CHAPTER II

PRELIMINARIES

The purpose of this chapter is to summarize necessary background on probability theory needed in this work.

2.1 Definitions and Well-Known Results

By a probability space we mean a measure space $(\Omega, \mathcal{F}, P)$ in which $P(\Omega) = 1$. Here Ω is a set known as a sample space, $\mathcal{F}$ is a Borel field of subsets of Ω, sets in $\mathcal{F}$ will be refered to as events and the measure P is call a probability measure.

By an m-dimensional random vector we mean a measurable function X from Ω into R^m. A 1-dimensional random vector is also called a random variable. If X is an m-dimensional random vector, then for any $w \in \Omega$, we have $X(w) = (X^1(w),\ldots,X^m(w))$, where $X^1,\ldots,X^m$ are the components of X. For any Borel measurable function $g : R^m \longrightarrow R$, the composition goX of g with an m-dimensional random vector X is always a random variable. If the integral $\int_\Omega goXdP$ exists, we call it the expectation of goX, and will be denoted by $E[g(X)]$. Let $g_j(X^1,\ldots,X^m) = X^j$. If the expectation $E\left[\{g_j(X) - E[g_j(X)]\}\{g_k(X) - E[g_k(X)]\}\right]$ exists, it will be called the covariance of X^j and X^k, written $\mathrm{cov}(X^j,X^k)$. In general, if $E\left[\prod_{j=1}^{m}(g_j(X))^{r_j}\right]$ exists, we call it a product moment of X of order $r_1+\ldots+r_m$, and will be denoted by $\mu'_{r_1,\ldots,r_m}(X)$.

Theorem 2.1.1 Let X be a random vector, and put

$$\lambda_k = \mu'_{k,0,\ldots,0}(X) + \mu'_{0,k,0,\ldots,0}(X) + \ldots + \mu'_{0,\ldots,0,k}(X) .$$

Then, if the series $\sum_{k=1}^{\infty} \lambda_{2k}^{-1/2k}$ is divergent, the distribution P of X is uniquely determined by its moments.

A proof of this theorem can be found in [1] .

For any random variable X , let $X^{[r]} = X(X-1)\ldots(X-r+1)$ and $X^{[0]} = 1$. If $E\left[\prod_{j=1}^{m}(g_j(X))^{[r_j]}\right]$ exists, it will be called a product factorial moment of X of order $r_1+\ldots+r_m$, and will be denoted by $\mu'_{[r_1]\ldots[r_m]}(X)$. It is well known that we can express

$$(2.1.1)\quad \mu'_{r_1,\ldots,r_m}(X) = \sum_{i_1=1}^{r_1} \ldots \sum_{i_m=1}^{r_m} a_{i_1\ldots i_m}\, \mu'_{[i_1]\ldots[i_m]}(X),$$

where $a_{i_1\ldots i_m}$ depend only on $r_1,\ldots,r_m$.

The characteristic function of a random vector X is defined as $\varphi(t_1,\ldots,t_m) = E(e^{it.X})$, where $t \in R^m$, $t.X = t_1X^1+\ldots+t_mX^m$. Since $\left| e^{it.X} \right| = 1$ for all $t \in R^m$, it follows that the characteristic function can be defined for every random vector. The generating function $G(t_1,\ldots,t_m)$, if it exists, is defined by $G(t_1,\ldots,t_m) = \varphi\left(\frac{\log t_1}{i},\ldots,\frac{\log t_m}{i}\right)$. It can be shown that, if $\mu'_{[r_1]\ldots[r_m]}(X)$ exists, then for any non-negative integers $r_1,\ldots,r_m$

$$(2.1.2)\quad \mu'_{[r_1]\ldots[r_m]}(X) = \lim_{\substack{t_j \to 1 \\ j=1,\ldots,m}} \frac{\partial^{r_1+\cdots+r_m}}{\partial t_m^{r_m} \cdots \partial t_1^{r_1}} (G(t_1,\ldots,t_m)) .$$

2.2 Convergence of Distributions of Random Vectors

A sequence $\{P_n\}$ is said to converge to the distribution P , and write $P_n \longrightarrow P$, if $\int_\Omega f dP_n \longrightarrow \int_\Omega f dP$ for every bounded , continuous real function f on Ω . Let $\{X_n\}$ be a sequence of random vectors , we say that $\{X_n\}$ converges in distribution to the random vector X , and write $X_n \xrightarrow{\mathcal{D}} X$, if the distribution P_n of X_n converges to the distribution P of X .

Theorem 2.2.1 Let $\{X_n\}$ be a sequence of random vectors such that for any non-negative integers $r_1,\ldots,r_m$

(a) $\mu'_{r_1,\ldots,r_m}(X_n)$, exist for all n ;

(b) $\left|\mu'_{r_1,\ldots,r_m}(X_n)\right| \leq K(r_1,\ldots,r_m)$ for all n;

(c) $\lim_{n\to\infty} \mu'_{r_1,\ldots,r_m}(X_n) = \mu'_{r_1,\ldots,r_m}$ exists ;

(d) there exists a random vector X with distribution P such that $\mu'_{r_1,\ldots,r_m}(X) = \mu'_{r_1,\ldots,r_m}$ and P is uniquely determined by $\mu'_{r_1,\ldots,r_m}$.

Then $\{X_n\}$ converges to X .

A proof of this theorem can be found in [5] .

2.3 Multivariate Poisson Distribution

In this section we give a definition of multivariate Poisson distributions and some results about them .

First we introduce some notations . For any positive integer m , let

$$(2.3.1) \quad P_m = \left\{ s \,/\, \phi \neq s \subseteq \{1,2,\ldots,m\} \right\} .$$

Throughout our discussion , we shall use P_m as the index set of our parameters .

Definition 2.3.1 If a random vector X has the characteristic function

$$(2.3.2) \quad \varphi(t_1,\ldots,t_m) = \exp\left[\sum_{s \in P_m} (a(s) \prod_{i \in s} z_i) - \sum_{s \in P_m} a(s) \right] ,$$

where a(s) are non-negative real numbers and $z_j = \exp(it_j)$, we say that it has a multivariate Poisson with parameters a(s) .

This definition is the same as that used by Carol E. Fuchs and H.T. David [2] . From (2.3.2) , we have

$$(2.3.3) \quad G(t_1,\ldots,t_m) = \exp\left[\sum_{s \in P_m} (a(s) \prod_{i \in s} t_i) - \sum_{s \in P_m} a(s) \right] ,$$

from which the joint probability functions of $X^1,\ldots,X^m$ can be derived . We have

$$(2.3.4) \quad P(X^1=k_1,\ldots,X^m=k_m) = e^{-A_m} \sum_{\substack{\sum_{j \in s} \alpha(s)=k_j \text{ for all } j \\ \alpha(s) \geq 0}} \prod_{s \in P_m} \frac{(a(s))^{\alpha(s)}}{\alpha(s)!} ,$$

where $A_m = \sum_{s \in P_m} a(s)$.

It can be shown that the distribution of a multivariate Poisson is uniquely determined by its moments $\mu'_{r_1,\dots,r_m}$. A proof of this fact can be found in [2] .

The following theorem is an immediate consequence of Theorem 2.2.1 .

Theorem 2.3.1 Let X be a multivariate Poisson random vector . Let $\{X_n\}$ be a sequence of random vectors such that for any non-negative integers $r_1,\dots,r_m$

(a') $\mu'_{r_1,\dots,r_m}(X_n)$, exist for all n ;

(b') $\left|\mu'_{r_1,\dots,r_m}(X_n)\right| \leq K(r_1,\dots,r_m)$ for all n ;

(c') $\lim_{n\to\infty} \mu'_{r_1,\dots,r_m}(X_n) = \mu'_{r_1,\dots,r_m}(X)$.

Then X_n converges to X .

By applying (2.1.1) , we obtain a sufficient condition for convergence to a Poisson random vector in terms of factorial moments .

Theorem 2.3.2 Let X be a multivariate Poisson random vector . Let $\{X_n\}$ be a sequence of random vectors such that for any non-negative integers $r_1,\dots,r_m$

(a") $\mu'_{[r_1]\dots[r_m]}(X_n)$, exist for all n ;

(b") $\left|\mu'_{[r_1]\dots[r_m]}(X_n)\right| \leq K(r_1,\dots,r_m)$ for all n ;

$$\text{(c'')} \quad \lim_{n\to\infty} \mu'_{[r_1]\cdots[r_m]}(X_n) = \mu'_{[r_1]\cdots[r_m]}(X) .$$

Then X_n converges to X .

In order to apply the Theorem 2.3.2 , we need to know the factorial moments of Poisson random vectors . These are given in the following theorem .

Theorem 2.3.3 Let X be an m-dimensional Poisson random vector with parameters a(s) , $s \in P_m$. Then the factorial moments of X are given by

$$(2.3.5) \quad \mu'_{[r_1]\cdots[r_m]}(X) = \sum_{d\in D} \prod_{j=1}^{m} (r_j!) \prod_{s\in P_m} \frac{(\lambda(s))^{d(s)}}{(d(s))!} ,$$

where $r_1,\ldots,r_m$ are any non-negative integers, and

$$D = \left\{ d/d: P_m \longrightarrow \{0,1,\ldots\} \ \ni \sum_{j\in s} d(s) = r_j \right\} , \quad \lambda(s) = \sum_{s \subsetneq s' \in P_m} a(s') .$$

Proof Observe that

$$(2.3.6) \quad G(t_1,\ldots,t_m) = \exp\left[\sum_{s\in P_m} (a(s) \prod_{i\in s} t_i) - \sum_{s\in P_m} a(s) \right]$$

$$= \exp\left[\sum_{s\in P_m} a(s)(\prod_{i\in s} t_i - 1) \right] .$$

We shall show that for any $s \in P_m$

$$(2.3.7) \quad \prod_{i\in s} t_i - 1 = \sum_{\phi \neq s^* \subseteq s} \prod_{i\in s^*} (t_i - 1) .$$

We prove this by induction on the cardinality of $s \in P_m$. For $|s| = 1$, it is clear that (2.3.7) holds . Assume that (2.3.7) holds for all $s \in P_m$ such that $|s| = k \geq 1$. Let $s' \in P_m$ be such that $|s'| = k+1$. We assume that $s' = \{j_1,\ldots,j_{k+1}\}$ and let $s'' = \{j_1,\ldots,j_k\}$.

Observe that

$$(\prod_{j\in s'} t_j) - 1 = (\prod_{j\in s'} t_j) - (\prod_{j\in s''} t_j) + (\prod_{j\in s''} t_j) - 1$$

$$= (\prod_{j\in s''} t_j)(t_{j_{k+1}} - 1) + (\prod_{j\in s''} t_j) - 1$$

$$= (\prod_{j\in s''} t_j)(t_{j_{k+1}} - 1) - (t_{j_{k+1}} - 1) + (\prod_{j\in s''} t_j) - 1 + (t_{j_{k+1}} - 1)$$

$$= (t_{j_{k+1}} - 1)(\prod_{j\in s''} t_j - 1) + (\prod_{j\in s''} t_j - 1) + (t_{j_{k+1}} - 1)$$

$$= (t_{j_{k+1}} - 1)\left[\sum_{\phi \neq s^* \subseteq s''} \prod_{j\in s^*} (t_j - 1)\right] + \sum_{\phi \neq s^* \subseteq s''} \prod_{j\in s^*} (t_j - 1)$$

$$+ (t_{j_{k+1}} - 1)$$

$$= \sum_{\phi \neq s^* \subseteq s'} \prod_{j\in s^*} (t_j - 1) .$$

Hence (2.3.7) holds for all $s \in P_m$.

By using (2.3.7) it follows from (2.3.6) that

$$G(t_1, \ldots, t_m) = \exp\left[\sum_{s\in P_m} a(s) \left(\sum_{\phi \neq s^* \subseteq s} \prod_{i\in s^*} (t_i - 1)\right)\right]$$

$$= \exp\left[\sum_{s\in P_m} \sum_{\phi \neq s^* \subseteq s} a(s) \prod_{i\in s^*} (t_i - 1)\right]$$

$$= \exp\left[\sum_{s^*\in P_m} \sum_{s^* \subseteq s\in P_m} a(s) \prod_{i\in s^*} (t_i - 1)\right]$$

$$= \exp\left[\sum_{s^*\in P_m} \left(\sum_{s^* \subseteq s\in P_m} a(s)\right) \prod_{i\in s^*} (t_i - 1)\right]$$

$$= \exp\left\{\sum_{s^*\in P_m} \lambda(s^*) \prod_{i\in s^*} (t_i - 1)\right\}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{s^*\in P_m} \lambda(s^*) \prod_{i\in s^*} (t_i - 1)\right]^n \Big/ n!$$

$$= \sum_{n=0}^{\infty} \left[\sum_{\substack{\sum_{s^* \in P_m} \alpha(s^*) = n \\ \alpha(s^*) \geq 0}} \left\{ \prod_{s^* \in P_m} \left(\lambda(s^*) \prod_{i \in s^*} (t_i - 1) \right)^{\alpha(s^*)} n! \right\} \Big/ \prod_{s^* \in P_m} (\alpha(s^*)!) \right] \Big/ n!$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{\sum_{s^* \in P_m} \alpha(s^*) = n, \\ \alpha(s^*) \geq 0}} \prod_{s^* \in P_m} \frac{(\lambda(s^*))^{\alpha(s^*)}}{\alpha(s^*)!} \prod_{j=1}^{m} (t_j - 1)^{\sum_{j \in s^*} \alpha(s^*)}$$

$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \sum_{\substack{\sum_{j \in s^*} \alpha(s^*) = k_j, \text{ for all } j \\ \alpha(s^*) \geq 0}} \prod_{s^* \in P_m} \frac{(\lambda(s^*))^{\alpha(s^*)}}{\alpha(s^*)!} \prod_{j=1}^{m} (t_j - 1)^{k_j}.$$

Hence

$$\frac{\partial^{r_1 + \cdots + r_m}}{\partial t_m^{r_m} \cdots \partial t_1^{r_1}} (G(t_1, \ldots, t_m))$$

$$= \sum_{k_1=r_1}^{\infty} \cdots \sum_{k_m=r_m}^{\infty} \sum_{\substack{\sum_{j \in s} \alpha(s) = k_j \text{ for all } j, \\ \alpha(s) \geq 0}} \prod_{s \in P_m} \frac{(\lambda(s))^{\alpha(s)}}{\alpha(s)!} \prod_{j=1}^{m} k_j^{[r_j]} (t_j - 1)^{k_j - r_j}$$

$$|t_j| < 1 \; , \; j = 1, \ldots, m \; .$$

Therefore

$$\lim_{\substack{t_j \to 1 \\ j=1,\ldots,m}} \frac{\partial^{r_1 + \cdots + r_m}}{\partial t_m^{r_m} \cdots \partial t_1^{r_1}} (G(t_1, \ldots, t_m))$$

$$= \sum_{\substack{\sum_{j \in s} \alpha(s) = r_j \text{ for all } j, \\ \alpha(s) \geq 0}} \prod_{s \in P_m} \frac{(\lambda(s))^{\alpha(s)}}{\alpha(s)!} \prod_{j=1}^{m} r_j^{[r_j]}$$

$$= \sum_{\substack{\sum_{j \in s} \alpha(s) = r_j \text{ for all } j, \\ \alpha(s) \geq 0}} \prod_{j=1}^{m} (r_j!) \prod_{s \in P_m} \frac{(\lambda(s))^{\alpha(s)}}{\alpha(s)!}$$

$$= \sum_{d \in D} \prod_{j=1}^{m} (r_j!) \prod_{s \in P_m} \frac{(\lambda(s))^{d(s)}}{d(s)!} .$$

By (2.1.2) , we have (2.3.5) .

2.4 Factorial Moments of Zero-One Random Vectors

Our main theorem of chapter III , deals with random vectors whose components X_i^j can take only the values 0 or 1 . For such random variables we always have $X_i^j(X_i^j - 1) = 0$ identically . This identity gives us a useful identity which will be needed in the computation of factorial moments of such random vectors .

Lemma 2.4.1 If $X_i^j = 0$ or 1 for all $i = 1,\ldots,n$ $j = 1,\ldots,m$, then for any non-negative integers $r_1,\ldots,r_m$ such that $r_1+\ldots+r_m \geq 1$, we have

(2.4.1) $$\prod_{j=1}^{m} \left[\sum_{i=1}^{n} X_i^j \right]^{\{r_j\}}$$

$$= \sum_{\substack{i_{11}=1 \\ i_{jl} \text{ are distinct}, j=1,\ldots,m}}^{n} \cdots \sum_{i_{1r_1}=1}^{n} \cdots \sum_{i_{m1}=1}^{n} \cdots \sum_{i_{mr_m}=1}^{n} X_{i_{11}}^1 \ldots X_{i_{1r_1}}^1 \ldots X_{i_{m1}}^m \ldots X_{i_{mr_m}}^m .$$

Proof First we prove that , if $r_j \geq 1$, then

$$(2.4.2) \quad \left[\sum_{i=1}^{n} X_i^j\right]^{[r_j]} = \sum_{i_{j1}=1}^{n} \cdots \sum_{i_{jr_j}=1}^{n} X_{i_{j1}}^j \cdots X_{i_{jr_j}}^j \quad .$$

i_{jl} are distinct

Since $\left[\sum_{i=1}^{n} X_i^j\right]^{[1]} = \sum_{i=1}^{n} X_i^j$, hence (2.4.2) holds for $r_j=1$.

Assume that (2.4.2) holds for $r_j=k\geq 1$.

Observe that

$$\left[\sum_{i=1}^{n} X_i^j\right]^{[k+1]} = \left[\sum_{i=1}^{n} X_i^j\right]^{[k]} \left\{\sum_{i=1}^{n} X_i^j - k\right\}$$

$$= \left\{\sum_{i_{j1}=1}^{n} \cdots \sum_{i_{jk}=1}^{n} X_{i_{j1}}^j \cdots X_{i_{jk}}^j\right\} \left\{\sum_{i=1}^{n} X_i^j - k\right\}$$

i_{jl} are distinct

$$= \sum_{i_{j1}=1}^{n} \cdots \sum_{i_{jk}=1}^{n} \left\{(X_{i_{j1}}^j \cdots X_{i_{jk}}^j)\left(\sum_{i_{j(k+1)}=1}^{n} X_{i_{j(k+1)}}^j + X_{i_{j1}}^j + \cdots + X_{i_{jk}}^j - k\right)\right\}$$

i_{jl} are distinct $\qquad i_{j(k+1)} \neq i_{j1}, \ldots, i_{jk}$

$$= \sum_{i_{j1}=1}^{n} \cdots \sum_{i_{jk}=1}^{n} \left\{(X_{i_{j1}}^j \cdots X_{i_{jk}}^j)\left(\sum_{i_{j(k+1)}=1}^{n} X_{i_{j(k+1)}}^j + (X_{i_{j1}}^j - 1) + \cdots + (X_{i_{jk}}^j - 1)\right)\right\}$$

i_{jl} are distinct $\qquad i_{j(k+1)} \neq i_{j1}, \ldots, i_{jk}$

$$= \sum_{i_{j1}=1}^{n} \cdots \sum_{i_{j(k+1)}=1}^{n} X_{i_{j1}}^j \cdots X_{i_{j(k+1)}}^j + \sum_{i_{j1}=1}^{n} \cdots \sum_{i_{jk}=1}^{n} X_{i_{j1}}^j (X_{i_{j1}}^j - 1) X_{i_{j2}}^j \cdots X_{i_{jk}}^j$$

i_{jl} are distinct $\qquad i_{jl}$ are distinct

$$+ \cdots + \sum_{i_{j1}=1}^{n} \cdots \sum_{i_{jk}=1}^{n} X_{i_{j1}}^j \cdots X_{i_{jk}}^j (X_{i_{jk}}^j - 1) \quad .$$

i_{jl} are distinct

$$= \sum_{\substack{i_{j1}=1}}^{n} \cdots \sum_{i_{j(k+1)}=1}^{n} X^j_{i_{j1}} \cdots X^j_{i_{j(k+1)}} \quad .$$
i_{jl} are distinct

The last equality follows from the fact that $X^j_i(X^j_i-1) = 0$.

Hence (2.4.2) holds for any positive integer r_j .

Observe that for any non-negative integers $r_1,\ldots,r_m$ such that

$r_1+\ldots+r_m \geqq 1$

$$\prod_{j=1}^{m}\left\{\sum_{i=1}^{n} X^j_i\right\}^{[r_j]} = \prod_{j=1}^{m}\left\{\sum_{i_{j1}=1}^{n}\cdots\sum_{i_{jr_j}=1}^{n} X^j_{i_{j1}}\cdots X^j_{i_{jr_j}}\right\}$$
i_{jl} are distinct

$$= \sum_{i_{11}=1}^{n}\cdots\sum_{i_{1r_1}=1}^{n}\cdots\sum_{i_{m1}=1}^{n}\cdots\sum_{i_{mr_m}=1}^{n} X^1_{i_{11}}\cdots X^1_{i_{1r_1}}\cdots X^m_{i_{m1}}\cdots X^m_{i_{mr_m}} \quad .$$
i_{jl} are distinct $j=1,\ldots,m$

<u>Theorem 2.4.1</u> Let $X_i = (X^1_i,\ldots,X^m_i)$, $i=1,\ldots,n$ be m-dimensional random vectors where the components can take only the values 0 or 1. Let $S_n = X_1+\ldots+X_n$. Then for any non-negative integers $r_1,\ldots,r_m$ such that $r_1+\ldots+r_m \geqq 1$, we have

(2.4.3) $\mu'_{[r_1]\ldots[r_m]}(S_n)$

$$= \sum_{i_{11}=1}^{n}\cdots\sum_{i_{1r_1}=1}^{n}\cdots\sum_{i_{m1}=1}^{n}\cdots\sum_{i_{mr_m}=1}^{n} P(X^1_{i_{11}}=\ldots=X^1_{i_{1r_1}}=\ldots X^m_{i_{m1}}=\ldots X^m_{i_{mr_m}}=1) \quad .$$
i_{jl} are distinct $j=1,\ldots,m$

Proof For any non-negative integers $r_1, \ldots, r_m$

(2.4.4) $\mu'_{[r_1]\ldots[r_m]}(S_n) = E\left(\prod_{j=1}^{m}\left[\sum_{i=1}^{n} X_i^j\right]^{[r_j]}\right)$.

Applying the Lemma 2.4.1 to (2.4.4) , we have

(2.4.5) $\mu'_{[r_1]\ldots[r_m]}(S_n)$

$$= E\left(\sum_{i_{11}=1}^{n}\cdots\sum_{i_{1r_1}=1}^{n}\cdots\sum_{i_{m1}=1}^{n}\cdots\sum_{i_{mr_m}=1}^{n} X_{i_{11}}^1 \ldots X_{i_{1r_1}}^1 \ldots X_{i_{m1}}^m \ldots X_{i_{mr_m}}^m\right)$$
i_{jl} are distinct $j=1,\ldots,m$

$$= \sum_{i_{11}=1}^{n}\cdots\sum_{i_{1r_1}=1}^{n}\cdots\sum_{i_{m1}=1}^{n}\cdots\sum_{i_{mr_m}=1}^{n} E(X_{i_{11}}^1 \ldots X_{i_{1r_1}}^1 \ldots X_{i_{m1}}^m \ldots X_{i_{mr_m}}^m)$$
i_{jl} are distinct $j=1,\ldots,m$

$$= \sum_{i_{11}=1}^{n}\cdots\sum_{i_{1r_1}=1}^{n}\cdots\sum_{i_{m1}=1}^{n}\cdots\sum_{i_{mr_m}=1}^{n} P(X_{i_{11}}^1 = \ldots X_{i_{1r_1}}^1 = \ldots X_{i_{m1}}^m = \ldots X_{i_{mr_m}}^m = 1) .$$
i_{jl} are distinct $j=1,\ldots,m$