CHAPTER IV

KERNELS OF COMPLETE THEORIES

In chapter I we stated Tarski's theorem that under certain assumptions, a consistent set can be extended to only one complete and consistent extension. The following example shows that there is a consistent set of sentences which can be extended to more than one complete and consistent extensions.

4.1 Example. Let $X_0 = \{p \rightarrow (q \rightarrow p),\ (s \rightarrow (p \rightarrow q)) \rightarrow ((s \rightarrow p) \rightarrow (s \rightarrow q)),\ \sim p \rightarrow q\}$. By the following truth-tables

$\rightarrow$	0	1	2
0	0	1	2
1	0	0	0
2	0	0	0

	$\sim$
0	2
1	2
2	2

where 0 is the only designated value, we have $\sim p$ is not a theorem of X_0. Therefore X_0 is consistent. By Lindenbaum's Theorem (Theorem 1.10) X_0 can be extended to a complete and consistent theory, say E. Hence E is also a complete and consistent extension of $X_0 - \{\sim p \rightarrow q\}$. Since every sentence in $X_0 - \{\sim p \rightarrow q\}$ is a tautology (as in Definition 2.9), SC is a complete and consistent extension of $X_0 - \{\sim p \rightarrow q\}$. Since $\sim p \rightarrow q$ is not a tautology, $\sim p \rightarrow q$ is not a theorem of SC. But $\sim p \rightarrow q$ is in E. Thus E and SC are distinct complete and consistent extensions of $X_0 - \{\sim p \rightarrow q\}$.

In this chapter we study the structure of the set of subtheories of a theory which has this theory as their only one complete and consistent extension. We partially order this set by set-inclusion and show that it is an upper semilattice but is not necessarily a totally ordered set. We also discuss questions of maximality and degrees of completeness.

4.1 Definition. A theory B is a kernel of a complete and consistent theory T if and only if

(i) $B \subseteq T$,

(ii) T is the only complete and consistent extension of B.

4.2 Definition. A theory B is a core of a complete and consistent theory T if and only if

(i) B is a kernel of T,

(ii) for any kernel C of T, $B \subseteq C$.

4.3 Examples of kernels.

(i) From Theorem 2.14, we have that $Cn(X_0)$ where $X_0 = \{p \to {\sim}{\sim}p,\ q \to (p \to q),\ {\sim}p \to (p \to q),\ p \to ({\sim}q \to {\sim}(p \to q)),$ $p \to p \vee q,\ q \to p \vee q,\ {\sim}p \to ({\sim}q \to {\sim}(p \vee q)),\ p \to (q \to p \wedge q),$ ${\sim}p \to {\sim}(p \wedge q),\ {\sim}q \to {\sim}(p \wedge q)\}$ is a kernel of the complete and consistent theory SC.

(ii) From Theorem 3.11, we have $Cn(X_0)$ where

$X_0 = \{p \to (q \to p),\ p \to ((p \to q) \to q),\ (q \to r) \to ((p \to q) \to (p \to r))\}$

is a kernel of the complete and consistent theory I.

4.4 Definition. Let T be a complete and consistent theory. Define $Ker(T) = \{B \mid B \text{ is a kernel of } T\}$.

Then Ker(T) is not empty, because $T \in Ker(T)$.

4.5 Proposition. Let $B \in Ker(T)$, C be a consistent theory such that $B \subseteq C$. Then $C \in Ker(T)$.

Proof. From Lindenbaum Theorem, C has a complete and consistent extension. Let T' be any complete and consistent extension of C. Since $B \subseteq C$, T' is also a complete and consistent extension of B, and since further $B \in Ker(T)$, we have $T' = T$. Therefore T is the only complete and consistent extension of C and hence $C \in Ker(T)$.

It is clear that Ker(T) is partially ordered by set inclusion. Thus $< Ker(T), \subseteq >$ is a partially ordered set.

4.6 Theorem. $< Ker(T), \subseteq >$ is an upper semilattice.

Proof. Let B, $C \in Ker(T)$. Since $B \cup C \subseteq T$ and T is a consistent theory, $B \cup C$ is also consistent, and so is $Cn(B \cup C)$. By Proposition 4.5, $Cn(B \cup C) \in Ker(T)$, since it is clear that $B \subseteq Cn(B \cup C)$ and $C \subseteq Cn(B \cup C)$. Let $D \in Ker(T)$ such that $B \subseteq D$ and $C \subseteq D$. Thus $Cn(B \cup C) \subseteq Cn(D) = D$. Therefore $Cn(B \cup C)$ is the least upper bound of B and C. Thus $< Ker(T), \subseteq >$ is an upper semilattice.

4.7 Theorem. Let T be a complete and consistent theory. If T has a core, then $< Ker(T), \subseteq >$ is a lattice.

Proof. Assume T has a core, say C. From Theorem 4.6 we have $< Ker(T), \subseteq >$ is an upper semilattice. We need only to show it is a lower-semilattice as well. Let B, D ε Ker(T). Then $C \subseteq B$ and $C \subseteq D$ and so $C = Cn(C) \subseteq Cn(B \cap D)$. By Proposition 4.5, $Cn(B \cap D) \varepsilon Ker(T)$ and we have $Cn(B \cap D) \subseteq B$ and $Cn(B \cap D) \subseteq D$. Next, let $K \varepsilon Ker(T)$ such that $K \subseteq B$ and $K \subseteq D$. Hence $K \subseteq B \cap D$ and so $K = Cn(K) \subseteq Cn(B \cap D)$. Therefore $< Ker(T), \subseteq >$ is a lower semilattice. Consequently $< Ker(T), \subseteq >$ is a lattice.

4.8 Theorem. $< Ker(T), \subseteq >$ is not necessarily a totally ordered set.

Proof. Let X_0 be as in example 4.3 (i). We see that $Cn(X_0) \varepsilon Ker(SC)$. Let $X = X_0 \cup \{p \vee \sim p\}$ and $Y = X_0 \cup \{(s \rightarrow (p \rightarrow q)) \rightarrow ((s \rightarrow p) \rightarrow (s \rightarrow q))\}$. Since $Cn(X) \subseteq SC$ and $Cn(Y) \subseteq SC$ and SC is consistent, Cn(X), Cn(Y) are also consistent. Then by Theorem 4.5, we have Cn(X), Cn(Y) are in Ker(T). Claim that $Cn(X) \nsubseteq Cn(Y)$ and $Cn(Y) \nsubseteq Cn(X)$.

We have $p \vee \sim p \notin Cn(Y)$ by the following truth-tables of connectives in SC :

→	0	1	2	3	4
0	0	1	2	4	4
1	0	1	2	4	4
2	0	1	0	0	0
3	0	1	0	0	0
4	0	1	0	0	0

∧	0	1	2	3	4
0	1	1	4	4	4
1	1	1	4	4	4
2	4	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4

∨	0	1	2	3	4
0	1	1	1	1	1
1	1	1	1	1	1
2	1	1	4	4	4
3	1	1	4	4	4
4	1	1	4	4	4

	~
0	4
1	2
2	1
3	3
4	0

where 0, 1 are designated values.

We have $(s \longrightarrow (p \longrightarrow q)) \longrightarrow ((s \longrightarrow p) \longrightarrow (s \longrightarrow q)) \notin Cn(X)$ by the following truth-tables of connectives of SC :

→	0	1	2	3	4
0	0	1	2	4	4
1	0	1	2	4	4
2	0	1	0	0	1
3	0	1	2	1	0
4	0	1	0	0	1

∧	0	1	2	3	4
0	1	1	4	4	4
1	1	1	4	4	4
2	4	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4

∨	0	1	2	3	4
0	1	1	1	1	1
1	1	1	1	1	1
2	1	1	4	4	4
3	1	1	1	4	4
4	1	1	4	4	4

	~
0	4
1	2
2	1
3	2
4	0

where 0, 1 are designated values.

Then $Cn(X) \nsubseteq Cn(Y)$ and $Cn(Y) \nsubseteq Cn(X)$, i.e. $< Ker(SC), \subseteq >$ is not a totally ordered set.

4.9 Definition. A theory X is axiomatizable if and only if there exist sentences $\psi_1, \ldots, \psi_n$ in X such that $Cn(\{\psi_1, \ldots, \psi_n\}) = X$.

4.10 Definition. Let T be a complete and consistent theory. A kernel K of T is a maximal kernel if and only if $K \subset T$ and for any kernel K' of T such that $K \subseteq K'$, we have $K' = K$ or $K' = T$.

4.11 Theorem. Let T be a complete and consistent theory such that every subtheory of T is axiomatizable, and let K_1 be a kernel of T. If there is a kernel K_0 properly contained in K_1, then there is a kernel K properly contained in K_1 such that $K_0 \subseteq K$ and for all $K' \varepsilon Ker(T)$, $K \subseteq K' \subseteq K_1$ implies $K' = K$ or $K' = K_1$.

Proof. Assume that there is a kernel K_0 properly contained in K_1. Let $\mathcal{B} = \{K_\alpha \varepsilon Ker(T) | K_0 \subseteq K_\alpha \text{ and } K_\alpha \subset K_1\}$. $\mathcal{B} \neq \phi$, since $K_0 \varepsilon \mathcal{B}$. Partially order $\mathcal{B}$ by set inclusion. To show that $\mathcal{B}$ has a maximal element, let $\{K_i\}_{i\varepsilon\lambda}$ be a chain in $\mathcal{B}$. Since $Cn(\bigcup_{i\varepsilon\lambda} K_i) \subseteq T$ and T is consistent, $Cn(\bigcup_{i\varepsilon\lambda} K_i)$ is consistent. Then by Proposition 4.5, $Cn(\bigcup_{i\varepsilon\lambda} K_i) \varepsilon Ker(T)$. Since for each $i\varepsilon\lambda$, $K_0 \subseteq K_i$, $K_0 \subseteq Cn(\bigcup_{i\varepsilon\lambda} K_i)$.

Next claim that $Cn(\bigcup_{i\varepsilon\lambda} K_i) \subset K_1$. Suppose $Cn(\bigcup_{i\varepsilon\lambda} K_i) = K_1$. Since by assumption, K_1 is axiomatizable, there exist sentences ψ_1, ..., ψ_n in K_1 such that $Cn(\{\psi_1, \ldots, \psi_n\}) = K_1$. Thus for each $i \varepsilon \{1,2,\ldots,n\}$, there is a finite sequence of sentences in $\bigcup_{i\varepsilon\lambda} K_i$ which is a proof of ψ_i. Consequently there are n finite sequences of sentences in $\bigcup_{i\varepsilon\lambda} K_i$ which are proofs of $\psi_1, \ldots, \psi_n$. Since $\{K_i\}_{i\varepsilon\lambda}$ is a chain, there is a $j\varepsilon\lambda$ such that $\psi_1, \ldots, \psi_n$ can be proved from K_j, and so $K_1 = K_j$ which is a contradiction. Therefore $Cn(\bigcup_{i\varepsilon\lambda} K_i) \subset K_1$.

Therefore $Cn(\bigcup_{i\varepsilon\lambda} K_i) \varepsilon \mathcal{B}$ and is an upper bound of the chain $\{K_i\}i\varepsilon\lambda$. By Zorn's Lemma, $\mathcal{B}$ has a maximal element, say K. Then $K \subset K_1$. Next let $K' \varepsilon Ker(T)$ such that $K \subseteq K' \subseteq K_1$. If $K' \neq K_1$, then $K' \varepsilon \mathcal{B}$ and so by maximality of K we have $K' = K$.

4.12 <u>Corrollary</u>. Every proper kernel of a consistent, complete and axiomatizable theory T is contained in a maximal kernel of T.

<u>Proof</u>. Directly from Theorem 4.11 by setting T to be K_1.

4.13 <u>Definition</u>. Let $< A, \leq >$ be a partially ordered set. A chain $\{K_\alpha\}_{\alpha\varepsilon\lambda}$ of $< A, \leq >$ is a <u>maximal chain</u> if and only if for any $K \varepsilon A$, Such that for each $\alpha\varepsilon\lambda$ $K \neq K_\alpha$, there exists $\beta\varepsilon\lambda$ such that $K \not\leq K_\beta$ and $K_\beta \not\leq K$.

4.14 Definition. Let $\{K_\alpha\}_{\alpha\varepsilon\lambda}$ be a chain of a partially ordered set $< A, \leq >$. It is a discrete chain if and only if for all K_α, K_β such that $K_\alpha \leq K_\beta$ there exists K_γ such that $K_\alpha \leq K_\gamma \leq K_\beta$ and if $K_\gamma \leq K_\theta \leq K_\beta$, then $K_\theta = K_\gamma$ or $K_\theta = K_\beta$.

4.15 Theorem. Let T be a complete and consistent theory such that every subtheory of T is axiomatizable. Then $< Ker(T), \subseteq >$ contains a discrete maximal chain.

Proof. We will define K_α for each ordinal number α.

(i) $K_0 = T$

(ii) if α is a limit ordinal, let

$$K_\alpha = \begin{cases} Cn(\bigcap_{\beta<\alpha} K_\beta), & \text{if } Cn(\bigcap_{\beta<\alpha} K_\beta) \varepsilon Ker(T) \\ T, & \text{otherwise,} \end{cases}$$

(iii) suppose K_α is defined, now we define $K_{\alpha+1}$ as follows

- there is no kernel K_1 such that $K_1 \subset K_\alpha$, let $K_{\alpha+1} = K_\alpha$
- there is a kernel K_1 such that $K_1 \subset K_\alpha$
  - if $K_\alpha = K_\beta$ for some $\beta<\alpha$, let β_0 be the smallest such β, let $K_{\alpha+1} = K_{\beta_0+1}$
  - if not, get $K_{\alpha+1}$ from Theorem 4.11 such that $K_{\alpha+1} \subset K_\alpha$ and for any kernel K^* if $K_{\alpha+1} \subseteq K^* \subseteq K_\alpha$, then $K^* = K_{\alpha+1}$ or $K^* = K_\alpha$.

Then $\{K_\alpha\}_{\alpha\varepsilon Ord}$ is a discrete chain in Ker(T). We claim that it is a maximal chain. Let $K^* \varepsilon Ker(T)$ such that $K^* \neq K_\alpha$ for all $\alpha\varepsilon Ord$. Suppose for all $\beta\varepsilon Ord$ $K^*\subseteq K_\beta$ or $K_\beta\subseteq K^*$. Since the number of elements of Ker(T) is less than or equal $2^{\aleph_0}$, there exists an ordinal which is a cardinal γ greater than $2^{\aleph_0}$ such that $K_\gamma = K_{\gamma+1}$. Then it is impossible that $K^*\subset K_\beta$ for all $\beta\varepsilon Ord$. Hence there exists a $\beta\varepsilon Ord$ such that $K_\beta\subseteq K^*$. Let α be the smallest such β. If α is not a limit ordinal, then $K_\alpha\subseteq K^*\subseteq K_{\alpha-1}$ and so $K^* = K_\alpha$ or $K^* = K_{\alpha-1}$ which is a contradiction. For another case, α is a limit ordinal. Then $K_\alpha\subseteq K^*$. Since for $\beta < \alpha$ $K^*\subseteq K_\beta$, we have $K^*\subseteq Cn(\bigcap_{\beta<\alpha} K_\beta)$ and this implies $K^*\subseteq Cn(\bigcap_{\beta<\alpha} K_\beta) = K_\alpha$. Hence $K^* = K_\alpha$ which is a contradiction. Then there is a $\beta\,\varepsilon\,Ord$ such that $K^*\nsubseteq K_\beta$ and $K_\beta\nsubseteq K^*$. Hence $\{K_\alpha\}_{\alpha\varepsilon Ord}$ is a maximal chain.

4.16 Definition. Let X be a set of sentences. The degree of completeness of X, deg(X), is the largest number $n \geq 0$ with the following property :

there exists a sequence of sets X_i satisfying the formulas : $X_0 = X$, $X_i\subseteq X_j\subseteq S$ and $Cn(X_i) \neq Cn(X_j)$ for $j < i < n$

If there is no such natural number we say that $deg(X) = \infty$.

4.17 Theorem. Let T be a complete and consistent theory such that every subtheory of T is axiomatizable. Then $< Ker(T), \subseteq >$ has a finite maximal chain (and then has a minimal element) or for any $\gamma\varepsilon\omega$, there exists a kernel K of T such that $deg(K) \geq \gamma$.

Proof. From the proof of Theorem 4.15, we have $\{K_\alpha\}_{\alpha\varepsilon Ord}$

is a maximal chain. There are two cases to consider :

(i) There is an $\alpha<\omega$ such that $K_\alpha = K_{\alpha+1}$, let α_0 be the smallest such α. Then $\{K_\alpha\}_{\alpha=0}^{\alpha_0}$ is a finite maximal chain (K_{α_0} is a minimal element).

(ii) There is no $\alpha<\omega$ such that $K_\alpha = K_{\alpha+1}$. Then for any $\gamma\varepsilon\omega$, we have $\deg(K_\gamma) \geq \gamma$.

APPENDIX A

BOOLEAN ALGEBRA

The material in this appendix is from [1].

A1. Definition. Let X be a non-empty set, and $\leq$ be a binary relation on X

1. For any $x_1 \varepsilon X$, $x_1 \leq x_1$.
2. For any x_1, $x_2 \varepsilon X$, if $x_1 \leq x_2$, and $x_2 \leq x_1$, then $x_1 = x_2$.
3. For any x_1, x_2, $x_3 \varepsilon X$, if $x_1 \leq x_2$ and $x_2 \leq x_3$, then $x_1 \leq x_3$.

   If $< X, \leq >$ satisfies 1-3, then it is a partially ordered set.

4. For any x_1, $x_2 \varepsilon X$, there exists $x_3 \varepsilon X$ such that $x_3 \leq x_1$ and $x_3 \leq x_2$, and for any $x_4 \varepsilon X$, if $x_4 \leq x_1$ and $x_4 \leq x_2$, then $x_4 \leq x_3$. We call x_3, the greatest lower bound of x_1 and x_2 and denote it by $x_1 \wedge x_2$.
5. For any x_1, $x_2 \varepsilon X$, there exists $x_3 \varepsilon X$ such that $x_1 \leq x_3$ and $x_2 \leq x_3$ and for any $x_4 \varepsilon X$, if $x_1 \leq x_4$ and $x_2 \leq x_4$, then $x_3 \leq x_4$. We call x_3, the least upper bound of x_1 and x_2 and denote it by $x_1 \vee x_2$.

   If $< X, \leq >$ satisfies 1-4, then it is a lower semi-lattice.

   If $< X, \leq >$ satisfies 1-3, 5, then it is an upper semi-lattice.

   If $< X, \leq >$ satisfies 1-5, then it is a lattice.

6. There exist x_1, $x_2 \varepsilon X$ such that for any $x_3 \varepsilon X$, $x_3 \leq x_1$ and $x_2 \leq x_3$, and there exists $x_4 \varepsilon X$ such that $x_3 \wedge x_4 = x_2$ and $x_3 \vee x_4 = x_1$. We denote x_1 by 1, x_2 by 0 and x_4 by x_3^*.

If $< X, \leq >$ satisfies 1-6, then it is a complemented lattice.

7. For any x_1, x_2, $x_3 \varepsilon X$, $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$.

8. For any x_1, x_2, $x_3 \varepsilon X$, $x_1 \vee (x_2 \wedge x_3) = (x_1 \vee x_2) \wedge (x_1 \vee x_3)$.

If $< X, \leq >$ satisfies 1-5, 7, 8, then it is a distributive lattice.

A2. Definition. $< B, \leq >$ is a Boolean Algebra if and only if it is a distributive complemented lattice.

A3. Remark. $<\{0,1\}, \{(0,0), (0,1), (1,1)\}>$ is a Boolean Algebra, denote this Boolean Algebra by $\mathscr{B}^2$.

A4. Definition. Let L be a lattice and F be a non-empty subset of L. F is a filter if and only if

(i) if x, $y \varepsilon F$, then $x \wedge y \varepsilon F$

(ii) if $x \varepsilon F$, $y \varepsilon L$ and $x \leq y$, then $y \varepsilon F$.

If $F \subset L$, then F is a proper filter.

A5. Definition. Let $< X, \leq >$ be a partially ordered set and $A \subseteq X$. $b \varepsilon X$ is called the infimum of A if and only if for all $x \varepsilon A$, $b \leq x$, and if $\exists c \varepsilon X$ such that for all $x \varepsilon A$, $c \leq x$, then $c \leq b$.

A6. Definition. A filter F of a Boolean Algebra $< B, \leq >$ is an ultrafilter if and only if no proper extension of F is a proper filter.

A7. Theorem. Each non-zero element of a Boolean Algebra is contained in some ultrafilter.

Proof. See [1] p. 16.

A8. Definition. Let B_1, B_2 be Boolean Algebras. A map $f : B_1 \longrightarrow B_2$ is a homomorphism if and only if for any $x, y \in B_1$,

(i) $f(x \wedge y) = f(x) \wedge f(y)$

(ii) $f(x \vee y) = f(x) \vee f(y)$

(iii) $f(x^*) = f(x)^*$.

If f is 1-1, onto, we say that f is an isomorphism between B_1 and B_2 and write $B_1 \cong B_2$.

Let F be a filter in a Boolean Algebra B. We define a relation $\sim_F$ on B as follows : for any $x, y \in B$

$x \sim_F y$ if and only if for some $f \in F$, $x \wedge f = y \wedge f$.

Then $\sim_F$ is an equivalence relation.

Let $|x| = \{y \in B \mid y \sim_F x\}$ and $B/F = \{|x| \mid x \in B\}$.

Define relation $\leq_1$ on B/F as follows : for any $|x|, |y| \in B/F$

$|x| \leq_1 |y|$ if and only if $x \leq y$.

Then $< B/F, \leq >$ is a Boolean Algebra in which

$|x \wedge y| = |x| \wedge |y|$ and $|x \vee y| = |x| \vee |y|$ and $|x^*| = |x|^*$.

We call this Boolean Algebra the quotient algebra of B modulo F.

The map $h : B \longrightarrow B/F$ which send x onto $|x|$ is clearly a homomorphism. It is called the canonical homomorphism of B onto B/F.

A9. Lemma. $|x| = 1$ in B/F if and only if $x \in F$.

A10. Proposition. If U is an ultrafilter of a Boolean Algebra $\langle B, \leq \rangle$, then $B/U \cong \mathcal{B}^2$.

Proof. See [1] p. 20.

APPENDIX B

INDEPENDENCE PROOFS

Let X be a set of sentences. A sentence ψ is said to be independent of X if and only if ψ is not a theorem of X.

In the sentential logic a standard device for establishing the independence of a sentence ψ from a set X of sentences in the same sentential logic is given as follows :

Let 0, 1, ..., ν be a system of truth-values and the first μ of these, 0, 1, ...,μ (where $1 \leq \mu < \nu$) being called designated truth-values.

To each connective is assigned a generalized truth-table in these truth-values. The value of a sentence is defined by the given values of its variables, the possible values of the variables being values 0, 1, ..., ν.

If every rule of inference has the property of preserving designated truth-values (i.e. the conclusion must be a designated value when the premisses are designated values) and every sentence in the set X has a designated value for any assignment of truth-values of its sentence variables, then if for some assignment of truth-values ψ is undesignated, it follows that ψ is independent of X.

<u>Example</u>. Let $X = \{p \rightarrow (q \rightarrow p)\}$. Consider the following diagram

$\rightarrow$	0	1	2
0	0	2	2
1	0	2	0
2	0	0	0

where 0 is the only designated value.

We see that MP. preserves designated value (i.e. if A and $A \rightarrow B$ have value 0, then B is 0) and $p \rightarrow (q \rightarrow p)$ has the value 0 for all assignments of truth-values of its variables, but $p \rightarrow p$ has an assignment of its variable such that $p \rightarrow p$ is not 0 by giving the value 1 to p.

Then $p \rightarrow p$ is not a theorem of X.