หอสมุดกลาง จุฬาลงกรณมหาวิทยาลัย

CHAPTER III

EXAMPLE 2 : THE IMPLICATIONAL CALCULUS (I)

In chapter II we have emphasized the matter of full sentential calculus, in this chapter we study a partial theory of Sentential Calculus, the Implicational Calculus (I) which has implication as its only primitive connective and we choose some axioms and rules such that the theorems coincide with the tautologies in this connective. We also prove that I is consistent and state that I is complete and give some subtheories of I which have I as their only complete and consistent extension.

We first introduce the symbols of I which are the following :

(i) a denumerable set of sentence variables :

$p, q, r, s, p_1, q_1, r_1, s_1, \ldots\ldots\ldots\ldots,$

(ii) a logical connective : $\longrightarrow$,

(iii) parentheses : (,) .

We define sentences of I as in definition 1.1, and rules of inference are as 1.2.

The axioms of I are the following :

A1. $p \rightarrow (q \rightarrow p)$

A2. $(s \rightarrow (p \rightarrow q)) \rightarrow ((s \rightarrow p) \rightarrow (s \rightarrow q))$

A3. $((p \rightarrow q) \rightarrow p) \rightarrow p.$

Similarly as in SC, we say that the sentence ϕ is a theorem of I instead of saying ϕ is a theorem of the set of axioms of I and use the notation $\vdash_I \phi$, and instead of saying the set of all theorems of I, we say the theory I or I. The definition of a proof from a set of sentences Σ in I is given similarly as Definition 2.3 and use the notation $\Sigma \vdash_I \phi$ where ϕ is the last sentence of the proof.

From Lemma 2.4, we also have that for any sentence ϕ of I, $\vdash_I \phi \longrightarrow \phi$. Thus we also have the Deduction Theorem for I.

3.1 <u>Theorem.</u> (Deduction Theorem.) Let Σ be a set of sentences in I, and ϕ, ψ be sentences of I. If $\Sigma \cup \{\phi\} \vdash_I \psi$, then $\Sigma \vdash_I \phi \longrightarrow \psi$.

<u>Proof.</u> Same as Theorem 2.5.

3.2 <u>Definition.</u> The <u>truth-value of a sentence</u> in I is defined as follows :

(i) The truth-value of a sentence variable is either true (T) or false (F) but not both.

(ii) If ϕ is of the form $\psi \longrightarrow \theta$, then ϕ is false if and only if ψ is true and θ is false.

(iii) Every sentence is either true or false but not both.

3.3 <u>Definition.</u> A sentence ϕ of I is a <u>tautology</u> if and only if ϕ is true for all assignments of truth-values to the sentence variables in ϕ.

3.4 Lemma. If ϕ and $\phi \rightarrow \psi$ are tautologies, then ψ is a tautology.

Proof. Let $a_1, \ldots, a_n$ be all the sentence variables in ϕ and ψ. Assume ϕ and $\phi \rightarrow \psi$ are tautologies. Suppose ψ is not a tautology. Then there is an assignment of truth-values to $a_1, \ldots, a_n$ say $s_1, \ldots, s_n$ such that ψ is false. Since ϕ is a tautology, ϕ is true for the assignment $s_1, \ldots, s_n$ to $a_1, \ldots, a_n$. Then $s_1, \ldots, s_n$ is an assignment to $a_1, \ldots, a_n$ such that $\phi \rightarrow \psi$ is false which is a contradiction because $\phi \rightarrow \psi$ is a tautology. Therefore ψ is a tautology.

3.5 Theorem. If $\vdash_I \phi$, then ϕ is a tautology.

Proof. We prove this theorem by induction on the length of proofs in I. We have that each axiom is a tautology and Substitution preserves tautology. By Lemma 3.4 MP. preserves tautology. Therefore every theorem of I is a tautology.

3.6 Theorem. I is consistent.

Proof. Since a sentence variable is not a tautology, by Theorem 3.5 it is not a theorem of I. Thus I is consistent.

3.7 Note. The following are theorems of I :

(i) $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$

(ii) $((q \rightarrow r) \rightarrow r) \rightarrow (((p \rightarrow q) \rightarrow r) \rightarrow r)$

(iii) $((p \rightarrow r) \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow r))$

(iv) $(p \rightarrow r) \rightarrow (((p \rightarrow q) \rightarrow r) \rightarrow r)$

(v) $((p \to q) \to r) \to ((p \to r) \to r)$.

3.8 Lemma. Let ϕ be a sentence in I, and $a_1, \ldots, a_n$ be distinct variables among which are all the variables occurring in ϕ, and let $s_1, \ldots, s_n$ be truth-values of $a_1, \ldots, a_n$, respectively. Let a be a variable not occurring among $a_1, \ldots, a_n$. For i = 1, ..., n, let ψ_i be a_i or $a_i \to a$ according as s_i is true or false and let ϕ' be $(\phi \to a) \to a$ or $\phi \to a$ according as the value of ϕ for the values $s_1, \ldots, s_n$ of $a_1, \ldots, a_n$ is true or false. Then $\psi_1, \ldots, \psi_n \vdash_I \phi'$.

Proof. We will prove by induction on length of ϕ (number of occurrences of $\to$ in ϕ).

If there is no occurrences of $\to$ in ϕ, then ϕ is one of the variables $a_1, \ldots, a_n$, say a_i and so ϕ' is of the form $(a_i \to a) \to a$ or $a_i \to a$ according as s_i is true or false. Since ψ_i is a_i or $a_i \to a$ according as s_i is true or false, ϕ' is $(\psi_i \to a) \to a$ or ψ_i. If ϕ' is ψ_i, we have immediately $\psi_1, \ldots, \psi_n \vdash_I \phi'$. If ϕ' is $(\psi_i \to a) \to a$, since by Deduction Theorem, $\psi_i \vdash_I (\psi_i \to a) \to a)$, we have $\psi_1, \ldots, \psi_n \vdash_I (\psi_i \to a) \to a)$, that is $\psi_1, \ldots, \psi_n \vdash_I \phi'$.

Now suppose there are occurrences of $\to$ in ϕ. Then ϕ is $\phi_1 \to \phi_2$ for some ϕ_1, ϕ_2 and so by induction hypothesis we have $\psi_1, \ldots, \psi_n \vdash_I \phi_1'$ and $\psi_1, \ldots, \psi_n \vdash_I \phi_2'$ where ϕ_1' is $(\phi_1 \to a) \to a$ or $\phi_1 \to a$ according as the value of ϕ_1 for the values $s_1, \ldots, s_n$ of $a_1, \ldots, a_n$ is true or false, and ϕ_2' is $(\phi_2 \to a) \to a$ or $\phi_2 \to a$ according as the value of ϕ_2 for the values $s_1, \ldots, s_n$ of $a_1, \ldots,$

a_n is true or false. In case ϕ_2' is $(\phi_2 \rightarrow a) \rightarrow a$ we have that ψ_1, ..., $\psi_n \vdash_I (\phi_2 \rightarrow a) \rightarrow a$ and ϕ' is $((\phi_1 \rightarrow \phi_2) \rightarrow a) \rightarrow a$. From Note 3.7 (ii) we have $\vdash_I ((\phi_2 \rightarrow a) \rightarrow a) \rightarrow (((\phi_1 \rightarrow \phi_2) \rightarrow a) \rightarrow a)$ and so ψ_1, ..., $\psi_n \vdash_I ((\phi_1 \rightarrow \phi_2) \rightarrow a) \rightarrow a$, that is ψ_1, ..., $\psi_n \vdash_I \phi'$. Next in case ϕ_1' is $\phi_1 \rightarrow a$. Then ψ_1, ..., $\psi_n \vdash_I \phi_1 \rightarrow a$ and ϕ' is $((\phi_1 \rightarrow \phi_2) \rightarrow a) \rightarrow a$. By Lemma 3.7 (iv), $\vdash_I (\phi_1 \rightarrow a) \rightarrow (((\phi_1 \rightarrow \phi_2) \rightarrow a) \rightarrow a)$ and so ψ_1, ..., $\psi_n \vdash_I ((\phi_1 \rightarrow \phi_2) \rightarrow a) \rightarrow a$, that is ψ_1, ..., $\psi_n \vdash_I \phi'$. Now there remains only the case ϕ_1' is $(\phi_1 \rightarrow a) \rightarrow a$ and ϕ_2' is $\phi_2 \rightarrow a$. Then ψ_1, ..., $\psi_n \vdash_I \phi_2 \rightarrow a$ and ψ_1, ..., $\psi_n \vdash_I (\phi_1 \rightarrow a) \rightarrow a$ and ϕ' is $(\phi_1 \rightarrow \phi_2) \rightarrow a$. From Lemma 3.7 (iii), $\vdash_I ((\phi_1 \rightarrow a) \rightarrow a) \rightarrow ((\phi_2 \rightarrow a) \rightarrow ((\phi_1 \rightarrow \phi_2) \rightarrow a)$ and so ψ_1, ..., $\psi_n \vdash_I ((\phi_1 \rightarrow \phi_2) \rightarrow a)$, that is ψ_1, ..., $\psi_n \vdash_I \phi'$.

3.9 <u>Theorem.</u> If ϕ is a tautology, then $\vdash_I \phi$.

<u>Proof</u>. Assume that ϕ is a tautology. Let a_1, ..., a_n be sentence variables of ϕ. Let s_1, ..., s_n be truth-values of a_1, ..., a_n and a be a variable not occurring among a_1, ..., a_n; ψ_i be a_i or $a_i \rightarrow a$ according as s_i is true or false ; ϕ' be $(\phi \rightarrow a) \rightarrow a$ or $\phi \rightarrow a$ according as the value of ϕ for values s_1, ..., s_n of a_1, ..., a_n is true or false. Then from Lemma 3.8, we have ψ_1, ..., $\psi_n \vdash_I \phi'$ and this holds for either choice of s_n. Hence ψ_1, ..., ψ_{n-1}, $a_n \rightarrow a \vdash_I \phi'$ and ψ_1, ..., ψ_{n-1}, $a_n \vdash_I \phi'$. By Deduction Theorem, ψ_1, ..., $\psi_{n-1} \vdash_I (a_n \rightarrow a) \rightarrow \phi'$ and ψ_1, ..., $\psi_{n-1} \vdash_I a_n \rightarrow \phi'$. From Note 3.7 (v), we have $\vdash_I ((a_n \rightarrow a) \rightarrow \phi') \rightarrow ((a_n \rightarrow \phi') \rightarrow \phi')$ and then ψ_1, ..., $\psi_{n-1} \vdash_I \phi'$. Repeating this n-1 times we get $\vdash_I \phi'$.

Since ϕ is a tautology, ϕ' must be $(\phi \longrightarrow a) \longrightarrow a$ and therefore $\vdash_I (\phi \rightarrow a) \rightarrow a$. By Substitution, $\vdash_I (\phi \longrightarrow \phi) \longrightarrow \phi$. Since $\vdash_I \phi \longrightarrow \phi$, $\vdash_I \phi$.

3.10 <u>Theorem</u>. I is complete.

<u>Proof</u>. See [3], p. 52.

3.11 <u>Theorem</u>. If X is a consistent set of sentences of I which contains the three sentences : $p \longrightarrow (q \longrightarrow p)$, $p \longrightarrow ((p \longrightarrow q) \longrightarrow q)$, $(q \longrightarrow r) \longrightarrow ((p \longrightarrow q) \longrightarrow (p \longrightarrow r))$ as elements, then I is the only consistent and complete theory which includes the set X.

<u>Proof</u>. See [3], pp. 397-399.