CHAPTER III

THEORETICAL CONSIDERATIONS

3.1 Basic Equations and General Solutions

Consider axisymmetric deformations of a poroelastic half-space with respect to the conventional cylindrical coordinates system (r,z) as shown in Figure 1. Following Rice and Cleary (1976), the constitutive relation for a homogeneous poroelastic material with compressible constituents can be expressed as the following:

$$\sigma_{rr} = 2\mu\left[\frac{\partial u_r}{\partial r} + \frac{\nu}{1-2\nu}\varepsilon\right] - \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)}p \tag{3.1}$$

$$\sigma_{\theta\theta} = 2\mu\left[\frac{u_r}{r} + \frac{\nu}{1-2\nu}\varepsilon\right] - \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)}p \tag{3.2}$$

$$\sigma_{zz} = 2\mu\left[\frac{\partial u_z}{\partial z} + \frac{\nu}{1-2\nu}\varepsilon\right] - \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)}p \tag{3.3}$$

$$\sigma_{zr} = \mu\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right] \tag{3.4}$$

where σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{zr} are the total stress components of the bulk material; ε is the dilatation of the solid matrix; p is defined as the excess pore fluid pressure (suction is considered negative); μ, ν and ν_u denote the shear modulus, drained and undrained Poisson's ratio respectively. In addition, B is Skempton's (1954) pore pressure coefficient.

It is noted that $0 \le B \le 1$ and $\nu \le \nu_u \le 0.5$ for all poroelastic materials. The limiting cases of a poroelastic solid with incompressible constituents and a dry elastic material are obtained when $\nu_u = 0.5$ and $B = 1$, and $B \to 0$ respectively. The excess pore fluid pressure can be expressed as

$$p = -\frac{2\mu B(1+\nu_u)}{3(1-2\nu_u)}\varepsilon + \frac{2\mu B^2(1-2\nu)(1+\nu_u)^2}{9(1-2\nu_u)(\nu_u-\nu)}\zeta \tag{3.5}$$

where ζ denotes the variation of fluid volume per unit reference volume. In addition, u_i and ψ_i denote the average displacement of solid matrix and the fluid displacement relative to the solid matrix, in the i^{th} direction $(i = r, z)$ respectively. Then,

$$\psi_i = \int_0^t q_i dt \qquad \text{and} \qquad q_i = -\kappa \frac{\partial p}{\partial i} \tag{3.6}$$

where q_i is the fluid discharge in the i^{th} direction and κ is the coefficient of permeability of the medium.

The quasi-static governing equations for a poroelastic medium with compressible constituents, expressed in terms of stresses and pore pressure as basic variables, can be transformed into Navier equations with coupling terms and a diffusion equation by treating the displacements, u_i and the variation of fluid volume per unit reference volume, ζ, as the basic unknowns as

$$\nabla^2 u_r + \frac{1}{1-2\nu_u}\frac{\partial \varepsilon}{\partial r} - \frac{u_r}{r^2} - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{\partial \zeta}{\partial r} = 0 \tag{3.7}$$

$$\nabla^2 u_z + \frac{1}{1-2\nu_u}\frac{\partial \varepsilon}{\partial z} - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{\partial \zeta}{\partial z} = 0 \tag{3.8}$$

$$\nabla^2 \zeta = \frac{\partial \zeta}{c\partial t} \tag{3.9}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{3.10}$$

$$\varepsilon = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \tag{3.11}$$

$$c = \frac{2\mu\kappa B^2(1-\nu)(1+\nu_u)^2}{9(1-\nu_u)(\nu_u-\nu)} \tag{3.12}$$

The general solutions for equations (3.7) to (3.9) can be derived by applying Laplace-Hankel transform (order zero) with respect to time and radial coordinates respectively.

At this stage, it is convenient to nondimensionalize all quantities including the co-ordinate frame with respect to length and time by selecting a layer thickness "h" as a unit length and "h^2/c" as a unit time respectively. All variables will be replaced by appropriate nondimensional variables, but the previous notations will be used for convenience.

The Laplace-Hankel transform of a function $\phi(r,z,t)$ with respect to the variable t and r respectively, is defined by (Sneddon, 1951)

$$\tilde{\phi}(r,z,s)=\int_0^\infty\int_0^\infty \phi(r,z,t)e^{-st}J_0(\xi r)rdrdt \tag{3.13}$$

In equation (3.13), s and ξ denote the Laplace and Hankel transform parameters respectively, and $J_0(\xi r)$ denotes the Bessel function of the first kind of order zero. The inverse relationship is given by

$$\phi(r,z,t)=\frac{1}{2\pi i}\int_{\varpi-i\infty}^{\varpi+i\infty}\int_0^\infty \tilde{\phi}(r,z,s)\, e^{st}J_0(\xi r)d\xi ds \tag{3.14}$$

where ϖ is greater than the real part of all singularities of $\tilde{\phi}(r,z,s)$ and i is the imaginary number.

It can be shown that the general solution for the solid and fluid displacements, pore pressure and stresses in the Laplace-Hankel transform space can be expressed in the following matrix form (Senjuntichai and Rajapakse, 1995)

$$\mathbf{u}(\xi,z,s)=\mathbf{R}(\xi,z,s)\mathbf{X}(\xi,s) \tag{3.15}$$

$$\mathbf{f}(\xi,z,s)=\mathbf{S}(\xi,z,s)\mathbf{X}(\xi,s) \tag{3.16}$$

in which

$$\mathbf{u}(\xi,z,s)=\left[u_i(\xi,z,s)\right]^{\mathrm{T}} \quad ; i=1,2,3 \tag{3.17}$$

$$\mathbf{f}(\xi,z,s)=\left[f_i(\xi,z,s)\right]^{\mathrm{T}} \quad ; i=1,2,3 \tag{3.18}$$

$$u_1(\xi,z,s)=\tilde{u}_r \tag{3.19}$$

$$u_2(\xi,z,s)=\tilde{u}_z \tag{3.20}$$

$$u_3(\xi,z,s)=\tilde{p} \tag{3.21}$$

$$f_1(\xi,z,s)=\tilde{\sigma}_{zr} \tag{3.22}$$

$$f_2(\xi,z,s)=\tilde{\sigma}_{zz} \tag{3.23}$$

$$f_3(\xi,z,s)=\tilde{\psi} \tag{3.24}$$

$$\mathbf{X}(\xi,s)=\begin{bmatrix}A & B & C & D & E & F\end{bmatrix}^{\mathrm{T}} \tag{3.25}$$

and the matrices $\mathbf{R}(\xi,z,s)$ and $\mathbf{S}(\xi,z,s)$ in equations (3.15) and (3.16) are explicitly given by equations (A - 1) to (A - 10) in Appendix A. The arbitrary functions $A(\xi,s)$, $B(\xi,s)$, ..., $F(\xi,s)$ appearing in the vector $\mathbf{X}(\xi,s)$ are to be determined by employing appropriate boundary and/or continuity conditions.

3.2 Stiffness Matrices

A multi-layered system with a total of Nl poroelastic layers overlying a poroelastic half-space is considered in this section. Layers and interfaces are numbered as shown in Figure 1. A subscript "n" is applied to denote quantities associated with the n^{th} layer ($n=1,2,...,Nl$). For the n^{th} layer, the following relationships can be established by using equations (3.15) and (3.16) :

$$\mathbf{U}^{(n)}=\begin{bmatrix}\mathbf{R}^{(n)}(\xi,z_n,s)\\ \cdots\cdots\cdots\cdots\\ \mathbf{R}^{(n)}(\xi,z_{n+1},s)\end{bmatrix}\mathbf{X}^{(n)}(\xi,s) \tag{3.26}$$

$$\mathbf{F}^{(n)}=\begin{bmatrix}-\mathbf{S}^{(n)}(\xi,z_n,s)\\ \cdots\cdots\cdots\cdots\\ \mathbf{S}^{(n)}(\xi,z_{n+1},s)\end{bmatrix}\mathbf{X}^{(n)}(\xi,s) \tag{3.27}$$

$$\mathbf{U}^{(n)}=\begin{bmatrix}\mathbf{u}^{(n)}(\xi,z_n,s) & \mathbf{u}^{(n)}(\xi,z_{n+1},s)\end{bmatrix}^T \tag{3.28}$$

$$\mathbf{F}^{(n)}=\begin{bmatrix}-\mathbf{f}^{(n)}(\xi,z_n,s) & \mathbf{f}^{(n)}(\xi,z_{n+1},s)\end{bmatrix}^T \tag{3.29}$$

From equations (3.26) to (3.29), $\mathbf{U}^{(n)}$ denotes a vector of generalized displacements for the n^{th} layer whose elements related to the Laplace-Hankel transform of displacements and pore pressure of the top and bottom surfaces of the n^{th} layer. Similarly, $\mathbf{F}^{(n)}$ denotes a generalized force vector whose elements related to the Laplace-Hankel transform of the tractions and fluid displacement of the top and bottom surfaces of the n^{th} layer.

The matrices $\mathbf{R}^{(n)}$ and $\mathbf{S}^{(n)}$ in equations (3.26) and (3.27) are identical to $\mathbf{R}$ and $\mathbf{S}$ defined in Appendix A except that the material properties of the n^{th} layer are employed in the definition and $z = z_n$ or z_{n+1}. Equation (3.26) can be inverted to express $\mathbf{X}^{(n)}$ in terms of $\mathbf{U}^{(n)}$ and the substitution in equation (3.27) yields

$$\mathbf{F}^{(n)} = \mathbf{K}^{(n)}\mathbf{U}^{(n)} \tag{3.30}$$

where $\mathbf{K}^{(n)}$ can be considered as an exact stiffness matrix in the Laplace-Hankel transform space describing the relationship between the generalized displacement vector $\mathbf{U}^{(n)}$ and the generalized force vector $\mathbf{F}^{(n)}$ for the n^{th} layer. In equation (3.30), the layer stiffness matrix $\mathbf{K}^{(n)}$ is a 6x6 symmetric matrix and its element, k_{ij}, is function of layer thickness, layer material properties and Laplace and Hankel transform parameter respectively.

For the underlying half-space, the arbitrary function $A^{(Nl+1)}$, $C^{(Nl+1)}$ and $E^{(Nl+1)}$ are set to be zero to guarantee the regularity of the solutions at infinity. The stiffness matrix for the bottom half-space can be expressed as

$$\mathbf{F}^{(Nl+1)} = \mathbf{K}^{(Nl+1)}\mathbf{U}^{(Nl+1)} \tag{3.31}$$

where

$$\mathbf{U}^{(Nl+1)} = \left[\mathbf{u}^{(Nl+1)}\left(\xi, z_{Nl+1}, s\right)\right]^{\mathrm{T}} \tag{3.32}$$

$$\mathbf{F}^{(Nl+1)} = \left[-\mathbf{f}^{(Nl+1)}\left(\xi, z_{Nl+1}, s\right)\right]^{\mathrm{T}} \tag{3.33}$$

Due to the regularity condition at $z \to \infty$, the matrix $\mathbf{K}^{(Nl+1)}$ is a 3x3 symmetric matrix. It is noted that exponential terms of ξ and s are not involved in the expression of $\mathbf{K}^{(Nl+1)}$ and its elements depend only on the material properties of the underlying half-space and the Laplace and Hankel transform parameter respectively. The elements of $\mathbf{K}^{(n)}$ and $\mathbf{K}^{(Nl+1)}$ are explicitly given by Senjuntichai and Rajapakse (1995).

3.3 Global Stiffness Matrix

The global stiffness matrix of a multi-layered poroelastic half-space is assembled by using layer and half-space stiffness matrices together with the continuity conditions of tractions and fluid flow at the layer interfaces. For example, the continuity conditions at the n^{th} interface can be expressed as

$$\mathbf{f}^{(n-1)}(\xi, z_n, s) - \mathbf{f}^{(n)}(\xi, z_n, s) = \mathbf{T}^{(n)} \tag{3.34}$$

where $\mathbf{f}^{(n-1)}$ and $\mathbf{f}^{(n)}$ are defined in equation (3.18) and

$$\mathbf{T}^{(n)} = \begin{bmatrix} \widetilde{T}_r^{(n)} & \widetilde{T}_z^{(n)} & \dfrac{\widetilde{Q}^{(n)}}{s} \end{bmatrix}^{\mathrm{T}} \tag{3.35}$$

where $\widetilde{T}_i^{(n)}\,(i = r, z)$ and $\widetilde{Q}^{(n)}$ denote the Laplace-Hankel transform of tractions and fluid source which applied at the n^{th} interface respectively.

Consideration of equation (3.35) at each layer interface together with the layer and bottom half-space stiffness matrices defined in equations (3.30) and (3.31) results in the following global stiffness equation of order $3(Nl + 1)$:

$$\begin{bmatrix} \mathbf{K}^{(1)} & & & & \\ & \mathbf{K}^{(2)} & & & \\ & & \ddots & & \\ & & & \mathbf{K}^{(Nl)} & \\ & & & & \mathbf{K}^{(Nl+1)} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{(1)} \\ \mathbf{U}^{(2)} \\ \vdots \\ \mathbf{U}^{(Nl)} \\ \mathbf{U}^{(Nl+1)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \vdots \\ \mathbf{T}^{(Nl)} \\ \mathbf{T}^{(Nl+1)} \end{Bmatrix} \tag{3.36}$$

The solutions of equation (3.36) are the influence functions required to establish the flexibility equation for the derivation of the strain energy of a multi-layered poroelastic half-space.

3.4 Variational Formulation

Consider a multi-layered poroelastic half-space with an embedded elastic pile group as shown in Figure 1. The pile group is subjected to an axial load $V_0 H(t)$ when

$H(t)$ is a Heaviside step function and each pile in the group is assumed to be perfectly bonded to the surrounding medium. The pile cap is assumed to be rigid and is not in contact with the ground. Since we are concerned with the deformation of an elastic pile group where each pile has a large length-to-radius ratio, it is well justified (see Muki and Sternberg 1969, 1970) to assume one-dimensional behavior for each pile. It was also found that the deformation of each pile decreases gradually along its length towards the pile base (Poulos 1980). It is then well justified (see Selvadurai and Rajapakse 1990) to assume the distribution of deformation for each pile in the form of exponential function with a number of arbitrary coefficients.

$$w^i(z,t)=\sum_{m=1}^{Nt} e^{-(m-1)z/L}\alpha_m^i(t) \qquad ; i=1,2,...,Np \tag{3.37}$$

In the above equation, $\alpha_m^i(t)$ is the arbitrary coefficient, Nt denotes the number of terms used to represent the deformation of the pile group, Np and Ne are the total number of piles and elements used for discretizing the pile respectively. In addition, L is the total length of pile. The displacement profile along the i^{th} pile in equation (3.37) is indeterminate within the arbitrary coefficients $\alpha_1^i(t)$, $\alpha_2^i(t)$, ..., $\alpha_{Nt}^i(t)$. By using Laplace transformation, equation (3.37) can be transformed into the Laplace domain and can be written as

$$\overline{w}^i(z,s)=\sum_{m=1}^{Nt} e^{-(m-1)z/L}\overline{\alpha}_m^i(s) \qquad ; i=1,2,...,Np \tag{3.38}$$

where the superposed bar is used to denote the Laplace transform of a function.

The system in Figure 1 is decomposed into an extended poroelastic medium and a number of fictitious piles as shown in Figures 2(a) and 2(b) respectively. The system can be considered as a composite system of an extended poroelastic medium reinforced by fictitious piles with their Young's moduli being equal to the difference between the Young's moduli of the real piles and the medium. Then, the strain energy of each fictitious pile i^{th} corresponding to the assumed displacement function in the Laplace domain can be expressed as

$$Up^i = \int_L\int_A \frac{E^{i^*}\left(\partial\overline{w}^i/\partial z\right)^2}{2}dA\,dz \tag{3.39}$$

By substituting the deformation of the i^{th} pile equation (3.38), into equation (3.39, the strain energy of fictitious elastic pile i^{th} can be obtained in the following form

$$Up^i = \sum_{m=1}^{Nt}\sum_{n=1}^{Nt} \overline{\alpha}_m^i(s) D_{mn}^i \overline{\alpha}_n^i(s) \tag{3.40}$$

where $D_{mn}^i = \dfrac{\pi(a^i)^2(m-1)(n-1)}{2L(m+n\text{-}2)} \sum_{k=1}^{Ne} E_k^{i^*}\left[\left(e^{-(m+n-2)(z_k^i-\Delta t_k^i/2)} - e^{-(m+n-2)(z_k^i+\Delta t_k^i/2)}\right)\right]$

, for $m + n \neq 2$

$$D_{m1}^i = D_{1n}^i = 0 \quad \text{for} \quad m,n = 1,2,...,Nt \tag{3.41}$$

$$E_k^{i^*} = Ep^i - Es_k \qquad \text{for} \quad k = 1,2,...,Ne \tag{3.42}$$

Finally, the strain energy of the fictitious pile group can be expressed as

$$Up_g = \sum_{i=1}^{Np} Up^i \tag{3.43}$$

where Up_g and Up^i denote the strain energy functional of the fictitious pile group and the i^{th} pile, respectively in the Laplace transform space; a^i is radius of the i^{th} pile; Ep^i and Es_k are the modulus of elasticity of the i^{th} pile and the modulus of elasticity of the layer that is at the level of the k^{th} element respectively. In addition, Δt_k^i denotes the thickness of the k^{th} element of the i^{th} pile as shown in Figure 2(b). The derivation of Up^i is given in detail in Appendix B.

The solutions for the vertical body force over a cylindrical region identical to the real pile and the vertical displacement of the extended half-space are determined by discretizing the pile into a total of Ne elements. It is assumed that the vertical body force corresponding to the k^{th} element of the i^{th} pile is constant with each element as shown in Figure 2(a). The relationship between the unknown body forces and the vertical displacements of the extended half-space can be expressed in terms of the following flexibility equations

$$\begin{bmatrix} \mathbf{F}^{1,1} & \mathbf{F}^{1,2} & \cdots & \mathbf{F}^{1,j} & \cdots & \mathbf{F}^{1,Np} \\ \mathbf{F}^{2,1} & \mathbf{F}^{2,2} & \cdots & \mathbf{F}^{2,j} & \cdots & \mathbf{F}^{2,Np} \\ \vdots & \vdots & \ddots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ \mathbf{F}^{i,1} & \mathbf{F}^{i,2} & \cdots & \mathbf{F}^{i,j} & \cdots & \mathbf{F}^{i,Np} \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & \vdots \\ \mathbf{F}^{Np,1} & \mathbf{F}^{Np,2} & \cdots & \mathbf{F}^{Np,j} & \cdots & \mathbf{F}^{Np,Np} \end{bmatrix} \begin{Bmatrix} \mathbf{B}^1 \\ \mathbf{B}^2 \\ \vdots \\ \mathbf{B}^j \\ \vdots \\ \mathbf{B}^{Np} \end{Bmatrix} = \begin{Bmatrix} \mathbf{w}^1 \\ \mathbf{w}^2 \\ \vdots \\ \mathbf{w}^j \\ \vdots \\ \mathbf{w}^{Np} \end{Bmatrix} \tag{3.44}$$

where $\mathbf{F}^{i,j}$ is an $NexNe$ symmetric matrix defined as

$$\mathbf{F}^{i,j} = \begin{bmatrix} \bar{f}^{i,j}_{11} & \bar{f}^{i,j}_{12} & \cdots & \bar{f}^{i,j}_{1Ne} \\ \bar{f}^{i,j}_{21} & \bar{f}^{i,j}_{22} & \cdots & \bar{f}^{i,j}_{2Ne} \\ \vdots & \vdots & \bar{f}^{i,j}_{kl} & \vdots \\ \bar{f}^{i,j}_{Ne1} & \bar{f}^{i,j}_{Ne2} & \cdots & \bar{f}^{i,j}_{NeNe} \end{bmatrix} \quad \text{for} \quad i,j = 1,2,\ldots,Np \ \text{and} \ \ k,l = 1,2,\ldots,Ne \tag{3.45}$$

In the above equation, the element $\bar{f}^{i,j}_{kl}$ of the matrix $\mathbf{F}^{i,j}$ denotes the Laplace transform of the vertical displacement at the nodal point of k^{th} element of the i^{th} pile due to a vertical body force of unit intensity acting over the disk element corresponding to the middle of the l^{th} element of the j^{th} pile. In addition, the vertical displacement of the j^{th} pile can the be expressed as

$$\mathbf{w}^j = \boldsymbol{\omega}^j \boldsymbol{\alpha}^j \tag{3.46}$$

where the vectors $\mathbf{w}^j$ and $\boldsymbol{\alpha}^j$ denote the vertical displacement and the arbitrary coefficient respectively of the j^{th} pile given by

$$\mathbf{w}^j = \begin{bmatrix} \bar{w}^j_1 & \bar{w}^j_2 & \cdots & \bar{w}^j_k & \cdots & \bar{w}^j_{Ne} \end{bmatrix}^{\mathrm{T}} \tag{3.47}$$

and $$\boldsymbol{\alpha}^j = \begin{bmatrix} \bar{\alpha}^j_1 & \bar{\alpha}^j_2 & \cdots & \bar{\alpha}^j_m & \cdots & \bar{\alpha}^j_{Nt} \end{bmatrix}^{\mathrm{T}} \tag{3.48}$$

In addition, $$\boldsymbol{\omega}^j = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1m} & \cdots & \omega_{1Nt} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2m} & \cdots & \omega_{2Nt} \\ \vdots & \vdots & \ddots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ \omega_{k1} & \omega_{k2} & \cdots & \omega_{km} & \cdots & \omega_{kNt} \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & \vdots \\ \omega_{Ne1} & \omega_{Ne2} & \cdots & \omega_{Nek} & \cdots & \omega_{NeNt} \end{bmatrix} \tag{3.49}$$

where $\omega_{km} = e^{-(m-1)z_k/L}$ for $m = 1, 2, ..., Nt$ and $k = 1, 2, ..., Ne$ (3.50)

The vertical body forces over a cylindrical region identical to the real j^{th} pile, denoted by $\mathbf{B}^j$, can be expressed in matrix form as

$$\mathbf{B}^j = \boldsymbol{\beta}^j \boldsymbol{\alpha}^j \tag{3.51}$$

where $\mathbf{B}^j = \begin{bmatrix} \overline{B}_1^j & \overline{B}_2^j & \cdots & \overline{B}_k^j & \cdots & \overline{B}_{Ne}^j \end{bmatrix}^T$ (3.52)

and
$$\boldsymbol{\beta}^j = \begin{bmatrix} \beta_{11}^j & \beta_{12}^j & \cdots & \beta_{1m}^j & \cdots & \beta_{1Nt}^j \\ \beta_{21}^j & \beta_{22}^j & \cdots & \beta_{2m}^j & \cdots & \beta_{2Nt}^j \\ \vdots & \vdots & \ddots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ \beta_{k1}^j & \beta_{k2}^j & \cdots & \beta_{km}^j & \cdots & \beta_{kNt}^j \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & \vdots \\ \beta_{Ne1}^j & \beta_{Ne2}^j & \cdots & \beta_{Nek}^j & \cdots & \beta_{NeNt}^j \end{bmatrix} \tag{3.53}$$

In addition, β_{km}^j denotes the intensity of the body force acting on the disk element corresponding to the k^{th} element of the j^{th} pile when the vertical displacements of the fictitious contact surface at the k^{th} element of the j^{th} pile equal to the m^{th} exponential term of equation (3.37).

In view of equations (3.44), (3.46) and (3.51), the relationship between the matrices $\boldsymbol{\beta}^j$ and $\boldsymbol{\omega}^j$ can be rewritten as

$$\sum_{i=1}^{Np} \mathbf{F}^{i,j} \boldsymbol{\beta}^j = \boldsymbol{\omega}^j \quad \text{for } j = 1,2,...,Np \tag{3.54}$$

The detailed derivation of the relationship between unknown $\boldsymbol{\beta}^j$ and $\boldsymbol{\omega}^j$ is explicitly given in Appendix C.

The strain energy of the extended half-space corresponding to the i^{th} pile, Uh^i, by using the body force can then be expressed as

$$Uh^i = \int_L \int_A \frac{\overline{B}^i \overline{w}^i}{2} dA dV \tag{3.55}$$

where $\overline{B}^i$ denotes the body force corresponding to the i^{th} pile. By using equations (3.38) and (3.51), the strain energy of the extended half-space corresponding of i^{th} pile, Uh^i, can be obtain in the following form

$$Uh^i = \frac{\pi a^2}{2} \sum_{m=1}^{Nt} \sum_{n=1}^{Nt} \overline{\alpha}_m^i(s) \beta_{km}^i \Delta t_k^i \omega_{kn}^i \overline{\alpha}_n^i(s) \tag{3.56}$$

The strain energy of the extended half-space corresponding to the i^{th} pile can be rewritten as

$$Uh^i = \sum_{m=1}^{Nt} \sum_{n=1}^{Nt} \overline{\alpha}_m^i(s) M_{mn}^i \overline{\alpha}_n^i(s) \tag{3.57}$$

where $$M_{mn}^i = \frac{\pi a^2}{2} \sum_{k=1}^{Ne} \beta_{km}^i \Delta t_k^i \omega_{kn}^i \tag{3.58}$$

Then, the strain energy of the extended half-space of the pile group can be expressed as

$$Uh_g = \sum_{i=1}^{Np} Uh^i \tag{3.59}$$

The potential energy of the vertical load $\overline{V}^i$ at the top of the i^{th} pile in Figure 2(b) due to the assumed displacement function in the Laplace transform space can be expressed as

$$W^i = -\overline{V}^i \sum_{m=1}^{Nt} \overline{\alpha}_m^i(s) \tag{3.60}$$

where $\overline{V}^i$ denotes the vertical load corresponding to the i^{th} pile. Then, the potential energy of the total of vertical load can be expressed as

$$W_g = \sum_{i=1}^{Np} W^i \tag{3.61}$$

In view of equations (3.43), (3.59) and (3.61), the total potential energy functional of the pile group-multi-layered medium system in the Laplace domain, π, can be written as

$$\pi = \sum_{i=1}^{Np} \left(Up^i + Uh^i + W^i \right) \tag{3.62}$$

substituting equations (3.40), (3.57) and (3.60) in equation (3.62) yields

$$\pi = \sum_{i=1}^{Np}\sum_{m=1}^{Nt}\sum_{n=1}^{Nt}\overline{\alpha}_m^i(s)D_{mn}^i\overline{\alpha}_n^i(s) + \sum_{i=1}^{Np}\sum_{m=1}^{Nt}\sum_{n=1}^{Nt}\overline{\alpha}_m^i(s)M_{mn}^i\overline{\alpha}_n^i(s) - \sum_{i=1}^{Np}\sum_{m=1}^{Nt}\overline{\alpha}_m^i(s)\overline{V}^i \tag{3.63}$$

Minimization of the above functional with respect to the arbitrary coefficient $\overline{\alpha}_m^i$ for the i^{th} pile yields a system of linear simultaneous equations

$$\sum_{n=1}^{Nt}\left\{\left(D_{mn}{}^i + D_{nm}{}^i\right) + \left(M_{mn}{}^i + M_{nm}{}^i\right)\right\}\overline{\alpha}_n^i = \overline{V}^i \quad \text{for } m = 1,\ 2,\ \ldots,\ Nt \tag{3.64}$$

The equilibrium of the vertical load acting on the pile cap in the Laplace domain can be expressed as

$$\frac{V_0}{s} = \sum_{i=1}^{Np}\overline{V}^i \tag{3.65}$$

Due to assumption that the pile cap is rigid, then the deformation at the top of each pile must be the same, i. e.,

$$\sum_{m=1}^{Nt}\overline{\alpha}_m^i(s) = \sum_{m=1}^{Nt}\overline{\alpha}_m^j(s) \quad ;\ i,j = 1,2,\ldots,Np \tag{3.66}$$

Let consider the pile group configuration P2, P3, P4 and P5 as shown in Figure 3(b). For P2, P3 and P4, each pile in those groups will have the same value of $\sum_{i=1}^{Np}\mathbf{F}^{i,j}$. Therefore, the load at the top of each pile, $\overline{V}^i$, and its arbitrary coefficients, $\boldsymbol{\alpha}^i$ will be the same. From the equilibrium of the vertical load given by equation (3.65), the load at the top of each pile in Laplace domain will be equal to $V_0/(sNp)$. Thereafter, the solution of $\boldsymbol{\alpha}^i$ can be obtained from the system of linear equations (3.64).

Next, let consider the pile group P5 as shown in Figure 3(b). In this case, $\overline{V}^1 = \overline{V}^2 = \overline{V}^3 = \overline{V}^4 \neq \overline{V}^5$ and $\boldsymbol{\alpha}^1 = \boldsymbol{\alpha}^2 = \boldsymbol{\alpha}^3 = \boldsymbol{\alpha}^4 \neq \boldsymbol{\alpha}^5$. Then, the equilibrium equation (3.65) can be written as

$$4\overline{V}^1 + \overline{V}^5 = \frac{V_0}{s} \tag{3.67}$$

and the equation (3.66) for the rigid cap assumption, can be expressed as

$$\sum_{m=1}^{Nt} \overline{\alpha}_m^1(s) = \sum_{m=1}^{Nt} \overline{\alpha}_m^5(s) \tag{3.68}$$

In view of equations (3.67) and (3.68), the arbitrary coefficient $\boldsymbol{\alpha}^i$ $(i = 1,\ 2,\ 3\ and\ 4)$ and $\boldsymbol{\alpha}^5$ can now be determined by solving equation (3.64)

The solutions of the system of linear equations (3.64) results in the value of arbitrary coefficients $\overline{\alpha}_m^i$ in the Laplace transform space. The inverse Laplace transformation is applied to transform those arbitrary coefficients from the Laplace domain to the time domain. Finally, back substitution of α_m^i into equation (3.37) results in the time histories of displacement profiles of the pile group.

3.5 The Inverse Laplace-Hankel Integral Transform

The solution of equation (3.36) yields the Laplace-Hankel transforms of displacement and pore pressure at layer interfaces for discrete value of ξ and s . The time domain response of a multi-layered half-space is determined by numerically evaluating the inverse relationship of the integral appearing in equation (3.14). The Laplace inversion will be carried out numerically and the inverse of the integral with respect to ξ in equation (3.14) will be numerically evaluated by employing the trapezoidal rule.

There are two Laplace inversion methods which are widely used in poroelastic problem. The first one was proposed by Stehfest (1970) and the other by Schapery (1962). The formula due to Stehfest is given by

$$f(t) \approx \frac{\ln 2}{t} \sum_{n=1}^{N} c_n \overline{f}\left(n \frac{\ln 2}{t}\right) \tag{3.69}$$

where $\overline{f}$ denotes the Laplace transform of $f(t)$ and

$$c_n = (-1)^{n+N/2} \sum_{k=[(n+1)/2]}^{\min(n,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(n-k)!(2k-n)!} \tag{3.70}$$

and N is even. It was found that the accurate time-domain solutions can be obtained from equation (3.70) with $N \geq 6$ for general poroelasticity problems. It is important to note that the Stehfest scheme is computationally quite demanding although it is accurate. A simpler and more computationally efficient scheme is given by Schapery which can be expressed as

$$f(t) \approx \left[s\overline{f} \right]_{s=0.5/t} \tag{3.71}$$

where $\overline{f}$ denotes the Laplace transform of $f(t)$ and s is the Laplace transform parameter.