CHAPTER III

THEORETICAL CONSIDERATIONS

3.1 Basic Equations and General Solutions

Consider a poroelastic medium with a conventional cylindrical polar coordinate system (r, θ, z) defined such that the z-axis is perpendicular to the free surface as shown in Figure 1. Let u_i and w_i denote the average displacement of the solid matrix and the fluid displacement relative to the displacement of the solid matrix, in the i direction $(i=r,z)$, respectively. Then, the constitutive relations for a homogeneous poroelastic material (Biot,1941) can be expressed as the following:

$$\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r} + \lambda e - \alpha p \tag{3.1}$$

$$\sigma_{zz} = 2\mu\frac{\partial u_z}{\partial z} + \lambda e - \alpha p \tag{3.2}$$

$$\sigma_{zr} = \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) \tag{3.3}$$

$$p = -\alpha M e + M\zeta \tag{3.4}$$

where

$$e = \frac{\partial u_r}{\partial r}+\frac{\partial u_z}{\partial z}+\frac{u_r}{r} \tag{3.5}$$

$$\zeta = -\left(\frac{\partial w_r}{\partial r}+\frac{\partial w_z}{\partial z}+\frac{w_r}{r}\right) \tag{3.6}$$

In the above equations σ_{rr}, σ_{zz}, σ_{zr} denote the total stress component of the bulk material; e is the dilatation of the solid matrix; μ is the shear modulus and λ is a constant of the bulk material, respectively; p is the excess pore fluid pressure (suction is considered negative) and ζ is the variation of fluid content per unit reference volume. In addition, α and M are Biot 's parameters accounting for compressibility of the two-phased material (Biot, 1941). It is noted that $0 \le \alpha \le 1$ and $0 \le M \le \infty$ for all poroelastic materials. For a completely dry material $M = 0$, whereas for a material with incompressible constituents $M \to \infty$ and $\alpha = 1$.

The equations of motion for a poroelastic medium undergoing axisymmetric deformations, in the absence of body forces (solid and fluid) and a fluid source, can be expressed according to Biot (1962) as

$$\mu\nabla^2 u_r + \left(\lambda + \alpha^2 M + \mu\right)\frac{\partial e}{\partial r} - \mu\frac{u_r}{r^2} - \alpha M\frac{\partial \zeta}{\partial r} = \rho\,\ddot{u}_r + \rho_f \ddot{w}_r \tag{3.7}$$

$$\mu\nabla^2 u_z + \left(\lambda + \alpha^2 M + \mu\right)\frac{\partial e}{\partial z} - \alpha M\frac{\partial \zeta}{\partial z} = \rho\,\ddot{u}_z + \rho_f \ddot{w}_z \tag{3.8}$$

$$\alpha M\frac{\partial e}{\partial r} - M\frac{\partial \zeta}{\partial r} = \rho_f \ddot{u}_r + m\ddot{w}_r + b\dot{w}_r \tag{3.9}$$

$$\alpha M\frac{\partial e}{\partial z} - M\frac{\partial \zeta}{\partial z} = \rho_f \ddot{u}_z + m\ddot{w}_z + b\dot{w}_z \tag{3.10}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{3.11}$$

In equations (3.7)-(3.10), an overdot denotes the derivative with respect to the time parameter t; ρ and ρ_f are the mass densities of the bulk material and the pore fluid, respectively, and $m = \rho_f / \beta$ (β=porosity), is a density-like parameter. In addition, b is a parameter accounting for the internal friction due to the relative motion between the solid matrix and the pore fluid. The parameter b is defined as the ratio between the fluid viscosity and the intrinsic permeability of the porous medium.

The axisymmetric motion under consideration is assumed to be time-harmonic with the factor of $e^{i\omega t}$, where ω is the frequency of the motion and i is the imaginary number. The term $e^{i\omega t}$ is henceforth suppressed from all expressions for brevity. The radius of the circular plate denoted by a is selected to non-dimentionalize all space dimensions including the co-ordinate frame. Stresses and pore pressure are non-dimentionalized with respect to the shear modulus μ of the bulk material. All variables are replaced by the non-dimentional ones, but the previous notations are used for convenience.

The governing partial differential equations, equations (3.7) to (3.10), can be solved by introducing the displacement decomposition based on Helmholtz representation for an axisymmetric vector field and the zeroth-order Hankel integral transform with respect to the radial coordinate. The displacement decompositions are as follows (Philippacopoulos, 1989):

$$u_r(r,z) = \frac{\partial\Phi_1(r,z)}{\partial r} - \frac{\partial\Psi_1(r,z)}{\partial z} \tag{3.12}$$

$$u_z(r,z) = \frac{\partial\Phi_1(r,z)}{\partial z} - \frac{1}{r}\frac{\partial(r\Psi_1(r,z))}{\partial r} \tag{3.13}$$

$$w_r(r,z) = \frac{\partial\Phi_2(r,z)}{\partial r} - \frac{\partial\Psi_2(r,z)}{\partial z} \tag{3.14}$$

$$w_z(r,z) = \frac{\partial\Phi_2(r,z)}{\partial z} - \frac{1}{r}\frac{\partial(r\Psi_2(r,z))}{\partial r} \tag{3.15}$$

where $\Phi_i\,(i=1,\,2)$ and $\Psi_i\,(i=1,\,2)$ are functions of scalar and vector fields, respectively. In addition, the nth-order Hankel integral transform of a function $f(r,z)$ is defined as (Sneddon, 1970)

$$\bar{f}(\xi,z) = \int_0^\infty rf(r,z)J_n(\xi r)dr \tag{3.16}$$

and the inverse relationship is given by

$$f(r,z) = \int_0^\infty \xi\ \bar{f}(\xi,z)J_n(\xi r)d\xi \tag{3.17}$$

where J_n is the Bessel function of the first kind of the nth order (Watson, 1944) and ξ is the Hankel transform parameter.

By applying Helmholtz representation for an axisymmetric vector field together with the zeroth-order Hankel integral transform to the governing partial differential equations as described above, it can be shown that the general solutions for the zeroth-order Hankel transforms of u_z, w_z, σ_{zz} and p, and the first-order Hankel transform of u_r, w_r and σ_{zr}, respectively, can be expressed as

$$\bar{u}_r = -\xi\left(Ae^{\gamma_1 z} + Be^{-\gamma_1 z} + Ce^{\gamma_2 z} + De^{-\gamma_2 z}\right) - \gamma_3\left(Ee^{\gamma_3 z} - Fe^{-\gamma_3 z}\right) \tag{3.18}$$

$$\bar{u}_z = \gamma_1\left(Ae^{\gamma_1 z} - Be^{-\gamma_1 z}\right) + \gamma_2\left(Ce^{\gamma_2 z} - De^{-\gamma_2 z}\right) + \xi\left(Ee^{\gamma_3 z} + Fe^{-\gamma_3 z}\right) \tag{3.19}$$

$$\bar{w}_z = \gamma_1\chi_1\left(Ae^{\gamma_1 z} - Be^{-\gamma_1 z}\right) + \gamma_2\chi_2\left(Ce^{\gamma_2 z} - De^{-\gamma_2 z}\right) - \xi\chi_3\left(Ee^{\gamma_3 z} + Fe^{-\gamma_3 z}\right) \tag{3.20}$$

$$\bar{\sigma}_{zr} = -2\xi\left(\gamma_1\left(Ae^{\gamma_1 z} - Be^{-\gamma_1 z}\right) + \gamma_2\left(Ce^{\gamma_2 z} - De^{-\gamma_2 z}\right)\right) - S_1\left(Ee^{\gamma_3 z} + Fe^{-\gamma_3 z}\right) \tag{3.21}$$

$$\bar{\sigma}_{zz} = \beta_1\left(Ae^{\gamma_1 z} + Be^{-\gamma_1 z}\right) + \beta_2\left(Ce^{\gamma_2 z} + De^{-\gamma_2 z}\right) + 2\xi\gamma_3\left(Ee^{\gamma_3 z} - Fe^{-\gamma_3 z}\right) \tag{3.22}$$

$$\bar{p} = \eta_1\left(Ae^{\gamma_1 z} + Be^{-\gamma_1 z}\right) + \eta_2\left(Ce^{\gamma_2 z} + De^{-\gamma_2 z}\right) \tag{3.23}$$

where $A(\xi,\delta)$, $B(\xi,\delta)$, ..., $F(\xi,\delta)$ are the arbitrary functions to be determined by using appropriate boundary and/or continuity conditions relevant to a given problem. The superposed bar denotes the Hankel transform of quantities with respect to the r-coordinate. In addition, the parameters γ_i, χ_i, etc., appearing in the above equations and the details for derivation of the general solutions are given in Appendix A.

3.2 Influence Functions

The formulation presented in an ensuring section for vertical vibrations of an elastic plate embedded in a homogeneous poroelastic half-space (Figure 1a) and a multi-layered poroelastic half-space (Figure 1b) require the influence functions for axisymmetric ring loads and fluid sources applied in the interior of a poroelastic half-space and a multi-layered poroelastic half-space, respectively. The loads are applied at a depth z' below the free surface of the half-space as shown in Figure 2.

3.2.1 Influence Functions of Homogeneous Poroelastic Half-Space

The required influence functions of a poroelastic half-space are obtained by solving the boundary-value problems related to the internally loaded half-space. Solutions can be derived by defining a fictitious plane at $z = z'$ and treating the half-space as a two-domain boundary-value problem. The general solutions for each domain are given by equations (3.18) to (3.23) together with arbitrary functions $A_i(\xi,\delta)$ to $F_i(\xi,\delta)$, where i $(i = 1,2)$ is used to identify the domain number. The domain "1" is bounded by $0 \le z \le z'$ and the domain "2" by $z' \le z \le \infty$. Note that for the domain "2", arbitrary functions $A_i(\xi,\delta)$ = $C_i(\xi,\delta)$ = $E_i(\xi,\delta)$ = 0 in order to satisfy the condition that the solutions vanish as $z \to \infty$. The boundary conditions corresponding to a fully permeable top surface $(z = 0,\ 0 \le r < \infty)$ can be expressed as

$$\sigma_{zn}^{(1)}(r,0) = 0, \quad n = r, z \tag{3.24}$$

$$p^{(1)}(r,0) = 0 \tag{3.25}$$

where a superscript (1) is used to denote the domain number.

The continuity conditions at the fictitious plane $(z = z', \ 0 \le r < \infty)$ corresponding to a homogeneous poroelastic half-space subjected to a buried vertical load are given by

$$u_n^{(1)}(r,z') - u_n^{(2)}(r,z') = 0, \ \ n = r,z \tag{3.26}$$

$$p^{(1)}(r,z') - p^{(2)}(r,z') = 0 \tag{3.27}$$

$$\sigma_{zz}^{(1)}(r,z') - \sigma_{zz}^{(2)}(r,z') = \frac{T_z(r)}{\mu} \tag{3.28}$$

$$\sigma_{zr}^{(1)}(r,z') - \sigma_{zr}^{(2)}(r,z') = 0 \tag{3.29}$$

$$w_z^{(1)}(r,z') - w_z^{(2)}(r,z') = 0 \tag{3.30}$$

where $T_z(r)$ denotes the intensity of the buried load in the $z-$direction.

The continuity conditions at the fictitious plane $(z = z', \ 0 \le r < \infty)$ corresponding to a homogeneous poroelastic half-space subjected to applied pore fluid pressure can be written as

$$u_n^{(1)}(r,z') - u_n^{(2)}(r,z') = 0, \ \ n = r,z \tag{3.31}$$

$$p^{(2)}(r,z') - p^{(1)}(r,z') = \frac{P(r)}{\mu} \tag{3.32}$$

$$\sigma_{zz}^{(1)}(r,z') - \sigma_{zz}^{(2)}(r,z') = \alpha\frac{P(r)}{\mu} \tag{3.33}$$

$$\sigma_{zr}^{(1)}(r,z') - \sigma_{zr}^{(2)}(r,z') = 0 \tag{3.34}$$

$$w_z^{(1)}(r,z') - w_z^{(2)}(r,z') = 0 \tag{3.35}$$

where $P(r)$ denotes the intensity of fluid pressure discontinuity at the depth $z = z'$. Note that according to equations (3.31) to (3.35), a discontinuity does not exist in the solid skeleton stress at $z = z'$.

Substitution of general solutions for displacements, stresses and pore pressure defined by equations (3.18) - (3.23) in the boundary conditions, equations (2.24) and (2.25), and the appropriate continuity conditions, i.e., equations (2.26) to (2.30) or equations (2.31) to (2.35), results in a set of linear simultaneous equations to determine arbitrary functions corresponding to the two domains. The explicit solutions for non-zero arbitrary functions corresponding to applied vertical loading and fluid pressure were presented by Zeng and Rajapakse (1999). These solutions are shown in Appendix B.

Solutions corresponding to equations (3.24) to (3.30) are the required influence functions of a poroelastic half-space when the excitation is represented by an axisymmetric vertical load. In addition, the solutions corresponding to equations (3.24), (3.25) and (3.31) to (3.35) are required to simulate pore pressure jumps that could exist under an impermeable plate.

3.2.2 Influence Functions of Multi-layered Poroelastic Half-Space

3.2.2.1 Stiffness Matrices

The required influence functions of a multi-layered poroelastic half-space can be obtained by using an exact stiffness matrix method, which was successfully developed to study the dynamic response of a multi-layered poroelastic half-plane (Rajapakse & Senjuntichai, 1995).

Consider a multi-layered system consisting of N poroelastic layers overlying a poroelastic half-space. Layers and interfaces are numbered as shown in Figure 1b. Following Section 3.2.1, the general solutions for solid and fluid displacements, pore pressure and stresses in the Hankel transform space of a homogeneous poroelastic medium, equations (3.18)-(3.23), can be expressed in the following matrix form.

$$\mathbf{u}(\xi,z,\omega) = \mathbf{R}(\xi,z,\omega)\,\mathbf{C}(\xi,\omega) \tag{3.36}$$

$$\mathbf{f}(\xi,z,\omega) = \mathbf{S}(\xi,z,\omega)\,\mathbf{C}(\xi,\omega) \tag{3.37}$$

where

$$\mathbf{u}(\xi,z,\omega) = [\,\overline{u}_r \quad \overline{u}_z \quad \overline{p}\,]^{\mathrm{T}} \tag{3.38}$$

$$\mathbf{f}(\xi,z,\omega) = [\,\overline{\sigma}_{zr} \quad \overline{\sigma}_{zz} \quad \overline{w}_z\,]^{\mathrm{T}} \tag{3.39}$$

$$\mathbf{C}(\xi,\omega) = [\,A \quad B \quad C \quad D \quad E \quad F\,]^{\mathrm{T}} \tag{3.40}$$

and the superscript T denotes the transpose of a vector or a matrix. The arbitrary functions $A(\xi,\omega)$, $B(\xi,\omega)$,..., $F(\xi,\omega)$ appearing in $\mathbf{C}(\xi,\omega)$ can be determined by employing appropriate boundary and/or continuity conditions. The matrices $\mathbf{R}(\xi,z,\omega)$ and $\mathbf{S}(\xi,z,\omega)$ in equations (3.36) and (3.37) are given by

$$\mathbf{R} = \begin{bmatrix} -\xi e^{\gamma_1 z} & -\xi e^{-\gamma_1 z} & -\xi e^{\gamma_2 z} & -\xi e^{-\gamma_2 z} & -\gamma_3 e^{\gamma_3 z} & \gamma_3 e^{-\gamma_3 z} \\ \gamma_1 e^{\gamma_1 z} & -\gamma_1 e^{-\gamma_1 z} & \gamma_2 e^{\gamma_2 z} & -\gamma_2 e^{-\gamma_2 z} & \xi e^{\gamma_3 z} & \xi e^{-\gamma_3 z} \\ \eta_1 e^{\gamma_1 z} & \eta_1 e^{-\gamma_1 z} & \eta_2 e^{\gamma_2 z} & \eta_2 e^{-\gamma_2 z} & 0 & 0 \end{bmatrix} \tag{3.41}$$

$$\mathbf{S} = \begin{bmatrix} -2\xi\gamma_1 e^{\gamma_1 z} & 2\xi\gamma_1 e^{-\gamma_1 z} & -2\xi\gamma_2 e^{\gamma_2 z} & 2\xi\gamma_2 e^{-\gamma_2 z} & -S_1 e^{\gamma_3 z} & -S_1 e^{-\gamma_3 z} \\ \beta_1 e^{\gamma_1 z} & \beta_1 e^{-\gamma_1 z} & \beta_2 e^{\gamma_2 z} & \beta_2 e^{-\gamma_2 z} & 2\xi\gamma_3 e^{\gamma_3 z} & -2\xi\gamma_3 e^{-\gamma_3 z} \\ \gamma_1\chi_1 e^{\gamma_1 z} & -\gamma_1\chi_1 e^{-\gamma_1 z} & \gamma_2\chi_2 e^{\gamma_2 z} & -\gamma_2\chi_2 e^{-\gamma_2 z} & \xi\chi_3 e^{\gamma_3 z} & \xi\chi_3 e^{-\gamma_3 z} \end{bmatrix} \tag{3.42}$$

and

$$\eta_i = (\alpha + \chi_i) M L_i^2, \quad i = 1,2 \tag{3.43}$$

$$S_1 = \mu(\xi^2 + \gamma_3^2) \tag{3.44}$$

$$\beta_i = 2\mu\gamma_i^2 - \lambda L_i^2 - \alpha\eta_i, \quad i = 1,2 \tag{3.45}$$

$$\chi_i = \frac{(\lambda + \alpha^2 M + 2\mu) L_i^2 - \rho\delta^2}{\rho_f \omega^2 - \alpha M L_i^2}, \quad i = 1,2 \tag{3.46}$$

$$\chi_3 = \frac{\rho_f \omega}{ib - m\omega} \tag{3.47}$$

In addition, μ, λ, α, M, ρ, ρ_f, m and b are the poroelastic material constants defined in Section 3.1 and $\gamma_i (i = 1,2,3)$ and $L_i^2 (i = 1,2,3)$ are given in equations (A.18) to (A.21) in the Appendix A with the following definitions of parameters w_i $(i = 1,2)$ and S^2.

$$w_1 = \frac{(m\omega^2 - ib\omega)(\lambda + \alpha^2 M + 2\mu) + \rho M \omega^2 - 2\alpha M \rho_f \omega^2}{(\lambda + 2\mu) M} \tag{3.48}$$

$$w_2 = \frac{(m\omega^2 - ib\omega)\rho\omega^2 - \rho_f^2 \omega^4}{(\lambda + 2\mu) M} \tag{3.49}$$

$$S^2 = \frac{\omega^2}{\mu}(\rho_f \chi_3 + \rho) \tag{3.50}$$

Let the superscript n denote quantities associated with the nth layer (n=1,2,..., N). Then, the following relationships can be established for the nth layer of the system shown in Figure 1b by using equations (3.36) and (3.37).

$$\mathbf{U}^{(n)} = \begin{bmatrix} \mathbf{R}^{(n)}(\xi, z_n, \omega) \\ \cdots\cdots\cdots\cdots \\ \mathbf{R}^{(n)}(\xi, z_{n+1}, \omega) \end{bmatrix} \mathbf{C}^{(n)} \tag{3.51}$$

$$\mathbf{F}^{(n)} = \begin{bmatrix} -\mathbf{S}^{(n)}(\xi, z_n, \omega) \\ \cdots\cdots\cdots\cdots \\ \mathbf{S}^{(n)}(\xi, z_{n+1}, \omega) \end{bmatrix} \mathbf{C}^{(n)} \tag{3.52}$$

where

$$\mathbf{U}^{(n)} = [\mathbf{u}^{(n)}(\xi, z_n, \omega) \;\; \mathbf{u}^{(n)}(\xi, z_{n+1}, \omega)]^{\mathrm{T}} \tag{3.53}$$

$$\mathbf{F}^{(n)} = [-\mathbf{f}^{(n)}(\xi, z_n, \omega) \;\; \mathbf{f}^{(n)}(\xi, z_{n+1}, \omega)]^{\mathrm{T}} \tag{3.54}$$

In the above equations, $\mathbf{U}^{(n)}$ denotes a column vector of generalized coordinates for the nth layer whose elements are the Hankel transforms of displacements and pore pressure of the top and bottom surfaces of the nth layer; $\mathbf{F}^{(n)}$ denotes a column vector of generalized force whose elements are the Hankel transforms of tractions and fluid displacements of the top and bottom surfaces of the nth layer. The matrices $\mathbf{R}^{(n)}$ and $\mathbf{S}^{(n)}$ in equations (3.51) and (3.52) are identical to $\mathbf{R}$ and $\mathbf{S}$ defined in equations (3.41) and (3.42), respectively, except that the material properties of the nth layer are employed in the definition and $z = z_n$ or $z = z_{n+1}$. The vector $\mathbf{C}^{(n)}$ is the arbitrary coefficient vector corresponding to the nth layer.

The equation (3.51) can be inverted to express $\mathbf{C}^{(n)}$ in terms of $\mathbf{U}^{(n)}$ and then substituted in equation (3.52). This results in the following matrix equation.

$$\mathbf{F}^{(n)} = \mathbf{K}^{(n)} \mathbf{U}^{(n)} \qquad , n=1,2,\ldots, N \tag{3.55}$$

where $\mathbf{K}^{(n)}$ is an exact stiffness matrix in the Hankel transform space describing the relationship between the generalized displacement vector $\mathbf{U}^{(n)}$ and the force vector $\mathbf{F}^{(n)}$ for the nth layer.

The explicit derivation of $\mathbf{K}^{(n)}$ corresponding to poroelastodynamics is extremely complicated since it involves the manipulation of fully populated 6 × 6 unsymmetric complex matrices. The computer algebra package *Mathematica* (Wolfram, 1988) is used to obtain $\mathbf{K}^{(n)}$ explicitly. *Mathematica* results in very lengthy and complicated expressions for elements of $\mathbf{K}^{(n)}$ which have to be manipulated and reduced extensively to obtain expressions which enhance the computational efficiency of the solution scheme. After lengthy manipulations, it is found that $\mathbf{K}^{(n)}$ is symmetric and its elements can be expressed as

1st Row:

$$k_{11}=\frac{\mu}{\varphi}\left(S_1-2\xi^2\right)\left[\left(\vartheta_{3n}^2+1\right)\phi_1-\left(\vartheta_{3n}^2-1\right)\xi^2\phi_3\right] \tag{3.56}$$

$$k_{12}=\frac{\mu}{\varphi}\xi\gamma_3\left(S_1+2\xi^2\right)\left[\left(\vartheta_{3n}^2+1\right)\phi_3-4\vartheta_{3n}\phi_4\right]-\xi\left(\vartheta_{3n}^2-1\right)\left[2\gamma_3\phi_1+S_1\phi_2\right] \tag{3.57}$$

$$k_{13}=\frac{1}{\varphi}\xi\left(S_1-2\xi^2\right)\left[4\vartheta_{3n}\phi_7-\left(\vartheta_{3n}^2-1\right)\phi_5-\left(\vartheta_{3n}^2+1\right)\phi_6\right] \tag{3.58}$$

$$k_{14}=\frac{2\mu}{\varphi}\left(S_1-2\xi^2\right)\left[\left(\vartheta_{3n}^2-1\right)\xi^2\phi_4-\vartheta_{3n}\phi_1\right] \tag{3.59}$$

$$k_{15}=\frac{2\mu\xi\gamma_3}{\varphi}\left(S_1-2\xi^2\right)\left[\vartheta_{3n}\phi_3-\left(\vartheta_{3n}^2+1\right)\phi_4\right] \tag{3.60}$$

$$k_{16}=\frac{2\xi}{\varphi}\left(S_1-2\xi^2\right)\left[\vartheta_{3n}\phi_6-\left(\vartheta_{3n}^2+1\right)\phi_7+\left(\vartheta_{3n}^2-1\right)\phi_8\right] \tag{3.61}$$

where

$$\vartheta_{in}=e^{-\gamma_i h_n}, \qquad i=1,2,3, \qquad n=1,2,\ldots,N \tag{3.62}$$

$$\phi_1=\gamma_3\left[\left(\vartheta_{1n}^2-1\right)\left(\vartheta_{2n}^2-1\right)\left(\eta_1\gamma_2-\eta_2\gamma_1\right)^2-4\left(\vartheta_{1n}-\vartheta_{2n}\right)^2\eta_1\eta_2\gamma_1\gamma_2\right] \tag{3.63}$$

$$\phi_2=\xi^2\left(\vartheta_{1n}^2-1\right)\left(\vartheta_{2n}^2-1\right)\left(\eta_1-\eta_2\right)^2 \tag{3.64}$$

$$\phi_3=\left(\eta_1-\eta_2\right)\left[\left(\vartheta_{1n}^2-1\right)\left(\vartheta_{2n}^2+1\right)\eta_1\gamma_2-\left(\vartheta_{1n}^2+1\right)\left(\vartheta_{2n}^2-1\right)\eta_2\gamma_1\right] \tag{3.65}$$

$$\phi_4=\left(\eta_1-\eta_2\right)\left[\vartheta_{2n}\left(\vartheta_{1n}^2-1\right)\eta_1\gamma_2-\vartheta_{1n}\left(\vartheta_{2n}^2-1\right)\eta_2\gamma_1\right] \tag{3.66}$$

$$\phi_5=\left(\eta_1-\eta_2\right)\xi^2\left[\left(\vartheta_{1n}^2-1\right)\left(\vartheta_{2n}^2+1\right)\gamma_2-\left(\vartheta_{1n}^2+1\right)\left(\vartheta_{2n}^2-1\right)\gamma_1\right] \tag{3.67}$$

$$\phi_6 = \gamma_3\left[\left(\vartheta_{1n}^2-1\right)\left(\vartheta_{2n}^2-1\right)(\gamma_1-\gamma_2)(\eta_1\gamma_2-\eta_2\gamma_1)+2\left(\vartheta_{1n}-\vartheta_{2n}\right)^2(\eta_1+\eta_2)\gamma_1\gamma_2\right] \tag{3.68}$$

$$\phi_7 = (\eta_1-\eta_2)\left(\vartheta_{1n}-\vartheta_{2n}\right)\left(\vartheta_{1n}\vartheta_{2n}-1\right)\gamma_1\gamma_2\gamma_3 \tag{3.69}$$

$$\phi_8 = (\eta_1-\eta_2)\xi^2\left[\vartheta_{2n}\left(\vartheta_{1n}^2-1\right)\gamma_2-\vartheta_{1n}\left(\vartheta_{2n}^2-1\right)\gamma_1\right] \tag{3.70}$$

$$\varphi = 2\gamma_3\xi^2\left[\left(\vartheta_{3n}^2+1\right)\phi_3-4\vartheta_{3n}\phi_4\right]-\left(\vartheta_{3n}^2-1\right)\left(\gamma_3\phi_1+\xi^2\phi_2\right) \tag{3.71}$$

2nd Row:

$$k_{22} = \frac{\mu\gamma_3}{\varphi}\left(S_1-2\xi^2\right)\left[\left(\vartheta_{3n}^2-1\right)\gamma_3\phi_3-\left(\vartheta_{3n}^2+1\right)\phi_2\right] \tag{3.72}$$

$$k_{23} = \frac{1}{\varphi}\left[\left(\vartheta_{1n}^2-1\right)\eta_1^2\chi_2\ell_1+\left(\vartheta_{2n}^2-1\right)\eta_2^2\chi_1\ell_3\right] -\frac{\eta_1\eta_2}{\varphi}(\chi_1+\chi_2)\left[\left(\vartheta_{1n}^2-1\right)\ell_2+\left(\vartheta_{2n}^2-1\right)\ell_4+\ell_5\right] \tag{3.73}$$

$$k_{24} = -k_{15} \tag{3.74}$$

$$k_{25} = \frac{2\gamma_3}{\varphi}\left(S_1-2\xi^2\right)\left[\vartheta_{3n}\phi_2-\left(\vartheta_{3n}^2-1\right)\gamma_3\phi_4\right] \tag{3.75}$$

$$k_{26} = \frac{2\gamma_3}{\varphi}\left(S_1-2\xi^2\right)\left[\left(\vartheta_{3n}^2+1\right)\phi_8-\vartheta_{3n}\phi_5-\left(\vartheta_{3n}^2-1\right)\phi_7\right] \tag{3.76}$$

where

$$\ell_1 = \left[\left(\vartheta_{2n}^2-1\right)\left(\vartheta_{3n}^2-1\right)\left(\gamma_2^2\gamma_3^2+\xi^4\right)+2\ell_2\right] \tag{3.77}$$

$$\ell_2 = \xi^2\gamma_2\gamma_3\left[4\vartheta_{2n}\vartheta_{3n}-\left(\vartheta_{2n}^2+1\right)\left(\vartheta_{3n}^2+1\right)\right] \tag{3.78}$$

$$\ell_3 = \left[\left(\vartheta_{1n}^2-1\right)\left(\vartheta_{3n}^2-1\right)\left(\gamma_2^2\gamma_3^2+\xi^4\right)+2\ell_4\right] \tag{3.79}$$

$$\ell_4 = \xi^2\gamma_1\gamma_3\left[4\vartheta_{1n}\vartheta_{3n}-\left(\vartheta_{1n}^2+1\right)\left(\vartheta_{3n}^2+1\right)\right] \tag{3.80}$$

$$\ell_5 = \left(\vartheta_{3n}^2 - 1\right)\left[\left(\vartheta_{1n}^2 - 1\right)\left(\vartheta_{2n}^2 - 1\right)\xi^4\right]$$
$$+\gamma_1\gamma_2\gamma_3^2\left(\vartheta_{3n}^2 - 1\right)\left[\left(\vartheta_{1n}^2 - 1\right)\left(\vartheta_{2n}^2 - 1\right) + 2\left(\vartheta_{1n} - \vartheta_{2n}\right)^2\right] \tag{3.81}$$

3[rd] Row:

$$k_{33} = \frac{1}{\varphi}(\chi_1 - \chi_2)\left[\left(\vartheta_{1n}^2 + 1\right)\eta_1\gamma_1\ell_1 - \left(\vartheta_{2n}^2 + 1\right)\eta_2\gamma_2\ell_2\right] - \frac{1}{\varphi}\left(\vartheta_{3n}^2 + 1\right)\ell_6 \tag{3.82}$$

$$k_{34} = k_{16} \tag{3.83}$$

$$k_{35} = -k_{26} \tag{3.84}$$

$$k_{36} = \frac{2}{\varphi}\left[(\chi_2 - \chi_1)\left(\vartheta_{2n}\eta_2\gamma_2\ell_3 - \vartheta_{1n}\eta_1\gamma_1\ell_1\right) + 2\vartheta_{3n}\ell_6\right] \tag{3.85}$$

where

$$\ell_6 = \xi^2\gamma_3\left(S_1 - 2\xi^2\right)\left[2\eta_1\gamma_2\left\{\left(\vartheta_{1n}^2 + 1\right)\left(\vartheta_{2n}^2 + 1\right) - 4\vartheta_{1n}\vartheta_{2n}\right\}\right.$$
$$\left. - \left(\vartheta_{1n}^2 - 1\right)\left(\vartheta_{2n}^2 - 1\right)\left(\gamma_1^2 + \gamma_2^2\right)\right] \tag{3.86}$$

4[th] Row:

$$k_{44} = k_{11}, \qquad k_{45} = -k_{12}, \qquad k_{46} = k_{13} \tag{3.87}$$

5[th] Row:

$$k_{55} = k_{22}, \qquad k_{56} = -k_{23} \tag{3.88}$$

6[th] Row:

$$k_{66} = k_{33} \tag{3.89}$$

The elements of the layer stiffness matrix $\mathbf{K}^{(n)}$ are functions of the layer thickness, the layer material properties and the Hankel transform parameter ξ. Only negative exponentials that decrease rapidly with increasing ξ are involved in k_{ij}.

For the underlying half-space, due to the conditions that the solutions vanish as $z \to \infty$, the general solutions involve only three arbitrary coefficients in the vector $\mathbf{C}^{(N+1)}$, i.e. $B^{(N+1)}$, $D^{(N+1)}$ and $F^{(N+1)}$. The stiffness matrix of the bottom half-space can be written as

$$\mathbf{F}^{(N+1)} = \mathbf{K}^{(N+1)}\,\mathbf{U}^{(N+1)} \tag{3.90}$$

where

$$\mathbf{U}^{(N+1)} = [\,\mathbf{u}^{(N+1)}(\xi, z_{N+1})\,]^{\mathrm{T}} \tag{3.91}$$

$$\mathbf{F}^{(N+1)} = [\,\text{-}\mathbf{f}^{(N+1)}(\xi, z_{N+1})\,]^{\mathrm{T}} \tag{3.92}$$

$$\mathbf{K}^{(N+1)} = \text{symm.}\left[\tilde{k}_{ij}\right]_{3\times 3} \tag{3.93}$$

The elements of the half-space stiffness matrix are given by

$$\tilde{k}_{11} = \frac{\mu}{\upsilon}(S_1 - 2\xi^2)(\eta_1\gamma_2 - \eta_2\gamma_1) \tag{3.94}$$

$$\tilde{k}_{12} = \frac{\mu\xi}{\upsilon}\left[(\eta_1 - \eta_2)S_1 + 2\gamma_3(\eta_1\gamma_2 - \eta_2\gamma_1)\right] \tag{3.95}$$

$$\tilde{k}_{13} = \frac{\xi}{\upsilon}(S_1 - 2\xi^2)(\gamma_1 - \gamma_2) \tag{3.96}$$

$$\tilde{k}_{22} = \frac{\mu\gamma_3}{\upsilon}(S_1 - 2\xi^2)(\eta_1 - \eta_2) \tag{3.97}$$

$$\tilde{k}_{23} = \frac{1}{\upsilon}\left[\eta_2\chi_1(\gamma_1\gamma_3 - \xi^2) - \eta_1\chi_2(\gamma_2\gamma_3 - \xi^2)\right] \tag{3.98}$$

$$\tilde{k}_{33} = \frac{1}{\upsilon}\left[\gamma_1\chi_1(\gamma_2\gamma_3 - \xi^2) - \gamma_2\chi_2(\gamma_1\gamma_3 - \xi^2) + (\gamma_1 - \gamma_2)\xi^2\chi_3\right] \tag{3.99}$$

where

$$\upsilon = \eta_1\left(\gamma_2\gamma_3 - \xi^2\right) - \eta_2(\gamma_1\gamma_3 - \xi^2) \tag{3.100}$$

It is noted that exponential terms of ξ are not involved in the expression of $\mathbf{K}^{(N+1)}$ and its elements depend on the material properties of the underlying half-space and the Hankel transform parameter ξ.

3.2.2.2 Global Stiffness Matrix

The global stiffness matrix of a multi-layered half-space is assembled by using the layer and half-space stiffness matrices together with the continuity conditions of tractions and fluid flow at the layer interfaces. For example, the continuity conditions at the nth interface can be expressed as

$$\mathbf{f}^{(n-1)}(x,z_n) - \mathbf{f}^{(n)}(x,z_n) = \mathbf{T}^{(n)} \tag{3.101}$$

where $\mathbf{f}^{(n)}$ is identical to $\mathbf{f}$ in equation (3.39) with a superscript n denoting the layer number and

$$\mathbf{T}^{(n)} = [\overline{T}_r^{(n)} \quad \overline{T}_z^{(n)} \quad \overline{Q}^{(n)}]^{\mathrm{T}} \tag{3.102}$$

where $\overline{T}_r^{(n)}$ and $\overline{T}_z^{(n)}$ are the first-order Hankel transform of the axisymmetric radial load and the zeroth-order Hankel transform of the vertical load, respectively, applied at the nth interface. In addition, $\overline{Q}^{(n)}$ is the zeroth-order Hankel transform of the fluid source applied at the nth interface.

The consideration of equation (3.101) at each layer interface together with equations (3.55) and (3.90) results in the following global equation system.

$$\begin{bmatrix} \mathbf{K}^{(1)} & & & & \\ & \mathbf{K}^{(2)} & & & \\ & & \ddots & & \\ & & & \mathbf{K}^{(N)} & \\ & & & & \mathbf{K}^{(N+1)} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{(1)} \\ \mathbf{U}^{(2)} \\ \vdots \\ \mathbf{U}^{(N)} \\ \mathbf{U}^{(N+1)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \vdots \\ \mathbf{T}^{(N)} \\ \mathbf{T}^{(N+1)} \end{Bmatrix} \tag{3.103}$$

The solutions of the above equation are the influence functions of a multi-layered half-space required to establish the flexibility equation for the derivation of the contact stresses and pore pressure jumps which are employed in the formulation presented in the next section.

3.3 Formulation of Interaction Problem

3.3.1 Strain and Kinetic Energy of Elastic Circular Plate

The deflection of a circular plate embedded in a homogeneous or a multi-layered poroelastic half-space in the z-direction, $w_p(r)$, is proposed to be

$$w_p(r) = \sum_{n=0}^{N} \alpha_n r^{2n}, \quad 0 \le r \le 1 \tag{3.104}$$

In equation (3.104), α_n (n=0,1,..., N) denotes a set of generalized coordinates.

The strain and kinetic energies of the plate denoted by U_p and T_p, respectively, can be expressed as (Timoshenko and Woinowsky-Krieger,1959)

$$U_p = \frac{1}{2}\int_0^1 2\pi D \left[\left(\frac{d^2 w_p}{dr^2} + \frac{1}{r}\frac{dw_p}{dr} \right)^2 - \frac{2(1-\nu_p)}{r}\frac{dw_p}{dr}\frac{d^2 w_p}{dr^2} \right] rdr \tag{3.105}$$

$$T_p = \frac{1}{2} h_p \int_0^1 2\pi \rho_p (\dot{w}_p)^2 rdr \tag{3.106}$$

where

$$D = \frac{E_p h_p^{\,3}}{12(1-\nu_p^{\,2})} \tag{3.107}$$

In addition, α_n (n=0,1,..., N) denotes a set of generalized coordinates; h_p is the thickness of the plate; E_p, ν_p and ρ_p are Young 's modulus, Poisson 's ratio and the mass density of the plate material, respectively

In view of equation (3.104), equations (3.105) and (3.106) can be expressed as

$$U_p = \alpha^T K^P \alpha \tag{3.108}$$

$$T_p = \dot{\alpha}^T M^P \dot{\alpha} \tag{3.109}$$

The elements K_{ij}^p and M_{ij}^p of square matrices K^p and M^p, respectively, of order $N+1$ are given by

$$K_{1j}^p = K_{j1}^p = 0 \tag{3.110}$$

$$K_{ij}^{p} = \frac{4(i-1)(j-1)\pi D}{(2i+2j-6)}\left[4(i-1)(j-1)-2(1-v_p)(2i-3)\right] \; ; \; 2 \le i,j \le (N+1) \quad (3.111)$$

$$M_{ij}^{p} = \frac{\pi h_p \rho_p}{2(i-1)+2(j-1)+2} \; ; \qquad 1 \le i,j \le (N+1) \quad (3.112)$$

3.3.2 Representation for Contact Stresses and Pore Pressure Jumps

The contact stresses and pore pressure jumps, denoted by $\overline{T}_z(r)$ and $\overline{T}_p(r)$, respectively, acting on the plate-soil contact surface S $(0 \le r \le 1)$ can be expressed in terms of generalized co-ordinates α_n as

$$\overline{T}_z(r) = \sum_{n=0}^{N} \alpha_n \overline{T}_{nz}(r) \quad (3.113)$$

$$\overline{T}_p(r) = \sum_{n=0}^{N} \alpha_n \overline{T}_{np}(r) \quad (3.114)$$

where $\overline{T}_{nz}(r)$ and $\overline{T}_{np}(r)$ denote the contact stress and pore pressure jump corresponding to the plate displacement field r^{2n}.

In the present study, solutions for contact stresses and pore pressure jumps are determined by discretizing the contact area S into a total of Ne ring elements. It is assumed that $\overline{T}_z$ and $\overline{T}_p$ are constant within each ring element as shown in Figure 3. The relationship between unknown contact stresses, pore pressure jumps and the displacements on the contact surface can be expressed in terms of the following flexibility equations.

$$\begin{bmatrix} \mathbf{G}^{zz} & \mathbf{G}^{zp} \\ \mathbf{G}^{pz} & \mathbf{G}^{pp} \end{bmatrix} \begin{Bmatrix} \overline{\mathbf{T}}_{zn} \\ \overline{\mathbf{T}}_{pn} \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_{zn} \\ \mathbf{u}_{pn} \end{Bmatrix}, \; n = 0, 1, 2, \ldots, N \quad (3.115)$$

The element G_{kl}^{ij}, where $k,l = 1,2,\ldots,N_e$, in $\mathbf{G}^{ij}$ denotes the influence function which is the vertical displacement $(i = z)$ and relative fluid displacement $(i = p)$ at the center of the kth ring element due to a vertical ring load $(j = z)$ and a pore pressure ring load $(j = p)$ over the lth ring element. $\mathbf{u}_{zn}$ is the vertical displacement at a point on S and $\mathbf{u}_{pn}$ is the relative fluid displacement normal to S. The elements $\overline{T}_{zni}$, $\overline{T}_{pni}$, u_{zni} and u_{pni} of $\overline{\mathbf{T}}_{zn}$, $\overline{\mathbf{T}}_{pn}$, $\mathbf{u}_{zn}$ and $\mathbf{u}_{pn}$, respectively, are given by

$$\overline{T}_{zni} = \overline{T}_{nz}(r_i) \quad (3.116)$$

$$\overline{T}_{pni} = \begin{cases} \overline{T}_{np}(r_i) & \text{, for an impermeable plate} \\ 0 & \text{, for a permeable plate} \end{cases} \tag{3.117}$$

$$u_{zni} = r_i^{2n} \tag{3.118}$$

Note that

$$u_{pni} = 0 \qquad \text{, for an impermeable plate} \tag{3.119}$$

For a fully permeable plate, $\overline{T}_{pn} = \mathbf{0}$, then the flexibility equation, equation (3.115), is reduced to

$$\mathbf{G}^{zz}\,\overline{\mathbf{T}}_{zn} = \mathbf{u}_{zn} \tag{3.120}$$

3.3.3 Equations of Motion

The Lagrangian function L_p of the plate is given by (Washizu, 1982)

$$L_p = T_p - U_p - \int_0^1 2\pi\left[\frac{1}{2}q(r) - f(r)\right] w_p(r) r dr \tag{3.121}$$

where

$$q(r) = \overline{T}_z(r) + \alpha \overline{T}_p(r) \tag{3.122}$$

In the above equations, $q(r)$ is the total vertical stress over the surface S and $f(r)$ is the given externally applied load. In the present study, it is assumed that the applied load $f(r)$ acting on the plate can be written as

$$f(r) = \sum_{n=0}^{\overline{N}} \psi_n r^n \tag{3.123}$$

where

ψ_n (n=0,1,2,..., $\overline{N}$) are coefficients of the loading function.

The Lagrange 's equations of motion for the plate can be expressed as

$$\frac{d}{dt}\left(\frac{\partial L_p}{\partial \dot{\alpha}_i}\right) - \frac{\partial L_p}{\partial \alpha_i} = 0, \quad i = 0, 1, 2, \ldots, N \tag{3.124}$$

In view of equations (3.104), (3.108), (3.109), (3.113) and (3.114), equation (3.121) can be expressed in terms of α and $\dot{\alpha}$. Substitution of equation (3.121) in equation (3.124) and subsequent differentiation together with the fact that motion is time-harmonic results in the following equations of motion to determine α_n.

$$\overline{K}\,\boldsymbol{\alpha}=\mathbf{X} \tag{3.125}$$

where

$$\overline{K}=-\omega^2\left(M^p+M^{p^T}\right)+K^p+K^{p^T}+K^s+K^{s^T} \tag{3.126}$$

The elements K_{ij}^s and X_i of K^s and $\mathbf{X}$, respectively, are given by

$$K_{ij}^s=\sum_{p=1}^{N_e}\pi q_{jzp}\frac{\left(r_p+\Delta r_p/2\right)^{2i}-\left(r_p-\Delta r_p/2\right)^{2i}}{2i} \tag{3.127}$$

$$X_i=2\pi\sum_{n=0}^{\overline{N}}\frac{\psi_n}{n+2i} \tag{3.128}$$

The solution of a system of linear simultaneous equation system given by equation (3.121) yields the numerical values of the generalized coordinates α_n (n=1,2,..., N) for a given plate-half-space system. Finally, the plate displacements, contact stresses and pore pressure can then be obtained by back substituting the generalized coordinates in equations (3.104), (3.113) and (3.114), respectively.