ is not integrable on $[0,1]$.

CHAPTER III

The Integral of Functions whose Values are in a Banach Space

In this chapter, we study the notion of a generalized Riemann integration of functions whose values are in a Banach space. The notion was introduced in 2001 by Jean Christophe Feauveau ([1],[2]).

3.1 Preliminaries

Jean Christophe Feauveau [1] introduced a definition of generalized-Riemann integration for functions whose values are in a Banach space. The idea is close to that of Henstock.

Definition 3.1.1. Let $[a,b]$ be a nondegenerate interval. Let δ be a gauge on $[a,b]$, and $\{[x_{i-1},x_i] \mid i = 1,2,...,n\}$ be a partition on $[a,b]$. A collection of interval-point pairs

$$D = \{([x_{i-1},x_i],t_i) \mid i = 1,2,...,n\}$$

is called *a tagged partition* and the points t_i are called *tags.* A tagged partition is said to be a δ*-fine partition* if

$$t_i \in [x_{i-1},x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i)),$$

and it is said to be a δ*-fine division* if for each $i = 1,2,...,n$,

$$[x_{i-1},x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i)).$$

Remark. The definition of a δ-fine division is very similar to that of δ-fine partition, but the condition

$$t_i \in [x_{i-1}, x_i]$$

for $i = 1, 2, ...n$ is not required. So, every δ-fine partition is a δ-fine division. The definition of a $\delta - fine\ \ subdivision$ can be given analogously.

Now, we present the definition of integral introduced by Jean-Christophe Feauveau.

Definition 3.1.2. [1] Let f be a function defined on a nondegenerate closed interval $I = [a, b]$ into a Banach space $(X, \|.\|)$. Then, f is said to be *integrable* on I if for each $\epsilon > 0$, there exists a gauge δ on I such that

$$\sum_{i=1}^{n} \|(x_i - x_{i-1})[f(t_i) - f(t'_i)]\| \leq \epsilon$$

whenever $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, ..., n\}$ and $D' = \{([x_{i-1}, x_i], t'_i) \mid i = 1, 2, ..., n\}$ are δ-fine divisions on I.

For an integrable function f, a gauge satisfying the above property for ϵ is said to be ϵ-adapted (to f).

Throughout this chapter, $(X, \|.\|)$ or shortly X, stands for a Banach space. The next theorem was given in [1] and the proof is sophisticated. Our results in the following chapter have drawn their inspiration from this theorem, so we present here the proof but in a modified manner.

Theorem 3.1.3. Let $f : [a, b] \to X$ be an integrable function. There exists a unique vector A in X satisfying the following condition : for every positive real number γ, there is an $\alpha > 0$ such that if ϵ is a positive real number less than α, then for each ϵ-adapted gauge δ_ϵ, we have

$$\left\| \sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i) - A \right\| < \gamma, \tag{3.1.1}$$

for every δ_ϵ-fine division $\{([x_{i-1}, x_i], t_i) \mid i = 1, 2, ..., n\}$ on $[a, b]$.

The vector A in the theorem is called the *integral of* f on $[a, b]$.

Proof. Since f is integrable, for each $\epsilon > 0$ there is an ϵ-adapted gauge, δ_ϵ for f.

Let $\Delta = \{\delta_\epsilon \mid \epsilon > 0 \ and \ \delta_\alpha \leq \delta_\beta \ for \ 0 < \alpha < \beta\}$, and D_ϵ be the family of all δ_ϵ-fine divisions, where $\delta_\epsilon \in \Delta$. For each δ_ϵ in Δ, if $\mathcal{P} = \{a = x_0, x_1, ..., x_n = b\}$ is a partition on $[a, b]$ and $D_\epsilon = \{([x_{i-1}, x_i], t_i) | i = 1, 2, ..., n\}$ is a δ_ϵ-fine division, we will denote D_ϵ by $(\mathcal{P}_\epsilon, t_\epsilon)$, and $\sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i)$ by $S(f, D_\epsilon)$.

For $0 < \alpha \leq \beta$, we can merge the partitions $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ into a finer one $\mathcal{P}_{\alpha,\beta}$. Then we build $(\mathcal{P}_{\alpha,\beta}, t_{\alpha,\beta})$ from $(\mathcal{P}_\alpha, t_\alpha)$ by repeating tags whenever necessary. The same work can be done from $(\mathcal{P}_\beta, t_\beta)$ to get $(\mathcal{P}_{\beta,\alpha}, t_{\beta,\alpha})$.

Thus $D_{\alpha,\beta} = (\mathcal{P}_{\alpha,\beta}, t_{\alpha,\beta})$ and $D_{\beta,\alpha} = (\mathcal{P}_{\beta,\alpha}, t_{\beta,\alpha})$ are δ_β-fine division, and

$$\|S(f, D_\alpha) - S(f, D_\beta)\| = \|S(f, D_{\alpha,\beta}) - S(f, D_{\beta,\alpha})\| \leq \beta. \tag{3.1.2}$$

We construct a sequence in X as follows : for each $n \in \mathbb{N}$, let $x_n = S(f, D_{\frac{1}{n}})$. Then the above discussion implies that (x_n) is a Cauchy sequence, and so convergent. Let A be the limit of $S(f, D_{\frac{1}{n}})$ as n tends to infinity. Then $A \in X$.

Now let γ be any positive real number. Since

$$\lim_{n \to \infty} S(f, D_{\frac{1}{n}}) = A,$$

we can choose an integer $N > \frac{2}{\gamma}$ such that

$$\left\| S(f, D_{\frac{1}{n}}) - A \right\| < \frac{\gamma}{2} \tag{3.1.3}$$

whenever $n \geq N$.

Now let $\alpha = \frac{1}{N}$. Then $\alpha > 0$ and for $0 < \epsilon \leq \alpha$, we have

$$\begin{aligned}\|S(f, D_\epsilon) - A\| &\leq \|S(f, D_\epsilon) - S(f, D_\alpha)\| + \left\|S(f, D_{\frac{1}{n}}) - A\right\| \\ &\leq \frac{1}{N} + \frac{\gamma}{2} \qquad \text{, by (3.1.2) and (3.1.3),} \\ &< \gamma.\end{aligned}$$

To obtain the uniqueness of A, suppose that A and A' satisfy (3.1.1). By the triangle inequality, we have for every $\gamma > 0$

$$\begin{aligned}\|A - A'\| &\leq \|A - S(f, D_\epsilon)\| + \|S(f, D_\epsilon) - A'\| \\ &< \frac{\gamma}{2} + \frac{\gamma}{2} \\ &= \gamma\end{aligned}$$

and D_ϵ is a δ_ϵ-fine division on $[a, b]$ where ϵ is as in (3.1.1). Thus $A = A'$ $\square$

3.2 Properties of the Integral

Let f and g be function on $[a,b]$ whose values are in a Banach space X. The proof of the following theorem is identical to the case of real-valued functions (see [5] for instance).

Theorem 3.2.1. Let f and g be integrable on $[a,b]$.

1) The function $f+g$ and λf are integrable for all scalars λ.

2) If $[c,d]$ is subinterval of $[a,b]$, then f is integrable on $[c,d]$.

Theorem 3.2.2. Let $a<c<b$ and $f:[a,b]\to X$ be such that the restrictions of f to $[a,c]$ and $[c,b]$ are integrable .Then f is integrable on $[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

The following property of the integral is an important tool to establish more advanced results.

Theorem 3.2.3. Let $f:[a,b]\to X$ be an integrable function. For any positive real number ϵ, if a gauge δ is such that

$$\left\| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{m} f(\hat{t}_j)(z_j - z_{j-1}) \right\| \leq \epsilon$$

for every δ-fine divisions $\{([x_{i-1},x_i],t_i)|i=1,2,...,n\}$ and $\{([z_{j-1},z_j],\hat{t}_j)|j=1,2,...,m\}$ of $[a,b]$, then

$$\sum_{i=1}^{n} \left\| f(t_i)(x_i - x_{i-1}) - \int_{x_{i-1}}^{x_i} f(x)dx \right\| \leq \epsilon$$

whenever $\{([x_{i-1},x_i],t_i)|i=1,2,...,n\}$ is a δ-fine division on $[a,b]$.

3.3 The Fundamental Theorem of Calculus

It is true that the fundamental theorem of calculus holds for the generalized Riemann integration of functions whose values are in a Banach space. Let us recall the definition of the derivative of a vector-valued function.

Let f be a function defined on a subset A of $\mathbb{R}$ with values in a Banach space X, and let c be a non-isolated point of A. The function f is said to have the *derivative at c* if the limit

$$lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists in X and the value of this limit is called the *derivative* of f at c, and denoted by $f'(c)$. The derivative f' of f is the function $x \mapsto f'(x)$ whose domain is the set of non-isolated points x of A at which $f'(x)$ exists. If $f'(x)$ exists for every $x \in B \subseteq A$, then we say that f is *differentiable* on B.

Let $f : [a,b] \to X$. The function f is said to be *absolutely continuous* on $[a,b]$ if for every positive real number ϵ there is a positive number $\delta > 0$ such that

$$\sum_{i=1}^{n} \|f(x_i) - f(y_i)\| \leq \epsilon$$

whenever $\{[x_i, y_i] | i = 1, 2, ..., n\}$ is a collection of subintervals of $[a,b]$ and such that $\sum_{i=1}^{n} |y_i - x_i| < \delta$.

It is easy to prove that if f is integrable on $[a,b]$, then $x \mapsto \int_a^x f(t)dt$ is continuous. In fact, we have a stronger result from the generalized Riemann theory.

Theorem 3.3.1. If $f : [a,b] \to X$ is integrable, then the function $F : x \mapsto \int_a^x f$ is absolutely continuous.

The following theorem is an important property of the generalized Riemann integral on the class of functions considered.

Theorem 3.3.2. ***[The Fundamental Theorem of Calculus]***

Let $f : [a, b] \to X$ be continuous. If f is differentiable on $[a, b]$ with f' integrable on $[a, b]$ then

$$\int_a^b f' = f(b) - f(a).$$

The following result is the reverse problem of Theorem 3.3.2.

Theorem 3.3.3. Let $f : [a, b] \to X$ be integrable and $F : [a, b] \to X$ defined by

$$F(x) = \int_a^x f$$

for each $x \in [a, b]$. Then the function F is differentiable a.e. on $[a, b]$ and $F' = f$ a.e. on $[a, b]$.

CHAPTER IV

A Generalized Riemann-Stieltjes Integral

4.1 Preliminaries

In this chapter, we develop an integral so-called a *generalized Riemann-Stieltjes integral* for functions whose values are in an L_2-space. The idea is based on the integral defined in chapter 3 and the Riemann-Stieltjes integral, a modification of Riemann integral which has proved to be of considerable utility in statistics. Let $(L_1, \|.\|_1)$ and $(L_2, \|.\|_2)$ denote an L_1-space and L_2-space, respectively.

Definition 4.1.1. Let $f, g : [a, b] \to (L_2, \|.\|_2)$. *We say that* f is *integrable with respect to* g on $[a, b]$ if for each $\epsilon > 0$, there exists a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \sum_{i=1}^{m} f(v_i)[g(u_i) - g(u_{i-1})] \right\|_1 \leq \epsilon \qquad (4.1.1)$$

whenever $D = \{([x_{i-1}, x_i], c_i) | i = 1, 2, .., n\}$ and $D' = \{([u_{j-1}, u_j], v_j) | j = 1, 2, ..., m\}$ are δ-fine partitions on $[a, b]$.

For a given function $g : [a, b] \to (L_2, \|.\|_2)$, if $f : [a, b] \to (L_2, \|.\|_2)$ is integrable with respect to g on $[a, b]$ and ϵ is a positive real number, then the gauge δ satisfying the condition (4.1.1) is called an ϵ-*adapted gauge for* f (over g) on $[a, b]$. If there is no confusion which function g is concerning, we may shortly say that δ is an ϵ-adapted for f on $[a, b]$.

In the above definition, a criteria of integrability is given, but for an integrable function, the integral is not given. The next theorem guarantees the existence of the integral of integrable function, which is an element in L_1.

Theorem 4.1.2. Let f and g be defined on $[a,b]$ with values in L_2. Then f is integrable with respect to g on $[a,b]$ if and only if there exists $A \in L_1$ such that for every $\epsilon > 0$, there corresponds a gauge δ on $[a,b]$ such that whenever $D = \{([x_{i-1}, x_i], c_i) | i = 1, ..., n\}$ is a δ-fine partition of $[a,b]$, we have

$$\|S_g(f,D) - A\|_1 \leq \epsilon, \tag{4.1.2}$$

where $S_g(f,D) = \sum_{i=1}^{n} f(t_i)[g(x_i) - g(x_{i-1})]$.

Proof. Suppose that f is integrable with respect to g on $[a,b]$. Then for each $n \in \mathbb{N}$, there is a gauge δ_n on $[a,b]$ which is $\frac{1}{n}$-adapted for f. That is if $D = \{([x_{i-1}, x_i], t_i) | i = 1, ..., n\}$ and $D' = \{([u_{j-1}, u_j], v_j) | j = 1, 2, ..., m\}$ are two δ_n-fine partitions on $[a,b]$, then

$$\|S_g(f,D) - S_g(f,D')\|_1 < \frac{1}{n}. \tag{4.1.3}$$

We may assume that for each $n \in \mathbb{N}$, $\delta_{n+1}(x) \leq \delta_n(x)$ for all $x \in [a,b]$. Otherwise, we let $\delta'_{n+1} = \min(\delta_n, \delta_{n+1})$. Consequently for $m \geq n$, every δ_m-fine partition is δ_n-fine partition.

For each $n \in \mathbb{N}$, let D_n be a fixed δ_n-fine partition. From (4.1.3), we note that $(S_g(f, D_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in L_1, hence there is an element $A \in L_1$ such that

$$\lim_{n\to\infty} S_g(f, D_n) = A$$

in L_1. Let $\epsilon > 0$ be arbitrary and let N be an integer greater than $\frac{2}{\epsilon}$ such that if $n \geq N$, then

$$\|S_g(f, D_n) - A\|_1 \leq \frac{\epsilon}{2}. \tag{4.1.4}$$

For $n \geq N$, if D'_n is any δ_n-fine division, then by the triangle inequality, we have

$$\begin{aligned}
\|S_g(f, D'_n) - A\|_1 &\leq \|S_g(f, D'_n) - S_g(f, D_n)\|_1 + \|S_g(f, D_n) - A\|_1\,, \\
&< \frac{1}{n} + \frac{\epsilon}{2} \quad \text{by } (4.1.4), \\
&\leq \frac{1}{N} + \frac{\epsilon}{2} \quad \text{for } n \geq N, \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{for } N \geq \frac{2}{\epsilon}, \\
&= \epsilon.
\end{aligned}$$

Now, assume the converse. We will show that f is integrable with respect to g on $[a, b]$. Let $\epsilon > 0$ be given. By (4.1.2), there is a gauge δ on $[a, b]$ such that whenever D is a δ - fine partitions on $[a, b]$, we have

$$\|S_g(f, D) - A\| \leq \frac{\epsilon}{2}.$$

Therefore, if D_1 and D_2 are any two δ-fine partitions on $[a, b]$

$$\|S_g(f, D_1) - S_g(f, D_2)\|_1 \leq |S_g(f, D_1) - A\|_1 + \|S_g(f, D_2) - A\|_1 \;\; < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, f is integrable with respect to g on $[a, b]$. □

It is obvious that the element A in the previous theorem is unique. It is called *the integral of* f (*with respect to* g) on $[a, b]$ and denoted by

$$\int_a^b f dg.$$

4.2 Elementary Properties of the Integral

In this section, we present the elementary properties of the Henstock-Stieltjes integral defined in the previous section.

Theorem 4.2.1. If f_1, f_2 and f are Henstock Stieltjes integrable with respect to g on $[a, b]$ and $\lambda \in \mathbb{R}$, then so are $f_1 + f_2$ and λf, and

$$\int_a^b (f_1 + f_2) dg = \int_a^b f_1 dg + \int_a^b f_2 dg,$$

$$\int_a^b \lambda f dg = \lambda \int_a^b f dg$$

Proof. Let f_1 and f_2 be Henstock-Stieltjes integrable with respect to g on $[a, b]$. Let $\epsilon > 0$ be arbitrary. Let δ_1 and δ_2 be gauges on $[a, b]$ such that

$$\left\| S_g(f_1, D_1) - \int_a^b f_1 dg \right\|_1 \leq \frac{\epsilon}{2} \tag{4.2.1}$$

for any δ_1-fine partition D_1 on $[a, b]$ and

$$\left\| S_g(f_2, D_2) - \int_a^b f_2 dg \right\|_1 \leq \frac{\epsilon}{2} \tag{4.2.2}$$

for any δ_2-fine partition D_2 on $[a, b]$.

Let $\delta : [a, b] \to \mathbb{R}^+$ defined by $\delta(t) = \min(\delta_1(t), \delta_2(t))$. Then δ is a gauge on $[a, b]$ and for any δ-fine partition D on $[a, b]$, we have

$$\left\| S_g(f_1 + f_2, D) - [\int_a^b f_1 dg - \int_a^b f_2 dg] \right\|_1 \leq \epsilon,$$

by the triangle inequality, (4.2.1) and (4.2.2).

Now, let f be Henstock-Stieltjes integrable with respect to g on $[a, b]$ and $\lambda \in \mathbb{R}$ be arbitrary. If δ is a gauge on $[a, b]$ and D is a δ-fine partition on $[a, b]$, then

$$\left\| S_g(\lambda f, D) - \lambda \int_a^b f dg \right\|_1 = |\lambda| \left\| S_g(f, D) - \int_a^b f dg \right\|_1 . \tag{4.2.3}$$

So it is obvious that the integrability of f implies the integrability of λf for any $\lambda \in \mathbb{R}$, and

$$\int_a^b \lambda f dg = \lambda \int_a^b f dg.$$

$\square$

Theorem 4.2.2. Let $f, g : [a,b] \to L_2$, and $c \in (a,b)$. Then f is Henstock-Stieltjes integrable with respect to g on $[a,b]$ if and only if the restrictions of f to $[a,c]$ and $[c,b]$ are both Henstock-Stieltjes integrable. If this is the case, we have

$$\int_a^b f dg = \int_a^c f dg + \int_c^b f dg. \tag{4.2.4}$$

Proof. Let f_1, the restriction of f to $[a,c]$, and f_2, the restriction of f to $[c,b]$, be Henstock-Stieltjes integrable with respect to g. Let $\epsilon > 0$ be given. Then there are gauges δ_1 on $[a,c]$ and δ_2 on $[c,b]$ such that if D_1 is a δ_1-fine partition on $[a,c]$ and D_2 is a δ_2-fine partition on $[c,b]$ then

$$\left\| S_g(f_1, D_1) - \int_a^c f dg \right\|_1 < \frac{\epsilon}{2} \quad \text{and} \quad \left\| S_g(f_2, D_2) - \int_c^b f dg \right\|_1 < \frac{\epsilon}{2} \tag{4.2.5}$$

We define a gauge δ on $[a,b]$ by

$$\delta(t) = \begin{cases} \min\{\delta_1(t), \frac{1}{2}(c-t)\} & \text{if } t \in [a,c), \\ \min\{\delta_1(c), \delta_2(c)\} & \text{if } t = c, \\ \min\{\delta_2(t), \frac{1}{2}(t-c)\} & \text{if } t \in (c,b]. \end{cases}$$

Then $\delta \leq \delta_1$ on $[a,c]$ and $\delta \leq \delta_2$ on $[c,b]$.

Let D be any δ-fine division on $[a,b]$, say $D = \{([x_{i-1}, x_i], t_i) | i = 1, 2, ..., n\}$. Then $c = t_{i_o}$ for some $i_o \in \{1, 2, ..., n\}$.

Let $D_1 = \{([x_{i-1}, x_i], t_i) | i = 1, ..., i_o - 1\} \cup \{([x_{i_o - 1}, t_{i_o}], t_{i_o})\}$, and

$D_2 = \{([x_{i-1}, x_i], t_i) | i = i_o + 1, ..., n\} \cup \{([t_{i_o}, x_{i_o}], t_{i_o})\}$. Then D_1 is a δ-fine partition, and hence a δ_1-fine partition on $[a, c]$. And also D_2 is a δ_2-fine partition on $[c, b]$. Note that

$$S_g(f, D) = S_g(f, D_1) + S_g(f, D_2). \tag{4.2.6}$$

Since D_1 is a δ_1-fine partition and D_2 is a δ_2-fine partition by (4.2.5) and (4.2.6), we conclude that

$$\begin{aligned}\left\| S_g(f, D) - \left(\int_a^c f dg + \int_c^b f dg\right) \right\|_1 &\leq \left\| S_g(f, D_1) - \int_a^c f dg \right\|_1 + \left\| S_g(f, D_2) - \int_c^b f dg \right\|_1 \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, f is Henstock-Stieltjes integrable with respect to g on $[a, b]$ and the equality in (4.2.4) holds.

Conversely, suppose that f is Henstock-Stieltjes integrable on $[a, b]$. Let ϵ be arbitrary. Then there is a gauge δ on $[a, b]$ such that for any δ-fine partitions D_1, D_2 on $[a, b]$,

$$\| S_g(f, D_1) - S_g(f, D_2) \|_1 \leq \epsilon. \tag{4.2.7}$$

Let $\delta' = \delta|_{[a,c]}$, D' and D'' be δ'-fine partitions on $[a, c]$. For $\delta'' = \delta|_{[c,b]}$, choose a partition $\widehat{D}$ on $[c, b]$ which is a δ''-fine partition.

Consider $D' \cup \widehat{D}$ and $D'' \cup \widehat{D}$. They are δ-fine partitions on $[a, b]$ and so by (4.2.6), we have

$$\begin{aligned}\| S_g(f, D') - S_g(f, D'') \|_1 &= \left\| S_g(f, D' \cup \widehat{D}) - S_g(f, D'' \cup \widehat{D}) \right\|_1 \\ &\leq \epsilon.\end{aligned}$$

This implies that f is Henstock-Stieltjes integrable on $[a, c]$.Similarly, f can be shown to be Henstock-Stieltjes integrable on $[c, b]$. Therefore, the equality (4.2.4) holds from the first part of the theorem. □

Corollary 4.2.3. If f is Henstock-Stieltjes integrable with respect to g on $[a, b]$ and $[c, d] \subseteq [a, b]$, then so is the restriction of f to $[c, d]$.

Proof. If f is Henstock-Stieltjes integrable with respect to g on $[a, b]$ and $c \in [a, b]$, then it follows from the previous theorem that the restriction of f to $[c, b]$ is also Henstock-Stieltjes integrable with respect to g. And if $d \in [c, d]$, another application of the theorem shows that the restriction to $[c, d]$ of f is also Henstock-Stieltjes integrable. □

The next corollary is immediately obtained from Theorem 4.2.2 by using mathematical induction.

Corollary 4.2.4. If f is Henstock-Stieltjes integrable with respect to g on $[a, b]$ and $a = c_0 < c_1 < \ldots < c_n = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are integrable and

$$\int_a^b f dg = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f dg.$$

Definition 4.2.5. If f is Henstock-Stieltjes integrable with respect to g on $[a, b]$ and $a \leq c \leq d \leq b$, we define

$$\int_d^c f dg = -\int_c^d f dg$$

and

$$\int_c^c f dg = 0.$$

Corollary 4.2.6. If f is Henstock-Stieltjes with respect to g on $[a, b]$ and c, d, e are any points in $[a, b]$, then

$$\int_c^e f dg = \int_c^d f dg + \int_d^e f dg, \tag{4.2.8}$$

in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality (4.2.8).

4.3 The Saks-Henstock Lemma

By the result of Theorem 4.1.2, we have as an equivalent condition for Henstock-Stieltjes integrability of f on $[a,b]$ that given any positive real number ϵ, there exists a gauge δ on $[a,b]$ such that if D is any δ-fine partition of $[a,b]$, then

$$\left\| S_g(f,D) - \int_a^b f dg \right\|_1 \leq \epsilon. \tag{4.3.1}$$

The inequality (4.3.1) can be referred as an approximation of the Henstock-Stieltjes integral of f on $[a,b]$ by the Riemann sum of f with respect to g and the δ-fine partition D. The Saks-Henstock Lemma asserts that the same degree of approximation is valid for

$$\left\| S_g(f,D') - \int_{\cup_{i=1}^n I_i} f dg \right\|_1 \leq \epsilon$$

for any δ-fine subpartition $D' = \{(I_i, t_i) | i = 1, 2, ..., n\}$ of $[a,b]$. This fact may not seem so surprising if the union of subintervals in D' is a subinterval of $[a,b]$. But it is not obvious that the result remains true for an arbitrary collection of subintervals. The next theorem is analogous to the important result known as the Saks-Henstock Lemma. The idea of the proof is due to this well known lemma.

Theorem 4.3.1. Let f be Henstock-Stieltjes integrable on $[a,b]$ and ϵ be any positive real number. Let δ be any gauge on $[a,b]$ such that for any δ-fine partition D on $[a,b]$

$$\left\| S_g(f,D) - \int_a^b f dg \right\|_1 \leq \epsilon.$$

Then for any δ-fine subpartition,

$$D' = \{([x_i, y_i], t_i) | i = 1, 2, ..., n\}$$

of $[a, b]$

$$\left\|\sum_{i=1}^{n}\{f(t_i)[g(y_i)-g(x_i)]-\int_{x_i}^{y_i} fdg\}\right\|_1 = \left\|S_g(f,D')-\int_{\cup_{i=1}^n[x_i,y_i]} fdg\right\|_1 \leq \epsilon \tag{4.3.2}$$

Proof. Let $K_1, ..., K_m$ be closed subintervals in $[a, b]$ such that $\{[x_i, y_i] | i = 1, 2, ..., n\} \cup \{K_1, ..., K_m\}$ forms a partition of $[a, b]$.

Now let $\gamma > 0$ be arbitrary. Since (by Theorem 4.2.2) the restriction of f to each subinterval K_j $(j = 1, 2, ..., m)$ is Henstock-Stieltjes integrable, there exists a gauge $\delta_{\gamma,j}$ on K_j such that if D_j is a $\delta_{\gamma,j}$-fine partition of K_j, then

$$\left\|S_g(f,D_j)-\int_{K_j} f\right\|_1 < \frac{\alpha}{m}. \tag{4.3.3}$$

We may assume that $\delta_{\gamma,j}(x) \leq \delta(x)$ for all $x \in K_j$. Let $D = D' \cup D_1 \cup ... \cup D_m$. Then D is a δ-fine partition of $[a, b]$ and

$$\left\|S_g(f,D)-\int_a^b fdg\right\|_1 \leq \epsilon.$$

Furthermore,

$$S_g(f,D) = S_g(f,D') + S_g(f,D_1) + ... + S_g(f,D_m)$$

and

$$\int_a^b fdg = \int_{\cup_{i=1}^n[x_{i-1},x_i]} fdg + \int_{K_1} fdg + ... + \int_{K_m} fdg.$$

Consequently, we obtain

$\left\|S_g(f,D')-\int_{\cup_{i=1}^n[x_{i-1},x_i]} fdg\right\|_1$

$= \left\|\{S_g(f,D)-[S_g(f,D_1)+...+S_g(f,D_m)]\}-\{\int_a^b fdg-[\int_{K_1} fdg+...+\int_{K_m} fdg]\}\right\|_1$

$\leq \left\|S_g(f,D)-\int_a^b fdg\right\|_1 + \left\|S_g(f,D_1)-\int_{K_1} fdg\right\|_1 + ... + \left\|S_g(f,D_m)-\int_{K_m} fdg\right\|_1$

$\leq \epsilon + m\frac{\gamma}{m}$

$= \epsilon + \gamma.$

Since $\gamma > 0$ is arbitrary, then $\left\|S_g(f,D')-\int_{\cup_{i=1}^n[x_{i-1},x_i]} fdg\right\|_1 \leq \epsilon$ as required. □

Corollary 4.3.2. Let f be Henstock-Stieltjes integrable with respect to g on $[a,b]$ and $\epsilon > 0$ be arbitrary. Let δ be a gauge on $[a,b]$ such that for any δ-fine partition $D = \{([x_{i-1}, x_i], t_i) | i = 1, 2, ..., n\}$ of $[a,b]$,

$$\left\| S_g(f, D) - \int_a^b f dg \right\|_1 \leq \epsilon.$$

Then for each $i = 1, 2, ..., n$,

$$\left\| f(t_i)[g(x_i) - g(x_{i-1})] - \int_{x_{i-1}}^{x_i} f dg \right\|_1 \leq \epsilon.$$

Proof. The conclusion is immediately obtained from Theorem 4.3.1 and the fact that $\{([x_{i-1}, x_i], t_i)\}$ is a δ-fine subpartition on $[a,b]$. $\square$

4.4 The Fundamental Theorem of Calculus

Now, we investigate an analogous result of the fundamental of calculus. In this section, we give an important application of Sakes-Henstock lemma (Theorem 4.3.1), by establishing the continuity of the indefinite integral. For simplicity, we consider here only the indefinite integral with the point a as the left endpoint, since any other indefinite integral differs from this by a constant (in L_1). However, in our work, the continuity of the integrator g is assumed.

Consider a function $f : [a,b] \to L_2$ which is Henstock-Stieltjes integrable with respect to g on $[a,b]$ $(g : [a,b] \to L_2)$. For any $x \in [a,b]$, the integral of f (with respect to g) on $[a,x]$ exists, so we can define a function $F : [a,b] \to L_1$ by

$$F(x) = \int_a^x f dg$$

for $x \in [a,b]$.

Theorem 4.4.1. If f is Henstock-Stieltjes integrable with respect to a continuous function g on $[a,b]$, then the indefinite integral

$$F(x) := \int_a^x f dg,$$

for $x \in [a,b]$, is continuous on $[a,b]$.

Proof. Let $c \in [a,b)$ be arbitrary. We will show that F is continuous from the right at c. Let $\epsilon > 0$ be given. Since g is continuous at c, there is an $\alpha > 0$ such that for any $t \in [a,b]$ with $|t-c| < \alpha$,

$$\|g(t) - g(c)\|_2 < \frac{\epsilon}{\|f(c)\|_2 + 1}.$$

Let δ be a gauge on $[a,b]$ such that for any δ-fine partition D on $[a,b]$,

$$\left\| S_g(f,D) - \int_a^b f dg \right\|_1 \le \epsilon.$$

Now, define a gauge δ' on $[a, b]$ by for each $t \in [a, b]$

$$\delta'(t) = \begin{cases} \min\{\delta(t), \frac{1}{2}|t-c|\} & \text{if} \quad t \neq c, \\ \delta(c) & \text{if} \quad t = c. \end{cases}$$

let $0 < h < \min\{\delta'(c), \alpha\}$ and D' be the δ'-fine subpartition consisting of only one member; that is $D' = \{([c, c+h], c)\}$. Then by applying the Corollary 4.3.2, we have

$$\left\| f(c)[g(c+h) - g(c)] - \int_c^{c+h} f dg \right\|_1 \leq \epsilon.$$

Hence, it follows from $h \leq \alpha$ that

That

$$\begin{aligned} \|F(c+h) - F(c)\|_1 &= \left\| \int_c^{c+h} f dg \right\|_1 \\ &\leq \|f(c)[g(c+h) - g(c)]\|_1 + \epsilon \\ &\leq \|f(c)\|_2 \, \|[g(c+h) - g(c)]\|_2 + \epsilon \quad \text{(by Cauchy Schwarz inequality)} \\ &\leq \|f(c)\|_2 \frac{\epsilon}{\|f(c)\|_2 + 1} + \epsilon \\ &< 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then F is continuous at $c$from the right. We can show that F is continuous from the left at any point in $(a, b]$ by the same manner. □

To establish the fundamental theorem of calculus, it is useful to confine our consideration of the integrator g to be *nice* enough. We consider only the function g which does not *oscillate too much* in the sense we will now define.

Definition 4.4.2. Let $g : [a, b] \to L_2$. We define the *variation of* g over the interval $I = [a, b]$ to be

$$Var(g, I) = sup\{\sum_{i=1}^{n} \|g(x_i) - g(x_{i-1})\|_2 \, |\mathcal{P} = \{x_o, ..., x_n\}\}$$

where the supremum is taken over all partition $\mathcal{P}$ of I. We say that g *has (or is of) bounded variation on* I if $Var(g, I) < \infty$. The collection of all functions defined on I with values in L_2 that have bounded variation on is denoted by $BV(I, L_2)$.

In order to investigate the fundamental theorem of calculus we recall the definition of the derivative with respect to g of a function [8].

Definition 4.4.3. Let $F : [a, b] \to L_1$ and $g : [a, b] \to L_2$. Then F is said to be *differentiable* with respect to g on $[a, b]$ if there is a function $f : [a, b] \to L_2$ satisfying the following condition: for any $\epsilon > 0$, there is a gauge δ on $[a, b]$ such that for any $x \in [a, b]$, if $u, v \in [a, b]$ are such that $|u - x| < \delta(x)$ and $|v - x| < \delta(x)$ then

$$\|F(v) - F(u) - f(x)[g(v) - g(u)]\|_1 \leq \epsilon \|g(v) - g(u)\|_2 .$$

The function f is called the *derivative* (with respect to g) of F on $[a, b]$, and denoted by $\frac{dF}{dg}$.

Theorem 4.4.4. Let $g : [a, b] \to L_2$ belong to $BV([a, b], L_2)$ of F on $[a, b]$.If f is the derivative with respect to g of F on $[a, b]$, then f is Henstock-Stieltjes integrable with respect to g on $[a, b]$ and

$$\int_a^b f dg = F(b) - F(a).$$

Proof. Let $\alpha = Var(g, I)$. Since $g \in BV([a, b], L_2)$, $0 \leq \alpha < \infty$. Let $\epsilon > 0$ be given. There is a gauge δ on $[a, b]$ such that for any δ-fine partition $D = \{([x_{i-1}, x_i], t_i) | i = 1, 2, ..., n\}$ on $[a, b]$ we have

$$\|F(x_i) - F(x_{i-1}) - f(t_i)[g(x_i) - g(x_{i-1})]\|_1 \leq \frac{\epsilon}{\alpha + 1} \|g(x_i) - g(x_{i-1})\|_2 .$$

By using the telescoping sum $F(b) - F(a) = \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})]$, we have

$$\begin{aligned}\|F(b) - F(a) - S_g(f, D)\|_1 &= \left\|\sum_{i=1}^{n}[F(x_i) - F(x_i - 1)] - \sum_{i=1}^{n} f(t_i)[g(x_i) - g(x_{i-1})]\right\|_1 \\ &\leq \sum_{i=1}^{n} \|F(x_i) - F(x_i - 1) - f(t_i)[g(x_i) - g(x_{i-1})]\|_1 \\ &\leq \frac{\epsilon}{\alpha - 1} \sum_{i=1}^{n} \|[g(x_i) - g(x_{i-1})]\|_2 \\ &< \epsilon.\end{aligned}$$

This implies that $\int_a^b f dg = F(b) - F(a)$. □

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