CHAPTER VI

FIELDS WHICH ARE IRREDUNDANT UNIONS OF SUBFIELDS

1 Introduction.

Let a field F be the irredundant union of its subfields F_α's. If we consider F as a group under addition, then

$$F = \bigcup_\alpha F_\alpha$$

is the irredundant union of its subgroups F_α. Also, if we consider $F \setminus \{o\}$ as a group under multiplication, then

$$F \setminus \{o\} = \bigcup_\alpha (F_\alpha \setminus \{o\})$$

is the irredundant union of its subgroups $F_\alpha \setminus \{o\}$. Thus in both cases, they must first be groups which are irredundant unions of subgroups as in Chapter III.

First, we will consider finite fields as follows:

2 Finite Fields can not be Irredundant Unions of Subfields.

Let F be a finite field. We have known that $F \setminus \{o\}$ is a cyclic group under multiplication, so it follows from 1 that F can not be an irredundant union of its subfields. Then we have

2.1 Theorem. No finite field can be an irredundant union of its subfields.

3 Groups of Mappings Defined on n-fields Which are n-groups.

For convenience, we will call a field which is an irredundant union of n subfields an n-field. Similar for an n-gro

Let F be a field. Define

$$G = \{ I_b^a \,/\, a \neq o,\ a,\ b \in F \text{ and } I_b^a(x) = ax+b \text{ for all } x \text{ in } F \},$$

$$\bar{G} = \{ I_b^a \,/\, a,\ b \in F \text{ and } I_b^a(x) = ax+b \text{ for all } x \text{ in } F \}$$

and define the operations o and $*$ on G and $\bar{G}$, respectively, as follows:

$$(I_b^a \circ I_d^c)(x) = I_b^a(I_d^c(x)) = I_b^a(cx+d) = acx+(ad+b)$$

and

$$(I_b^a * I_d^c)(x) = I_b^a(x) + I_d^c(x) = (a+c)x+(b+d)$$

for all x in F.

3.1 Theorem. If F is an n-field ($n > 2$), then $(G, \circ)$ and $(\bar{G}, *)$ are n-groups.

Proof. Let $F = \bigcup_{i=1}^{n} F_i$ be the irredundant union of the F_i.

First, we will show that $(G, \circ)$ is a group.

(i) Let I_b^a, I_d^c be in G. For any x in F we have

$$(I_b^a \circ I_d^c)(x) = acx+(ad+b) = I_{ad+b}^{ac}(x).$$

Thus $I_b^a \circ I_d^c = I_{ad+b}^{ac}$ which belongs to G.

(ii) Let I_b^a, I_d^c and I_f^e be in G. Then for any x in F, we have

$$((I^a_b \circledcirc I^c_d) \circledcirc I^e_f)(x) = (I^a_b \circledcirc I^c_d)(I^e_f(x))$$

$$= I^a_b\,(I^c_d(I^e_f(x)))$$

$$= I^a_b\,((I^c_d \circledcirc I^e_f)(x))$$

$$= (I^a_b \circledcirc (I^c_d \circledcirc I^e_f))(x),$$

so that $(I^a_b \circledcirc I^c_d) \circledcirc I^e_f \;=\; I^a_b \circledcirc (I^c_d \circledcirc I^e_f)$.

(iii) For any I^a_b in G. We have

$$I^a_b \circledcirc I^1_o = I^{a1}_{oa+b} = I^a_b = I^1_o \circledcirc I^a_b.$$

Then I^1_o is the identity of G.

(iv) For any I^a_b in G, $a \neq o$, then a^{-1} exists. Since

$$I^{a^{-1}}_{-a^{-1}b} \circledcirc I^a_b = I^{a^{-1}a}_{a^{-1}b-a^{-1}b} = I^1_o = I^a_b \circledcirc I^{a^{-1}}_{-a^{-1}b},$$

I^a_b and $I^a_{-a^{-1}b}$ are inverses in G.

Hence $(G, \circledcirc)$ is a group.

Similarly, we can show that $(\bar{G}, *)$ is a group with I^o_o is the identity of $\bar{G}$ and I^a_b and I^{-a}_{-b} are inverses in $\bar{G}$. Set

$$G_i = \left\{ I^a_b \;/\; a \neq o,\; a \in F_i,\; b \in F \right\}$$

and

$$\bar{G}_i = \left\{ I^a_b \;/\; a \in F_i,\; b \in F \right\}$$

for $i = 1, 2, \ldots n$. As above, $(G_i, \circledcirc)$ and $(\bar{G}_i, *)$ are subgroups of $(G, \circledcirc)$ and $(\bar{G}, *)$, respectively. It is clear that

$$G = \bigcup_{i=1}^{n} G_i$$

and

$$\bar{G} = \bigcup_{i=1}^{n} \bar{G}_i \quad .$$

Now, we want to show that $G = \bigcup_{i=1}^{n} G_i$ is the irredundant union of the G_i. Since $F_j \setminus \bigcup_{\substack{i=1 \\ i \neq j}}^{n} F_i \neq \emptyset$ for all j in $\{1, 2, \ldots n\}$, we can let x_j be in $F_j \setminus \bigcup_{\substack{i=1 \\ i \neq j}}^{n} F_i$, $j = 1, 2, \ldots n$. Suppose that $I_o^{x_j}$ belongs to G_k for some $k \neq j$. Then $I_o^{x_j} = I_b^a$ for some $a \neq o$ in F_k and b in F, so that $x_j = I_o^{x_j}(1) = I_b^a(1) = a+b$ and $o = I_o^{x_j}(o) = I_b^a(o) = b$, which implies that $x_j = a$, which contradicts the choice of x_j. Then $I_o^{x_j}$ belongs to $G_j \setminus \bigcup_{\substack{i=1 \\ i \neq j}}^{n} G_i$, $j = 1, 2, \ldots n$. This proof also show that $\bar{G} = \bigcup_{i=1}^{n} \bar{G}_i$ is the irredundant union of subgroups $\bar{G}_i$.

Hence the theorem is proved completely.

3.2 Remark. Theorem 3.1 is still true, if F is only assumed to be a division ring.

A partial converse of Theorem 3.1 is given by the following theorem:

3.3 <u>Theorem</u>. Assuming the notations preceding 3.1 and if further that

$$G = \bigcup_{i=1}^{n} G_i , \quad \bar{G} = \bigcup_{i=1}^{n} \bar{G}_i$$

are irredundant unions of their subgroups G_i and $\bar{G}_i$; respectively and $\bar{G}_i = G_i \cup \{ I^o_b / b \in F \}$,

then F is an n-field.

<u>Proof</u>. For i = 1, 2,...n, we define

$$F_i = \{ a \in F / I^a_b \in \bar{G}_i \} .$$

It is clear that $F = \bigcup_{i=1}^{n} F_i$. We will show that each F_i is a subfield of F.

(i) Let a, b be in F_i. Then there exist c and d in F such that I^a_c and I^b_d are in $\bar{G}_i$. Since $I^a_c * I^b_d = I^{a+b}_{c+d}$ is in $\bar{G}_i$ and if a ≠ o, b ≠ o, then $I^a_c \odot I^b_d = I^{ab}_{ad+c}$ is in $G_i \subset \bar{G}_i$. Then we have a+b and ab are in F_i for any a, b in F_i.

(ii) Since I^o_o is in $\bar{G}_i$ and I^1_o is in $G_i \subset \bar{G}_i$, o, 1 are in F_i.

(iii) Let a be in F_i. Then there exists b in F such that I^a_b is in $\bar{G}_i$. Since the inverse of I^a_b in $\bar{G}_i$ is I^{-a}_{-b}, -a is in F_i. If a ≠ o, then the inverse of I^a_b in G is $I^{a^{-1}}_{-a^{-1}b}$ which is in $G_i \subset \bar{G}_i$ and therefore a^{-1} is in F_i.

Hence F_i is a subfield of F.

Finally, we will show that

$$F = \bigcup_{i=1}^{n} F_i$$

is the irredundant union of subfields F_i. For each i in $\{1,2,\ldots n\}$, $\bar{G}_j \setminus \bigcup_{\substack{i=1 \\ i\neq j}}^{n} \bar{G}_i \neq \emptyset$, let $I_{y_j}^{x_j}$ be in $\bar{G}_j \setminus \bigcup_{i=1}^{n} \bar{G}_i$. Suppose that $I_o^{x_j}$ belongs to $\bar{G}_k$ for some $k \neq j$. Then $I_o^{x_j} * I_{y_j}^{o} = I_{y_j}^{x_j}$ belongs to $\bar{G}_k$, which contradicts the choice of $I_{y_j}^{x_j}$ in $\bar{G}_j \setminus \bigcup_{\substack{i=1 \\ i\neq j}}^{n} \bar{G}_i$. If x_j is in F_l for some $l \neq j$, then $I_y^{x_j}$ is in $\bar{G}_l$ for some y in F. Therefore $I_y^{x_j} * I_{-y}^{o} = I_o^{x_j}$ belongs to $\bar{G}_I$, which is a contradiction. Then x_j belongs to $F_j \setminus \bigcup_{\substack{i=1 \\ i\neq j}}^{n} F_i$.

Hence the theorem is proved.