# CHAPTER III

# CONVOLUTIONS

The materials of this chapter are drawn from references [5], [8], [13].

3.1 Definition. Let f and g be real valued measurable functions on $R^n$. The convolution $h = f*g$ is defined by

$$h(x) = (f*g)(x) = \int f(y)g(x-y)dy$$

where $\int$ is a lebesgue integral and integrate over whole space.

Convolution is a kind of multiplication. $L^1$ is not closed under pointwise multiplication. For instance,

$$f(x) = g(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise .} \end{cases}$$

It shows that f,g belongs to $L^1$ but fg does not. However, $L^1$ is closed under the " convolution product ". To show this claim, we have a theorem as fallows :

3.2 Lemma. Let X, Y, Z be topological spaces, and let $\emptyset : X \longrightarrow Y$ be continuous function. If $g : Y \longrightarrow Z$ is Borel measurable, then so is $g \circ \emptyset : X \longrightarrow Z$.

Proof : Since $\emptyset$ is a continuous function. If E is open, then $\emptyset^{-1}(E)$ is open. Let $\mathcal{B}$ be the collection of all $E \subset Y$, for which $\emptyset^{-1}(E)$ is a Borel set. We claim that $\mathcal{B}$ is $\sigma$-algebra in Y.

i) $Y \in \mathcal{B}$ , since $X = \emptyset^{-1}(Y)$ is a Borel set.

ii) If $A \in \mathcal{B}$ , i.e. $\emptyset^{-1}(A)$ is a Borel set, and $\emptyset^{-1}(A^c) = [\emptyset^{-1}(A)]^c$ is a Borel set,
then $A^c \in \mathcal{B}$ .

iii) If $A_i \in \mathcal{B}$ , $i = 1,2,\ldots$, i.e. $\emptyset^{-1}(A_i)$ are Borel set for all i,
and $\emptyset^{-1}(\bigcup_i A_i) = \bigcup_i \emptyset^{-1}(A_i)$ is a Borel set,
then $\bigcup_i A_i \in \mathcal{B}$ .

Hence $\mathcal{B}$ is a $\sigma$-algebra in Y, and contains all Borel sets in Y. Then $\emptyset^{-1}(E)$ is a Borel set in X, for any Borel set E in Y.

Now let V be any open set in Z, then $g^{-1}(V)$ is a Borel set in Y, and so is $\emptyset^{-1}(g^{-1}(V))$. Accordingly $g \circ \emptyset$ is a Borel measurable.

3.3 Theorem. If f, $g \in L^1(R^n)$, then $h = f*g$ is finite a.e. and belongs to $L^1$. Also,

$$\|h\|_1 \leqslant \|f\|_1 \|g\|_1 .$$

Proof : We shall first prove that the function F defined by

$$F(x,y) = f(y)\, g(x-y)$$

is a Borel function on $R^n \times R^n$.

Consider a function $\emptyset : R^n \times R^n \longrightarrow R^n$ is defined by $\emptyset : (x,y) \longmapsto x - y$. It is easily seen that $\emptyset$ is continuous.

Since $g(x-y) = g \circ \emptyset(x,y)$ , and g is a Borel function. By Lemma(3.2) $(x,y) \longmapsto g(x-y)$ is a Borel function. Similary $(x,y) \longmapsto f(y)$ is a Borel function. Since the product of two Borel functions is a Borel function, our assertion concerning F is proved.

Next, we observe that

$$\int dy \int |F(x,y)|\, dx = \int dy \int |f(y)|\,|g(x-y)|\, dx$$

$$= \int |f(y)|\, dy \int |g(x-y)|\, dx$$

$$= \|f\|_1 \, \|g\|_1 < \infty \quad ,$$

where $\int |g(x-y)|\, dx = \|g\|_1$ for every $y \in R^n$ which devied from the translation invarance of lebesgue measure. Thus $h = f*g$ is finite a.e. and $h \in L^1(R^n)$. Consequently, by virtue of Fubini's Theorem,

$$\|h\|_1 = \int |h(x)|dx \leqslant \int dx \int |F(x,y)|\, dy$$

$$= \int dy \int |F(x,y)|\, dx$$

$$= \|f\|_1 \, \|g\|_1 \, .$$

Convolution is commutative, that is $f*g = g*f$.

Since $(f*g)(x) = \int f(g)\, g(x-y)dy$.

If we change the variable by writing $x-y = z$, then

$$(f*g)(x) = \int f(x-z)\, g(z)dz$$

$$= \int g(z)f(x-z)dz$$

$$= (g*f)(x).$$

Convolution is associative in $L^1$, that is $(f*g)*h = f*(g*h)$ where $f, g, h \in L^1$. Since

$$[(f*g)*h](x) = \int (f*g)(y)\, h(x-y)dy$$

$$= \int \left[\int f(z)\, g(y-z)dz\right] h(x-y)dy$$

$$= \int \left[\int f(z)g(y-z)h(x-y)dz\right] dy.$$

Substituting $y = z+t$, we obtain

$$[(f*g)*h](x) = \int \left[\int f(z)g(t)h(x-z-t)dz\right] dt.$$

Since $f(z)g(t)h(x-z-t)$ is measurable and

$$\int \left[\int |f(z)g(t)h(x-z-t)|\, dt\right] dz = \int \left[\int |g(t)h(x-z-t)|dt\right] |f(z)|dz$$

$$= \|f\|_1 \|g*h\|_1 < +\infty,$$

by Fubini's Theorem,

$$\int \left[\int f(z)g(t)h(x-z-t)dz\right]dt = \int \left[\int f(z)g(t)h(x-z-t)\, dt\right] dz$$

$$= \int \left[\int g(t)h(x-z-t)dt\right] f(z)dz$$

$$= \int f(z)\, (g*h)(x-z)dz$$

$$= [f*(g*h)](x).$$

3.4 <u>Theorem</u>. (W.H. Yong). Let $1 \leqslant p \leqslant \infty$. If $f \in L^p$ and $g \in L$, then $h = f*g$ is finite a.e., belongs to $L^p$, and satisfies the inequality

$$\|h\|_p \leqslant \|f\|_p \|g\|_1 .$$

Proof : The case $p = 1$ was proved in Theorem 3.3. To prove the case that p equal to $\infty$ , let $M = \|f\|_\infty$ , then for almost every x,

$$|h(x)| \leqslant \int |f(y)||g(x-y)|\,dy \leqslant M \int |g(x-y)|\,dy = \|f\|_\infty \|g\|_1 \quad .$$

which implies the conclusion.

Let $1 < p < \infty$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Using Hölder's inequality we have

$$|h(x)| \leqslant \int |f(y)||g(x-y)|\,dy = \int |f(y)||g(x-y)|^{1/p}|g(x-y)|^{1/q}\,dy$$

$$\leqslant \left\{\int |f(y)|^p |g(x-y)|\,dy\right\}^{1/p} \left\{\int |g(x-y)|\,dy\right\}^{1/q} .$$

Hence, $|h(x)|^p \leqslant \|g\|_1^{p/q} \left(\int |f(y)|^p |g(x-y)|\,dy\right)$.

Integrating with respect to x and interchanging the order of integration, we obtain

$$\int |h(x)|^p dx \quad < \|g\|_1^{p/q} \int dx\left(\int |f(y)|^p |g(x-y)|\,dy\right)$$

$$= \|g\|_1^{p/q} \int dy\left(\int |f(y)|^p |g(x-y)|\,dx\right)$$

$$= \|g\|_1^{p/q} \int |f(y)|^p dy\left(\int |g(x-y)|\,dx\right).$$

Therefore, $\|h\|_p^p \leqslant \|g\|_1^{p/q} \|f\|_p^p \|g\|_1$

$= \|g\|_1^p \|f\|_p^p$ , since $\frac{p}{q} + 1 = P(\frac{1}{q}+\frac{1}{p}) = p$.

Taking $p^{th.}$ roots the conclusion follows.

More generally, we have the following result.

Consider a function $H_\varepsilon$ which depending on the parameter $\varepsilon > 0$ and satisfies conditions :

a) $\int |H_\varepsilon(x)|\,dx \leqslant A$, for all $\varepsilon$ (A is non-negative constant and independent of $\varepsilon$ )

b) $\int H_\varepsilon(x)dx = 1$, for all $\varepsilon$

c) For any fixed $\delta > 0$, $\int_{|x| \geqslant \delta} |H_\varepsilon(x)|dx \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Here $H_\varepsilon$ is called a Kernel.

When does $f*H_\varepsilon \longrightarrow f$ as $\varepsilon \longrightarrow 0$ ?

Before we illustrate the meaning of these conditions we shall show that such Kernels $H_\varepsilon$ are easily manufactured.

3.5 Lemma. Suppose that $H \in L^1(R^n)$ is normalized so that $\int H(x)dx = 1$. Let $H_\varepsilon(x) = \frac{1}{\varepsilon^n} H(\frac{x}{\varepsilon})$. Then $H_\varepsilon$ satisfies conditions a), b), c).

Proof : $$\int |H_\varepsilon(x)|\,dx = \int \frac{1}{\varepsilon^n} |H(\tfrac{x}{\varepsilon})|\,dx.$$

Let $y = \frac{x}{\varepsilon}$ , so $dy = \frac{dx}{\varepsilon^n}$ and

$$\int |H_\varepsilon(x)|\,dx = \int |H(y)|dy = A \text{ for all } \varepsilon .$$

Likewise, $$\int H_\varepsilon(x)dx = \int H(y)dy = 1.$$

So a) and b) hold.

For any fixed $\delta > 0$,

$$\int_{|x| \geqslant \delta} |H_\varepsilon(x)|\,dx = \int_{|x| \geqslant \delta} \frac{1}{\varepsilon^n} |H(\tfrac{x}{\varepsilon})|\,dx = \int_{|y| \geqslant \delta/\varepsilon} |H(y)|\,dy \longrightarrow 0$$

as $\varepsilon \longrightarrow 0$, since $\delta/\varepsilon \longrightarrow \infty$ and integrating outside an expanding sphere. So c) also holds.

3.6 <u>Theorem</u>. If $f \in L^{\infty}(R^n)$ and $H_\varepsilon$ satisfies conditions a), b) and c). Then $f*H_\varepsilon(x) \longrightarrow f(x)$, as $\varepsilon \longrightarrow 0$, at every point of continuity of f and the convergence is uniform for x belonging to any compact set of points of continuity of f.

<u>Proof</u> : Let x be a point of continuity of f. It can be given that $\gamma > 0$ and there exists $\delta > 0$ such that

(1) $$|f(x-y)-f(x)| < \gamma \quad \text{whenever} \quad |y| < \delta .$$

By condition b),

$$f*H_\varepsilon(x)-f(x) = \int f(x-y)H_\varepsilon(y)dy - \int f(x)H_\varepsilon(y)dy$$
$$= \int [f(x-y)-f(x)]\, H_\varepsilon(y)dy .$$

Hence

$$|f*H_\varepsilon(x)-f(x)| \leqslant \int |f(x-y)-f(x)|\,|H_\varepsilon(y)|\,dy = \int_{|y|<\delta} + \int_{|y|\geqslant\delta} = I_1 + I_2 .$$

By (1) and condition (a),

$$I_1 \leqslant \gamma \int_{|y|<\delta} |H_\varepsilon(y)|\,dy \leqslant \gamma \int |H_\varepsilon(y)|\,dy \leqslant A\gamma .$$

If $m = \|f\|_\infty < \infty$ , then by condition (c) ,

$$I_2 \leqslant 2m \int_{|y|\geqslant\delta} |H_\varepsilon(y)|\,dy \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Therefore,

$$|f*H_\varepsilon(x)-f(x)| \leqslant I_1 + I_2 \leqslant A\gamma + o(1) < 2A\gamma$$

if $\varepsilon$ is sufficiently small. Since A is a positive constant and $\gamma$ is arbitrary, $f*H_\varepsilon(x) \longrightarrow f(x)$, as $\varepsilon \longrightarrow 0$.

Let K be any compact set of points of continuity of f. By Theorem (2.6) f is uniform continuous on K; i.e. for any giver $\gamma > 0$ there exists $\delta(\gamma) > 0$ such that $|f(x-y)-f(x)| < \gamma$ whenever $|y| < \delta$ for all $x \in K$. Hence $(f*H_\varepsilon)(x)$ is uniform convergence to f(x), as $\varepsilon \longrightarrow 0$.

3.7 <u>Theorem</u>. Let $x \in R^n$, if $f \in L^\infty(R^n)$, $H_\varepsilon$ satisfies conditions b) and c) above, and $H_\varepsilon \geqslant 0$ then

(1) $$\limsup_{\varepsilon \to 0} (f*H_\varepsilon)(x_0) \leqslant \limsup_{x \to x_0} f(x) ;$$

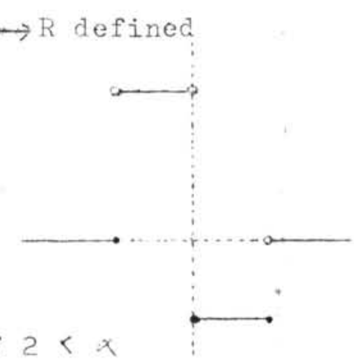
and

(2) $$\liminf_{\varepsilon \to 0} (f*H_\varepsilon)(x_0) \geqslant \liminf_{x \to x_0} f(x) .$$

The Theorem is false, if $H(x) < 0$ for some x; as the following example shows.

3.8 <u>Example</u>.* Consider the function $H : R \longrightarrow R$ defined as follows

$$H(x) = \begin{cases} 2 & \text{if } x \in (-1,0) \\ -1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$



Since $$H^{-1}((\alpha, \infty)) = \begin{cases} \emptyset & \text{if } 2 < \alpha \\ (-1,0) & \text{if } 0 < \alpha \leqslant 2 \\ (-\infty,0)\cup(1,\infty) & \text{if } -1 < \alpha \leqslant 0 \\ R & \text{if } \alpha \leqslant -1 \end{cases}$$ is open,

$H^{-1}((\alpha,\infty))$ is measurable, H is measurable function. So $H \in L^{\infty}(R)$, since $\|H\|_{\infty} = 2 < +\infty$,

$$\int_R H(x)dx = \int_{-\infty}^{-1} H(x)dx + \int_{-1}^{0} H(x)dx + \int_{0}^{1} H(x)dx + \int_{1}^{+\infty} H(x)dx = 0+2-1+0 = 1.$$

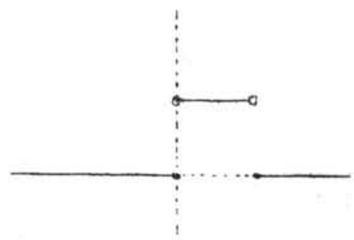
Letting $H_{\varepsilon}(x) = \frac{1}{\varepsilon} H(\frac{x}{\varepsilon})$ , we have

$$H_{\varepsilon}(x) = \begin{cases} \frac{2}{\varepsilon} & \text{if } x \in (-\varepsilon,0) \\ -\frac{1}{\varepsilon} & \text{if } x \in [0,\varepsilon] \\ 0 & \text{otherwise} \end{cases} .$$

In accordance with Lemma (3.6), $H_{\varepsilon}(x)$ satisfies conditions a), b) and c).

Consider the function $f : R \longrightarrow R$ which is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases} ,$$



then $f \in L^{\infty}(R)$.

Let $x_0 = 0$. We want to show that

$$\limsup_{\varepsilon \to 0} (f*H_{\varepsilon})(0) > \limsup_{x \to 0} f(x).$$

Since

$$(f*H_{\varepsilon})(0) = \int f(0-y)H_{\varepsilon}(y)dy = \int_{-1}^{0} H_{\varepsilon}(y)dy.$$

Let $\mathcal{N}_0$ be a collection of all neighborhoods of 0. For all $V \in \mathcal{N}_0$, given $\gamma > 0$, there exists $\delta \in V$, $0 < \delta \leq 1$, such that

$$(f*H_{\delta})(0) = \int_{-\delta}^{0} H_{\delta}(y)dy = \frac{2}{\delta}\delta = 2 > 2-\gamma,$$

then $\sup_{\varepsilon \in V} (f * H_\varepsilon)(0) = 2$ ; for all $V \in \mathcal{N}_0$,

and hence $\inf_{V \in \mathcal{N}_0} [\sup_{\varepsilon \in V} (f * H_\varepsilon)(0)] = 2$,

i.e. $\lim_{\varepsilon \to 0} \sup (f * H_\varepsilon)(0) = 2$.

For all $V \in \mathcal{N}_0$, given $\beta > 0$, there exists $y \in V$ such that

$$f(y) = 1 > 1 - \beta,$$

then $\sup_{x \in V} f(x) = 1$, for all $V \in \mathcal{N}_0$,

and hence $\inf_{V \in \mathcal{N}_0} [\sup_{x \in V} f(x)] = 1$,

i.e. $\lim_{x \to 0} \sup f(x) = 1$.

Therefore $\lim_{\varepsilon \to 0} \sup (f * H_\varepsilon)(0) > \lim_{x \to 0} \sup f(x)$.

3.9 Lemma. Let $x_0 \in R^n$, if $f \in L^\infty(R^n)$, $H_\varepsilon$ satisfies conditions b) and c) above, $H_\varepsilon \geqslant 0$, and if there is constant k such that $f \leqslant k$ a.e. in a neighborhood of $x_0$, then

$$\lim_{\varepsilon \to 0} \sup (f * H_\varepsilon)(x_0) \leqslant k.$$

Proof*: Choose $\delta > 0$ such that $f(x) \leqslant k$ a.e. for $x \in B_{x_0, \delta}$.

Then

$$(f * H_\varepsilon)(x_0) = \int_{|y| < \delta} f(x_0 - y) H_\varepsilon(y)dy + \int_{|y| \geqslant \delta} f(x_0 - y) H_\varepsilon(y)dy$$

$$= I_1(x_0) + I_2(x_0).$$

Since, $\| x_0-(x_0-y)\| = \| y \| < \delta \longrightarrow f(x_0-y) \leqslant k$ ,

$$I_1(x_0) \leqslant k \int_{|y|<\delta} H_\varepsilon(x)dy \leqslant k \int H_\varepsilon(x)dy = k .$$

$$|I_2(x_0)| \leqslant \int_{|y|\geqslant\delta} |f(x_0-y)| H_\varepsilon(y)\, dy$$

$$\leqslant m \int_{|y|\geqslant\delta} H_\varepsilon(y)\, dy \longrightarrow 0 \text{ , as } \varepsilon \longrightarrow 0 ,$$

where $m = \| f \|_\infty < \infty$ .

Therefore $\limsup_{\varepsilon\to 0} (f*H_\varepsilon)(x_0) \leqslant \limsup_{\varepsilon\to 0} I_1(x) + \limsup_{\varepsilon\to 0} I_2(x) \leqslant k.$

<u>Proof of Theorem</u> (3.7). We can assume that the right side of (1) is not $+\infty$ , for otherwise there is nothing to prove. If k is any number greater than the right member of (1). i.e. $\limsup_{x\to x_0} f(x) < k.$ We claim that there exists $\delta > 0$ such that $f(x) < k$ for all $x \in B_{x_0,\delta}$ . Let $\mathcal{N}_{x_0}$ be a collection of all neighborhood of $x_0$. Therefore

$$\limsup_{x\to x_0} f(x) = \inf_{V\in\mathcal{N}_{x_0}} [\sup_{x\in V} f(x)] < k .$$

Suppose, for all $V \in \mathcal{N}_{x_0}$ , there exists $x \in V$ such that $f(x) \geqslant k$, then

$$\sup_{x\in V} f(x) \geqslant k ,$$

and $\inf_{V\in\mathcal{N}_{x_0}} [\sup_{x\in V} f(x)] \geqslant k$ , which is a contradiction.

Hence there exists $V_0 \in \mathcal{N}_{x_0}$ such that $f(x) < k$ for all $x \in V_0$. Since $x \in V_0$, there exists $\delta > 0$ such that $x \in B_{x_0,\delta} \subset V_0$, i.e., there exists $\delta > 0$ such that $f(x) < k$ for all $x \in B_{x_0, \delta}$. By Lemma (3.9),

$$\lim_{\varepsilon \to 0} \sup (f*H_\varepsilon)(x_0) \leq k ,$$

and k is any number greater than $\lim_{x \to x_0} \sup f(x)$. We claim that

$$\lim_{\varepsilon \to 0} \sup (f*H_\varepsilon)(x_0) \leq \lim_{x \to x_0} \sup f(x) .$$

Suppose $\lim_{\varepsilon \to 0} \sup (f*H_\varepsilon)(x_0) > \lim_{x \to x_0} \sup f(x)$ , there exists $k_0$ such that $\lim_{\varepsilon \to 0} \sup (f*H_\varepsilon)(x_0) > k_0 > \lim_{x \to x_0} \sup f(x)$. Since $k_0 > \lim_{x \to x_0} \sup f(x)$ from above we replace k by $k_0$ follows that $\lim_{\varepsilon \to 0} \sup (f*H_\varepsilon)(x_0) \leq k_0$, which is a contradiction. Hence $\lim_{\varepsilon \to 0} \sup (f*H_\varepsilon)(x_0) \leq \lim_{x \to x_0} \sup f(x)$.

Since $(-f*H_\varepsilon) = -(f*H_\varepsilon)$ , we also have

$$\lim_{\varepsilon \to 0} \sup \left[-(f*H_\varepsilon)(x_0)\right] \leq \lim_{x \to x_0} \sup [-f(x)]$$

$$-\lim_{\varepsilon \to 0} \inf (f*H_\varepsilon)(x_0) \leq -\lim_{x \to x_0} \inf f(x)$$

$$\lim_{\varepsilon \to 0} \inf (f*H_\varepsilon)(x_0) \geq \lim_{x \to x_0} \inf f(x)$$

3.10 <u>Corollary</u>: The hypothesis as in Theorem (3.7).

1) If f is upper semicontinuous at a point $x_0$, then

$$\lim_{\varepsilon \to 0} \sup (f*H_\varepsilon)(x_0) \leq f(x_0).$$

2) If f is lower semicontinuous at a point $x_0$, then

$$\lim_{\varepsilon \to 0} \inf (f*H_\varepsilon)(x_0) \geq f(x_0).$$

Let $f(x)$ be a measurable function of $x \in R^n$. Then , for any vector $u \in R^n$, the function $f(x+u)$ is a " translation " of $f(x)$ by the vector $-u$. The following lemma shows that this operation of Translating function is continuous in the metric of the spaces $L^p$, $1 \leqslant p < \infty$ .

3.11 Lemma. If $f \in L^p(R^n)$, $1 \leqslant p < \infty$ then

(a) $$\| f(x+u)-f(x) \|_p = \left( \int |f(x+u)-f(x)|^p dx \right)^{1/p} \longrightarrow 0 \quad \text{as } |u| \longrightarrow 0.$$

Proof : We will prove the Lemma in series of steps as :

1) If $f = g+h$ where $g$ satisfies (a) and $h$ has arbitrary small norm, then $f$ satisfies (a). Since for any $\varepsilon > 0$, $\|h\|_p < \varepsilon$ . Then

$$\begin{aligned} \|f(x+u)- f(x)\|_p &= \|g(x+u)-g(x)+h(x+u)-h(x)\|_p \\ &\leqslant \|g(x+u)-g(x)\|_p + \|h(x+u)\|_p + \|h(x)\|_p \\ &< \|g(x+u)-g(x)\|_p + 2\varepsilon < 3\varepsilon \end{aligned}$$

for any sufficiently small $|u|$.

2) The Lemma holds, if $f$ is continuous function with compact support. Since $f$ is a continuous function with compact support, by Theorem (2.6), $f$ must be uniformly continuous. If $S$ is the support of $f$ then

$$\|f(x+u)-f(x)\|_p \leqslant C \max_{x \in S} |f(x+u)-f(x)| \longrightarrow 0 \quad \text{as } |u| \longrightarrow 0 .$$

3) Since $C_c(R^n)$ is dense in $L^p(R^n)$ [13, P68] , any $f \in L^p(R^n)$ is the limit of function in 2). So using 1) the conclusion holds.

Moreover, by Theorem (3.4), if $f \in L^p$, $1 \leq p \leq \infty$, then $f*H_\varepsilon$ is also in $L^p$ for all $\varepsilon$. The next Theorem we will shows that $f*H_\varepsilon$ converges to f in the metric of $L^p$.

3.12 Theorem. If $f \in L^p$, $1 \leq p < \infty$, and $H_\varepsilon$ satisfies conditions a), b) and c), then

$$\| f*H_\varepsilon - f \|_p \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Proof : Case I : $1 < p < \infty$ ; let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality with properties a) and b),

$$\begin{aligned}
|f*H_\varepsilon(x) - f(x)| &\leq \int |f(x-y)-f(x)| \, |H_\varepsilon(y)| \, dy \\
&= \int |f(x-y)-f(x)| \, |H_\varepsilon(y)|^{1/p} |H_\varepsilon(y)|^{1/q} \, dy \\
&\leq \left(\int |f(x-y)-f(x)|^p |H_\varepsilon(y)| dy\right)^{1/p} \left(\int |H_\varepsilon(y)| dy\right)^{1/q}. \\
&\leq A^{1/q} \left(\int |f(x-y)-f(x)|^p \, |H_\varepsilon(y)| dy\right)^{1/p}.
\end{aligned}$$

Hence

$$\int |f*H_\varepsilon(x)-f(x)|^p dx \leq A^{p/q} \int dx \left( \int |f(x-y)-f(x)|^p |H_\varepsilon(y)| dy \right)$$

and, interchanging the order of integration, we have

$$\begin{aligned}
\| f*H_\varepsilon - f \|_p^p &\leq A^{p/q} \int dy \left( \int |f(x-y)-f(x)|^p |H_\varepsilon(y)| dx \right) \\
&= A^{p/q} \int \left( \int |f(x-y)-f(x)|^p dx \right) |H_\varepsilon(y)| \, dy.
\end{aligned}$$

If we set $\quad g(y) = \int |f(x-y)-f(x)|^p dx,$

then g is everywhere defined and bounded by $2 \| f \|_p^p$ ; moreover, by Lemma (3.11), $g(y) \longrightarrow 0$ as $y \longrightarrow 0$. Therefore, by the same argument as in Theorem (3.6) we conclude that

$$\|f*H_\varepsilon - f\|_p^p \leqslant A^{p/q} \int g(y)\,|H_\varepsilon(y)|\,dy \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Case II : $P = 1$.

$$|f*H_\varepsilon(x)-f(x)| \leqslant \int |f(x-y)-f(x)|\,|H_\varepsilon(y)|\,dy .$$

Since $f(x,y) \longmapsto (x-y)$ is measurable function. Integrating with respect to x and interchanging the order of integration.

$$\int |f*H_\varepsilon(x)-f(x)|\,dx \leqslant \int dx\left(\int |f(x-y)-f(x)|\,|H_\varepsilon(y)|\,dy\right)$$
$$= \int \left(\int |f(x-y)-f(x)|\,dx\right)|H_\varepsilon(y)|\,dy$$

If we set $g(y) = \int |f(x-y)-f(x)|dx$ , then g is everywhere defined and bounded by $2\|f\|_1$, moreover, by Lemma 3.11, $g(y) \longrightarrow 0$ as $y \longrightarrow 0$. Therefore, by the same argument as in Theorem 3.6 we conclude that

$$\|f*H_\varepsilon - f\|_1 \leqslant \int g(y)\,|H_\varepsilon(y)|\,dy \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Suppose that f is a locally integrable function, and let $B(x,\varepsilon) = \{y : |y-x| \leqslant \varepsilon\}$ . By Lebesgue's Theorem on the differentiability of integrals [13 p157] we have that, as $\varepsilon \longrightarrow 0$,

$$\frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} f(y)dy \longrightarrow f(x)$$

at almost all points $x \in R^n$, where $|B(x,\varepsilon)|$ denote the measure of $B(x,\varepsilon)$. Lemma (3.14) below is a strengthening of the result.

3.13 Definition. Let f be a locally integrable function. x is Lebesgue point of f, if

$$\frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} |f(y)-f(x)|\,dy \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0.$$

Note that $|f(x)|$ must be finite, otherwise the integral is divergent. Clearly, any point of continuity of f is also a Lebesgue point of f.

3.14 Lemma. Suppose f is a locally integrable function. Then almost every point in $R^n$ is a Lebesgue point of f.

Proof : Let r be a rational number. Then by Lebesgue's Theorem, with $B = B(x,\varepsilon)$, as $\varepsilon \longrightarrow 0$

$$(1) \qquad \frac{1}{|B|}\int_B |f(y) - r|\,dy \longrightarrow |f(x) - r|$$

except possibly for x belonging to a set $Z_r$ of measure zero. Let Z be the union of sets $Z_r$, for all rational numbers r ; then Z has also measure zero. We show that if $x \notin Z$, then x is a lebesgue point of f

Let $\delta > 0$ then, for some rational number r, $|f(x)-r| < \delta/_2$ . Now,

$$(2) \qquad \frac{1}{|B|}\int_B |f(y)-f(x)|\,dy \leqslant \frac{1}{|B|}\int_B |f(y)-r|\,dy + \frac{1}{|B|}\int_B |r-f(x)|\,dy$$

but the second member of the right-side of (2) is less than $\delta/_2$ , and the first member tends to $|f(x)-r| < \delta/_2$ , as $\varepsilon \longrightarrow 0$, by virtue of (1). Hence the left-side of (2) is less than $\delta$, for all sufficiently small $\varepsilon$ since $\delta$ was an arbitrary positive number, it follows that x is a Lebesgue point of f.

3.15 Theorem. Suppose that H is bounded measurable function and $H(x) = O(|x|^{-(x+1)})$ for $|x| \geqslant 1$, (hence H is integrable). Suppose also that $\int H(x)dx = 1$, and $H_\varepsilon(x) = \varepsilon^{-n}H(\frac{x}{\varepsilon})$. Then for any $f \in L^p$, $1 \leqslant p < \infty$ ,

$$(f*H_\varepsilon)(x) \longrightarrow f(x) \quad \text{a.e.,} \quad \text{as } \varepsilon \longrightarrow 0.$$

Proof : By Lemma (3.14), it suffices to prove convergence at every Lebesgue point of f. We claim that there exists a positive constant A such that

$$\text{(a)} \qquad |H(x)| \leqslant \frac{A}{1+|x|^{n+1}} \qquad \text{for all } x \in R^n.$$

Since H is bounded and $H(x) = O(|x|^{-(n+1)})$ for $|x| \geqslant 1$, there exists $M > 0$, $N > 0$, $R > 0$ such that

$$|H(x)| \leqslant M \quad \text{and} \quad |H(x)||x|^{(n+1)} \leqslant N \quad \text{for all } |x| \geqslant R \text{ respectively.}$$

Therefore $\quad |H(x)| + |H(x)||x|^{(n+1)} \leqslant M + N$

$$|H(x)| \leqslant \frac{M+N}{1+|x|^{n+1}} \qquad \text{for all } |x| \geqslant R.$$

For all $|x| \leqslant R$,

$$\frac{1}{1+|x|^{n+1}} \geqslant \frac{1}{1+R^{n+1}}$$

$$\frac{1+R^{n+1}}{1+|x|^{n+1}} \geqslant 1.$$

Therefore $\quad |H(x)| \leqslant M \leqslant \dfrac{M(1+R^{n+1})}{1+|x|^{n+1}}$ .

Then there exists $A = \max[M+N, M(1+R^{n+1})]$ such that (a) holds.

Hence $\quad |H_\varepsilon(y)| = |\varepsilon^{-n}H(\frac{y}{\varepsilon})| \leqslant \varepsilon^{-n}\dfrac{A}{1+|\frac{y}{\varepsilon}|^{n+1}} = \dfrac{A\varepsilon}{\varepsilon^{n+1}+|y|^{n+1}}$ .

Let x be a Lebesgue point of f, and let

$$E(u) = \int_{|y|\leqslant u} |f(x-y) - f(x)|\,dy.$$

Since x is a Lebesgue point of f,

$$\frac{E(u)}{u^n} = \frac{V_n E(u)}{V_n u^n} = \frac{V_n}{V_n u^n} \int_{|y| \leqslant u} |f(x-y) - f(x)|\,dy$$

$$= \frac{V_n}{|B(x,u)|} \int_{B(x,u)} |f(z) - f(x)|\,dz \longrightarrow 0 \text{ , as } u \longrightarrow 0$$

where $z = x-y$, and $V_n$ is the volume of unit ball. Hence, given an arbitrary $\delta > 0$, there exists a (small) $w > 0$ such that $E(u) < \delta u^n$ where $0 < u \leqslant w$. Since

$$\int H_\varepsilon(y)\,dy = \int H(y)\,dy = 1,$$

$$|(f*H_\varepsilon)(x) - f(x)| = \left|\int f(x-y)H_\varepsilon(y)\,dy - f(x)\right|$$

$$= \left|\int f(x-y)H_\varepsilon(y)\,dy - f(x)H_\varepsilon(y)\,dy\right|$$

$$= \left|\int [f(x-y) - f(x)]H_\varepsilon(y)\,dy\right|$$

$$\leqslant \int |f(x-y) - f(x)||H_\varepsilon(y)|\,dy = \int_{|y| \leqslant \varepsilon} + \int_{|y| \geqslant \varepsilon}$$

$$= I_1 + I_2, \text{ where } 0 < \varepsilon \leqslant w.$$

Now,

$$I_1 = \int_{|y| \leqslant \varepsilon} |f(x-y) - f(x)||H_\varepsilon(y)|\,dy$$

$$\leqslant \int_{|y| \leqslant \varepsilon} |f(x-y) - f(x)| \frac{A\varepsilon}{\varepsilon^{n+1} + y^{n+1}}\,dy$$

$$\leqslant \int_{|y| \leqslant \varepsilon} |f(x-y) - f(x)| \frac{A\varepsilon}{\varepsilon^{n+1}}\,dy = \frac{A}{\varepsilon^n} \int_{|y| \leqslant \varepsilon} |f(x-y) - f(x)|\,dy \longrightarrow 0$$

as $\varepsilon \longrightarrow 0$, since x is a Lebesgue point.

On the other hand,

$$I_2 \leqslant \int_{|y|\geqslant\varepsilon} |f(x-y) - f(x)| \frac{A\varepsilon}{\varepsilon^{n+1} + |y|^{n+1}} dy$$
$$\leqslant A\varepsilon \int_{|y|\geqslant\varepsilon} |f(x-y) - f(x)| \frac{dy}{|y|^{n+1}} .$$

Let $\varepsilon \int_{|y|\geqslant\varepsilon} |f(x-y) - f(x)| \frac{dy}{|y|^{n+1}} = \varepsilon \int_{\varepsilon\leqslant|y|\leqslant w} + \varepsilon \int_{y \geqslant w}$ .

We claim that the last term tends to zero as $\varepsilon \to 0$ .

It is enought to show that $\int_{|y|\geqslant w}$ is finite.

$$\int_{|y|\geqslant w} |f(x-y) - f(x)| \frac{dy}{|y|^{n+1}} \leqslant \int_{|y|\geqslant w} |f(x-y)| \frac{dy}{|y|^{n+1}} + |f(x)| \int_{|y|\geqslant w} \frac{dy}{|y|^{n+1}} .$$

Since x is a lebesgue point of f, we have that $|f(x)|$ is finite,

and

$$(3.16) \qquad \int_{|y|\geqslant w} \frac{dy}{|y|^{n+1}} = \int_w^{+\infty} \frac{1}{\rho^{n+1}} \left( \int_{\partial B(0,\rho)} ds \right) d\rho$$
$$= \int_w^{+\infty} \frac{1}{\rho^{n+1}} S_n \rho^{n+1} d\rho$$
$$= S_n \int_w^{+\infty} \frac{1}{\rho^2} d\rho = S_n \frac{1}{w} < +\infty ,$$

where $S_n$ is the surface area of unit ball, so the last integral is finite.

If $p = 1$, then $\int_{|y|\geqslant w} |f(x-y)| \frac{dy}{y^{n+1}} \leqslant \frac{1}{w^{n+1}} \int_{|y|\geqslant w} |f(x-y)| dy$

$$\leqslant \frac{1}{w^{n+1}} \|f\|_1 < +\infty .$$

If $1 < p < \infty$ , let q be such that $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder inequality,

$$\int_{|y|\geq w} \frac{|f(x-y)|}{|y|^{n+1}}\,dy \leq [\int_{|y|\geq w} |f(x-y)|^p dy]^{\frac{1}{p}} [\int_{|y|\geq w} (\frac{1}{|y|^{n+1}})^q\, dy]^{\frac{1}{q}}$$

$$\leq \|f\|_p [\int_{|y|\geq w} \frac{dy}{|y|^{n+1}}]^{\frac{1}{q}} = \|f\|_p [S_n \frac{1}{w}]^{\frac{1}{q}} < +\infty \quad .$$

Then the last term tends to zero as $\varepsilon \to 0$ .

Since $E(u) = \int_{|y|\leq u} |f(x-y) - f(x)|dy$, we claim that

$$\int_{\varepsilon\leq|y|\leq w} |f(x-y) - f(x)|\frac{dy}{|y|^{n+1}} = \int_{\varepsilon}^{w} \frac{1}{u^{n+1}}\, dE(u) \ .$$

Since $\frac{1}{u^{n+1}}$ is uniformly continuous on $[\varepsilon, w]$ for $n = 1,2,\ldots$

i.e. for any given $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that

$|\frac{1}{u^n} - \frac{1}{z^n}| < \frac{\varepsilon_0}{\delta w^n}$ for all $u, z \in [\varepsilon, w]$ for which $|u-z| < \delta_0$.

Let $\varepsilon \leq u_0 \leq u_1 \leq \ldots \leq u_m = w$, and let $u_{i-1} \leq \xi_i \leq u_i$ . Since

$$\int_{\varepsilon}^{w} \frac{1}{u^{n+1}} dE(u) = \lim_{\Delta u_i \to 0} \sum_{i=1}^{m} [\frac{1}{\xi_i^{n+1}}][E(u_i) - E(u_{i-1})] \ ,$$

where $u_{i-1} \leq \xi_i \leq u_i$. The existence of Riemann-Stieltjes Integral is **supperted** by [12:P.108]. Then we must show that

$$\lim_{\substack{\Delta u_i \to 0 \\ m \to \infty}} \sum_{i=1}^{m} [\frac{1}{\xi_i^{n+1}}][E(u_i) - E(u_{i-1})] = \int_{\varepsilon\leq|y|\leq w} |f(x-y) - f(x)|\frac{dy}{|y|^{n+1}} \ .$$

We choose $N > 0$ such that $\Delta u_i = |u_{i-1} - u_i| < \delta_0$, $i = 1,2,\ldots,m$, for all $m \geq N$.

For $m \geq N$,

$$\left|\sum_{i=1}^{m}\left[\frac{1}{\rho_i^{n+1}}\right][E(u_i)-E(u_{i-1})] - \int_{\varepsilon\leqslant|y|\leqslant w}|f(x-y)-f(x)|\frac{dy}{|y|^{n+1}}\right|$$

$$= \left|\sum_{i=1}^{m}\left[\frac{1}{\rho_i^{n+1}}\right]\int_{u_{i-1}\leqslant|y|\leqslant u_i}|f(x-y)-f(x)|\,dy - \sum_{i=1}^{m}\int_{u_{i-1}\leqslant|y|\leqslant u_i}\frac{1}{|y|^{n+1}}|f(x-y)-f(x)|\,dy\right|$$

$$\leqslant \sum_{i=1}^{m}\int_{u_{i-1}\leqslant|y|\leqslant u_i}\left|\frac{1}{\rho_i^{n+1}}-\frac{1}{|y|^{n+1}}\right||f(x-y)-f(x)|\,dy$$

$$< \frac{\varepsilon_o}{\delta w^n}\sum_{i=1}^{m}\int_{u_{i-1}\leqslant|y|\leqslant u_i}|f(x-y)-f(x)|\,dy$$

$$= \frac{\varepsilon_o}{\delta w^n}\int_{\varepsilon\leqslant|y|\leqslant w}|f(x-y)-f(x)|\,dy < \frac{\varepsilon_o}{\delta w^n}E(w) < \frac{\varepsilon_o}{\delta w^n}\delta w^n = \varepsilon_o.$$

Hence

$$(3.17) \qquad \varepsilon\int_{\varepsilon\leqslant|y|\leqslant w}|f(x-y)-f(x)|\frac{dy}{|y|^{n+1}} = \varepsilon\int_{\varepsilon}^{w}\frac{1}{u^{n+1}}\,dE(u).$$

Integrating by parts

$$\varepsilon\int_{\varepsilon}^{w}\frac{1}{u^{n+1}}dE(u) = \varepsilon\left\{\left[\frac{E(u)}{u^{n+1}}\right]_{\varepsilon}^{w} + (n+1)\int_{\varepsilon}^{w}\frac{E(u)}{u^{n+2}}\,du\right\}$$

$$\leqslant \varepsilon\frac{E(w)}{w^{n+1}} + \varepsilon(n+1)\int_{\varepsilon}^{w}\frac{E(u)}{u^{n+2}}\,du.$$

As $\varepsilon\to o$, $\varepsilon E(w)w^{-(n+1)}$ tends to zero, because $w$ is fixed. Finally,

$$\varepsilon(n+1)\int_{\varepsilon}^{w}\frac{E(u)}{u^{n+2}}du < \varepsilon(n+1)\int_{\varepsilon}^{w}\frac{\delta}{u^2}du < \varepsilon\delta(n+1)\int_{\varepsilon}^{\infty}\frac{1}{u^2}du = \delta(n+1),$$

whence, as $\varepsilon\to o$, $\limsup I_2 < A(n+1)\delta$ for any arbitrarily small $\delta$. Therefore, at any Lebesgue point of $f$,

$$|f*H_\varepsilon(x)-f(x)| \leqslant I_1 + I_2 \longrightarrow 0 \text{ as } \varepsilon\longrightarrow 0.$$