CHAPTER III

ON LEAST HARMONIC MAJORANTS IN HALF-SPACES

In this chapter, we will discuss on the existence of harmonic majorant of a subharmonic function s on $\mathbb{R}^+_{n+1}$.

3.1 The Poisson Kernel

Let n be a positive integer and $\mathbb{R}_{n+1}$ be the euclidean space of dimension n+1; an arbitrary point in $\mathbb{R}_{n+1}$ is represented by

$$M = (X,y) = (x_1, x_2, \ldots, x_n, y)$$

we put

$$X^2 = |X|^2 = x_1^2 + x_2^2 + \ldots + x_n^2, \quad dX = dx_1 dx_2 \ldots dx_n .$$

For each real number a, let D_a be the open half-space

$$D_a = \{M \varepsilon \mathbb{R}_{n+1} : y > a\}$$

where a = 0, D_a is simply replaced by $\mathbb{R}^+_{n+1}$.

In $\mathbb{R}^+_{n+1}$, the Poisson kernel P is given by

$$P(X,y) = \frac{y}{c_n(|X|^2+y^2)^{\frac{1}{2}(n+1)}} ,$$

where

$$c_n = \frac{\pi^{\frac{1}{2}(n+1)}}{\Gamma[\frac{1}{2}(n+1)]}$$

This function has following three basic properties

3.1.1 $P(X,y) \geqslant 0$.

3.1.2) $\int_{\mathbb{R}_n} P(X,y)dX = 1$ for all $y > 0$.

This property is obtained immediately as we can show that

$$\int_{\mathbb{R}_n} \frac{y}{(|X|^2+y^2)^{\frac{1}{2}(n+1)}} dx = c_n = \frac{\pi^{\frac{1}{2}(n+1)}}{\Gamma[\frac{1}{2}(n+1)]} .$$

And this equality can be shown as follows (see Helms [4] for all notation used).

Let $r^2 = x_1^2 + x_2^2 + \dots + x_n^2 = |X|^2$

$\Theta = (\Theta_1, \Theta_2, \dots, \Theta_n)$ and

$\Theta_i = \frac{x_i}{r}$, $i = 1,2,\dots,n$

then $|\Theta| = 1$

$$\int_{\mathbb{R}_n} \frac{y}{(|X|^2+y^2)^{\frac{1}{2}(n+1)}} dX = y \int_0^\infty \int_{\partial B(0,1)} \frac{1}{(r^2+y^2)^{\frac{1}{2}(n+1)}} \frac{r^{n-1}}{|\Theta_n|} d\Theta_1 d\Theta_2 \dots d\Theta_{n-1} dr$$

$$= y \int_0^\infty \frac{r^{n-1}}{(r^2+y^2)^{\frac{1}{2}(n+1)}} \int_{\partial B(0,1)} d\sigma(\Theta) dr$$

$$= \sigma_n \int_0^\infty \frac{r^{n-1}/y^{n-1}}{(r^2/y^2+1)^{\frac{1}{2}(n+1)}} dr/_y$$

$$= \sigma_n \int_0^\infty \frac{s^{n-1}}{(s^2+1)^{\frac{1}{2}(n+1)}} ds \; ; \quad s = r/_y .$$

Let $s^2 = x/(1-x)$, the last quantity equals to

$$\frac{\sigma_n}{2} \int_0^1 x^{(\frac{n}{2}-1)} (1-x)^{(\frac{1}{2}-1)} dx$$

$$= \frac{\sigma_n}{2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma[\frac{1}{2}(n+1)]} .$$

Since σ_n = surface area of n-dimensional unit sphere $= \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$,

we get $$\frac{\sigma_n}{2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma[\frac{1}{2}(n+1)]} = c_n .$$

This means $\int_{\mathbb{R}_n} P(X,y)dx = 1 , \quad y > 0 .$

3.1.3) If $r > 0$, then $\int_{|X| \geq r} P(X,y)dX \to 0$ as $y \to 0$.

To show this, it suffices to prove that

$$\lim_{y \to 0} \int_{|X| \geq r} \frac{y}{(|X|^2+y^2)^{\frac{1}{2}(n+1)}} dX = 0 .$$

Since $$\int_{|X| \geq r} \frac{y}{(|X|^2+y^2)^{\frac{1}{2}(n+1)}} dX = \int_{|X| \geq r} \frac{y}{y^{n+1}(\frac{|X|^2}{y^2}+1)^{\frac{1}{2}(n+1)}} dX$$

$$= \int_{|X| \geq r} \frac{1}{y^n(\frac{|X|^2}{y^2}+1)^{\frac{1}{2}(n+1)}} dX .$$

Let $T = (t_1, t_2, \ldots, t_n)$; $t_i = \dfrac{x_i}{y}$

$$\int_{|X|\geq r} \frac{y}{(|X|^2+y^2)^{\frac{1}{2}(n+1)}}\, dX = \int_{|T|\geq\frac{r}{y}} \frac{1}{(|T|^2+1)^{\frac{1}{2}(n+1)}}\, dT$$

$$= \int_{\mathbb{R}_n} \frac{1}{(|T|^2+1)^{\frac{1}{2}(n+1)}} \chi_{[\frac{r}{y},\infty)}(|T|)dT ,$$

where χ_A being the characteristic function on A.

Let m be the greatest integer less than or equal to $\frac{r}{y}$, then

$$\chi_{[\frac{r}{y},\infty)} \leq \chi_{[m,\infty)} .$$

Let $$F_m(T) = \frac{1}{(|T|^2+1)^{\frac{1}{2}(n+1)}} \chi_{[m,\infty)}(|T|) ,$$

thus, $\{F_m\}$ is a decreasing sequence of nonnegative measurable functions. And $F_1(T) \leq \frac{1}{(|T|^2+1)^{\frac{1}{2}(n+1)}}$ which is integrable on $\mathbb{R}_n$, i.e., $\{F_m\}$ is a decreasing sequence of nonnegative integrable functions, and $0 \leq \lim\limits_{m\to\infty} F_m \leq \lim\limits_{m\to\infty} \chi_{[m,\infty)} = 0.$

By Monotone Convergence Theorem, we get

$$0 \leq \lim_{y\to 0} \int_{|X|\geq r} \frac{y}{(|X|^2+y^2)^{\frac{1}{2}(n+1)}}\, dX \leq \lim_{m\to\infty} \int_{\mathbb{R}_n} \frac{1}{(|T|^2+1)^{\frac{1}{2}(n+1)}} \chi_{[m,\infty)}(|T|)dT$$

$$= \int_{\mathbb{R}_n} \frac{1}{(|T|^2+1)^{\frac{1}{2}(n+1)}} \lim_{m\to\infty} \chi_{[m,\infty)}(|T|)dT$$

$$= 0 .$$

Thus $$\lim_{y\to 0} \int_{|X|\geq r} P(X,y)dX = 0 .$$

3.2 The existence of the least harmonic majorant.

Let f be a function defined at least on the hyperplane of equation y = a, such that the function

$$X \longmapsto (X^2+1)^{-\frac{1}{2}(n+1)} f(X,a)$$

is Lebesque integrable in $\mathbb{R}^n$. The Poisson integral I_f^a in D_a of the restriction of f to the hyperplane is given by

$$I_f^a (X,y) = \int_{\mathbb{R}_n} P(X-Z, y-a)f(Z,a)dZ$$

for all (X,y) belonging to D_a. We note that I_f^a is harmonic in D_a (Nualtaranee [6]). The following theorem is proved by Nualtaranee [6]. It guarantees that the least harmonic majorant of s exists if $K(|s|, \bullet)$ is bounded.

We then construct the classes S^p of subharmonic functions in half-spaces. For $p \geqslant 1$, the subharmonic function $s : \mathbb{R}_{n+1}^+ \to \tilde{\mathbb{R}}$ is said to belong to S^p if the function

$$K(|s|^p,y) = \int_{\mathbb{R}_n} \{(X^2+(1+y)^2\}^{-\frac{1}{2}(n+1)} |s(X,y)|^p \, dX$$

is bounded in the open inteval $(0,+\infty)$.

In fact, by Hölder's inequality and the fact that

$$\int_{\mathbb{R}_n} \{X^2+ (1+y)^2\}^{-\frac{1}{2}(n+1)} dX = \frac{1}{c_n}(y+1)$$

we have, for any $p > q \geqslant 1$, that

$$K(|s|^q,y) \leqslant \{K(|s|^p,y)\}^{q/p} \{\frac{1}{c_n(y+1)}\}^{1-\frac{q}{p}} .$$

This means that if $p > q \geq 1$, then

$$S^p \subseteq S^q \subseteq S^1 .$$

Theorem 3.2.1 Suppose that $s \in S^1$ and $c \geq 0$ such that

$$K(|s|, y) \leq c \qquad (y > 0).$$

Then the least harmonic majorant h_s of s in $\mathbb{R}^+_{n+1}$ exists and is given by

$$h_s(X,y) = ky + \int_{\mathbb{R}_n} P(X-Z,y)d\nu(Z) \qquad ((X,y) \in \mathbb{R}^+_{n+1})$$

where $k = \frac{1}{c_n} \lim_{y \to +\infty} K(s,y)$, and

ν is a signed Radon measure in $\mathbb{R}_n$ such that

$$\int_{\mathbb{R}_n} (Z^2+1)^{-\frac{1}{2}(n+1)} d|\nu| (Z) \leq c .$$

If further $s \geq 0$ ($s \leq 0$) in $\mathbb{R}^+_{n+1}$ then ν is a positive (respectively negative) Radon measure in $\mathbb{R}_n$.

The following theorem will guarantee the existence of a least harmonic majorant of s if $K(|s|^p, \circ)$ is bounded. The proof of the theorem will be given in §.3.3.

Theorem 3.2.2 Suppose that $s \in S^p$ where $p > 1$, $a > 0$ and

$$K(|s|^p, y) \leq c \qquad (0 < y < a).$$

Then the least harmonic majorant h_s of s in $\mathbb{R}^+_{n+1}$ exists and is given by

$$h_s(X,y) = \int_{\mathbb{R}_n} P(X-Z,y)f(Z)dZ$$

$$= I_f(X,y), \quad \text{say} \quad ((X,y)\varepsilon \mathbb{R}^+_{n+1}),$$

where f is a measurable function in $\mathbb{R}_n$ such that

$$\int_{\mathbb{R}_n} (Z^2+1)^{-\frac{1}{2}(n+1)} |f(Z)|^p \, dZ \leqslant c$$

and almost everywhere in $\mathbb{R}_n$

$$f(X) = \lim_{y\to 0^+} h_s(X,y) \qquad (X \varepsilon \mathbb{R}_n).$$

If further $s \geqslant 0$ ($s \leqslant 0$) in a neighbourhood V in $\mathbb{R}^+_{n+1}$ then f can be taken to be non-negative (resp. non-positive) in $\mathbb{R}_n$.

Also Stien and Weiss [1] proved a similar following theorem.

<u>Theorem 3.2.3</u> Let $s(X,y) \geqslant 0$ be a subharmonic function defined in the region $\mathbb{R}^+_{n+1}$ satisfying

$$\int_{\mathbb{R}_n} [s(X,y)]^p \, dX \leqslant c^p < \infty$$

where $1 \leqslant p < \infty$ and c is independent of $y > 0$. Then s has a harmonic majorant h_s in $\mathbb{R}^+_{n+1}$ and

a) if $p > 1$, $h_s(X,y) = \int_{\mathbb{R}_n} P(X-Z,y)f(Z)dZ \qquad ((X,y)\varepsilon \mathbb{R}^+_{n+1})$

where $f \varepsilon L^p(\mathbb{R}_n)$ such that $f(X) = \lim_{y\to 0} h_s(X,y)$ a.e.

and $\|f\|_p \leqslant c$

b) if $p = 1$, $h_s(X,y) = \int_{\mathbb{R}_n} P(X-Z,y)d\mu(Z) \qquad ((X,y) \in \mathbb{R}^+_{n+1})$

where μ is a signed Radon measure such that $\int_{\mathbb{R}_n} d|\mu|(Z) \leq c$.

3.3 Proof of theorem 3.2.2.

Before we prove the theorem 3.2.2, we give the properties of some spaces in duality and some lemmas which will be needed to prove the theorem.

Let ν be a measure defined by

$$d\nu(X) = (X^2+1)^{-\frac{1}{2}} dX \qquad (x \in \mathbb{R}_n).$$

(Note that ν is a positive Radon measure in $\mathbb{R}_n$). For each p such that $1 \leq p < \infty$ we denote by $L^p(\nu)$ the space of measurable function f such that $|f|^p$ is ν-integrable in $\mathbb{R}_n$.

The space $L^p(\nu)$ is a Banach space, i.e., in the metric arising from the norm $\|.\|_p$ defined by

$$\|f\|_p = \left(\int_{\mathbb{R}_n} |f(X)|^p d\nu(X)\right)^{1/p} \qquad (f \in L^p(\nu)),$$

$L^p(\nu)$ is a complete metric space, where we agree that $f \equiv g$ means $f = g$ a.e. in $\mathbb{R}_n$.

Further the Banach space $L^p(\nu)$ and $L^q(\nu)$ are in duality when $p > 1$ and q is the Hölder's conjugate of p, i.e., $q = \frac{p}{(p-1)}$.

If $f \in L^p(\nu)$ and $g \in L^q(\nu)$ the bilinear form $\langle f,g \rangle$ is defined by

$$\langle f,g \rangle = \int_{\mathbb{R}_n} f(X)g(X)d\nu(X) .$$

It is well-known that for $1 < p < +\infty$, the space $L^p(\nu)$ is seperable.

<u>Lemma 3.3.1</u> Suppose that $1 < p < +\infty$, $0 < a < \infty$ and

$$A = \{f : f \in L^p(\nu), \|f\|_p \leq a\} ,$$

then every sequence $\{f_m\}$ in A has a subsequence $\{f_{m_i}\}$ converging weakly to a function f in A, i.e.,

$$\lim_{i\to\infty} < f_{m_i} , g > = <f,g> \qquad (g \in L^q(\nu))$$

where $q = p(p-1)^{-1}$.

<u>Proof</u> Let $V = \{g : g \in L^q(\nu), \|g\|_q \leq \frac{1}{a}\}$

Let $\{f_m\}$ be any sequence in A.

$$|< f_m,g >| \leq \int_{R_n} |f_m(X)||g(X)|d\nu(X)$$

$$\leq \|f_m\|_p \|g\|_q \leq a \cdot \frac{1}{a} = 1, \quad g \in V.$$

Hence, by Rudin [9], theorem 3.17, there is a subsequence $\{f_{m_i}\}$ of $\{f_m\}$ and there is a function f in A, such that

$$\lim_{i\to\infty} < f_{m_i}, g > = <f,g> \qquad (g \in L^q(\nu)).$$

<u>Theorem 3.3.2</u>. Let s be subharmonic in $\mathbb{R}^+_{n+1}$. Suppose that for each $y_0 > 0$, there exists a positive constant $c(s,y_0)$ such that

$$K(s^+,y) = \int_{R_n} \{X^2+(y+1)^2\}^{-\frac{1}{2}} s^+(X,y)\, dX \leq c(s,y_0)$$

for all $y \geq y_0$, then

i) $K(s,y)$ has a finite limit as $y \to +\infty$.

ii) for each $a > 0$, the least harmonic majorant $h_{s,a}$ of s in D_a exists and is given by

$$h_{s,a}(X,y) = k(y-a) + I_s^a(X,y) \qquad ((X,y) \in D_a)$$

where k is a constant indepent of a and given by

$$k = \frac{1}{c_n} \lim_{y\to+\infty} K(s,y).$$

If, further, $K(s^+, \circ)$ is bounded in $(0,+\infty)$, then the least harmonic majorant h_s of s in $\mathbb{R}^+_{n+1}$ exists and is given by

$$h_s(X,y) = ky + \lim_{a\to 0^+} I_s^a(X,y) \qquad ((X,y) \in \mathbb{R}^+_{n+1}).$$

(The proof of the theorem 3.3.2 is shown in Nualtaranee [6], theorem 2).

<u>Lemma 3.3.3</u>. Suppose that $1 \leq p < +\infty$ and $s \in S^p$.

Then

$$\lim_{a\to 0^+} I_s^a(X,y+a) = \lim_{a\to 0^+} I_s^a(X,y)$$

$$= H(X,y) \quad \text{say} ,$$

for all (X,y) belonging to $\mathbb{R}^+_{n+1}$.

To prove this, we first note that, by theorem 3.3.2 H exists and is harmonic in $\mathbb{R}^+_{n+1}$. So it is enough to show that for any null-sequence $\{a_i\}$, $i = 1,2,\ldots$ and for any

$$M_o = (X_o,y_o) \in \mathbb{R}^+_{n+1}$$

$$\lim_{i\to+\infty} I_s^{a_i}(X_o,y_o+a_i) = H(X_o,y_o) .$$

In fact, by the first part of theorem 3.3.2 and the fact that for $0 < a < b$, $h_s \geqslant h_{s,a} \geqslant h_{s,b}$ in D_b, we get

$$I_s^b(M) - |k|b \leqslant I_s^a(M) \leqslant H(M) + |k|b$$

for all M in D_b. (3.3.4)

Now, we put

$$B = \{M \in \mathbb{R}_{n+1} : M_oM < \frac{1}{3} y_o\}$$, where MM_o is

the usual distance from M_o to M. By using (3.3.4) with $0 < a_i < \frac{1}{3} y_o = b$, we get that the functions $I_s^{a_i}$ are uniformly bounded in B, hence $I_s^{a_i} \to H$ uniformly in B as $i \to +\infty$, i.e., for any $\varepsilon > 0$, there exists a positive number $\delta_1 \leqslant \frac{1}{3} y_o$ such that whenever $0 < a_i < \delta_1$,

$$|I_s^{a_i}(M) - H(M)| < \varepsilon/_2 \qquad (M \in B).$$

And hence, in particular,

$$|I_s^{a_i}(X_o,y_o+ a_i)- H(X_o,y_o+ a_i)| < \varepsilon/_2 \quad (0 < a_i< \delta_1).$$

On the other hand, by the continuity of H, there exists a $\delta_2 > 0$ such that

$$|H(M_o) - H(M)| < \varepsilon/_2 \qquad (MM_o < \delta_2)$$

and hence in particular

$$|H(X_o,y_o+ a_i)- H(X_o,y_o)| < \varepsilon/_2 \qquad (0 < a_i < \delta_2).$$

Collecting the results and putting $\delta = \min(\delta_1,\delta_2)$, we have

$$|I_s^{a_i}(X_o, y_o + a_i) - H(X_o, y_o)| < \varepsilon \qquad (0 < a_i < \delta)$$

this completes the proof of Lemma 3.3.3.

To prove theorem 3.2.2, we note that

$$\int_{\mathbb{R}_n} |s(X,y)|^p (X^2+1)^{-\frac{1}{2}(n+1)} dX \leq c(1+\varepsilon)^{n+1} \quad (0 < y \leq \varepsilon) \qquad (3.3.5)$$

which can be easily deduced from the inequality

$$\frac{X^2 + (y+1)^2}{X^2 + 1} \leq (1 + y)^2 .$$

For any $\varepsilon > 0$, we put

$$A_\varepsilon = \{f : f \in L^p(\nu), \ \|f\|_p \leq (1+\varepsilon)^{\frac{1}{p}(n+1)} c^{\frac{1}{p}}\}$$

and take a decreasing null-sequence $\{a_i\}$, $i = 1,2,\ldots$. By (3.3.5), we have that, from some i onwards ($a_i < \varepsilon$), the functions

$$X \longmapsto s(X, a_i)$$

belong to A_ε . Hence, by using Lemma 3.3.1, with $A = A_\varepsilon$, there exist $f \in A_\varepsilon$ and a subsequence $\{b_j\}$ $(j = 1,2,\ldots)$ of $\{a_i\}$ such that

$$\lim_{j\to+\infty} \int_{R_n} s(Z,b_j) g(Z) d\nu(Z) = \int_{R_n} f(Z) g(Z) d\nu(Z) \qquad (3.3.6)$$

for all $g \in L^q(\nu)$ whence in particular for

$$Z \longmapsto g(Z) = \frac{1}{c_n} y \left\{\frac{Z^2 + 1}{(X-Z)^2 + y^2}\right\}^{\frac{1}{2}(n+1)}$$

with $(X,y) \in \mathbb{R}^+_{n+1}$ (since, for each (X,y) fixed in $\mathbb{R}^+_{n+1}$, the function g is continuous and bounded in $\mathbb{R}_n$ and

$$\int_{\mathbb{R}_n} |g(Z)|^q d\nu(Z) \leq \sup_{Z\varepsilon\mathbb{R}_n} |g(Z)|^q \int_{\mathbb{R}_n} d\nu(Z) \quad < +\infty).$$

Hence

$$\lim_{j\to+\infty} I_s^{b_j}(X,g+b_j) = \frac{1}{c_n} y \int_{\mathbb{R}_n} f(Z)\left\{\frac{Z^2+1}{(X-Z)^2+y^2}\right\}^{\frac{1}{2}(n+1)} d\nu(Z)$$

$$= \frac{1}{c_n} y \int_{\mathbb{R}_n} f(Z)\{(X-Z)^2+y^2\}^{-\frac{1}{2}(n+1)} dZ$$

$$= \int_{\mathbb{R}_n} P(X-Z,y)f(Z)dZ \qquad ((X,y) \varepsilon \mathbb{R}^+_{n+1}) .$$

This together with Lemma 3.3.3 gives

$$I_f(X,y) = \int_{\mathbb{R}_n} P(X-Z,y)f(Z)dZ = \lim_{a\to 0^+} I_s^a(X,y) \qquad ((X,y) \varepsilon \mathbb{R}^+_{n+1})$$

and hence, by theorem 3.3.2, the fact that $S^p \subset S^1$ and $\lim_{y\to+\infty} K(|s|,y) = 0$ (see Nualtaranee [6], theorem 7), we obtain the first equality of theorem 3.2.2. To prove the inequality of theorem 3.2.2, we note that, since $f \varepsilon A_\varepsilon$

$$\int_{\mathbb{R}_n} |f(Z)|^p d\nu(Z) \leq (1+\varepsilon)^{n+1} c \qquad (\varepsilon > 0).$$

Hence

$$\int_{\mathbb{R}_n} |f(Z)|^p d\nu(Z) \leq c$$

and the result follows.

To prove the last equality of theorem 3.2.2, we note that

$$\int_{\mathbb{R}_n} (|X|^{n+1} + 1)^{-1} \, |f(X)| dX \quad < +\infty,$$

since, by using the inequality $(a+b)^m \leq 2^m(a^m + b^m)$ $(a \geq 0, b \geq 0)$ and Hölder inequality, we get

$$\int_{\mathbb{R}_n} (|X|^{n+1}+1)^{-1} |f(X)| dX \leq 2^{\frac{1}{2}(n+1)} \int_{\mathbb{R}_n} (X^2+1)^{-\frac{1}{2}(n+1)} |f(X)| dX$$

$$\leq 2^{\frac{1}{2}(n+1)} \left[\frac{1}{c_n}\right]^{\frac{1}{q}} \int_{\mathbb{R}_n} (X^2+1)^{-\frac{1}{2}(n+1)} |f(X)|^p dX$$

$$< +\infty .$$

Hence by the limit property of Poisson integral in $\mathbb{R}^+_{n+1}$ (see, Horváth [3], proposition 2.1), we have almost everywhere in $\mathbb{R}_n$

$$\lim_{y\to 0^+} I_f(X,y) = f(X) \qquad (X \in \mathbb{R}_n)$$

and the result follows.

We prove the last part of Theorem 3.2.2. in the case where $s \geq 0$ in V. The proof of the case where $s \leq 0$ in V is similar. Suppose that $s \geq 0$ in V and that by the contradiction the n-dimensional Lebesque measure of the set

$$E = \{X \in \mathbb{R}_n : f(X) < 0\}$$

is positive. Hence there is a compact subset F of E with positive Lebesque measure. Since F is compact, there exists a > 0 such that

F x (0,a) $\subset$ V and consequencetly from some j onwards ($b_j < a$ and)

$$\int_{R_n} s(Z,b_j)\chi_F(Z)d\nu(Z) \geq 0$$

since $\chi_F \in L^q(\nu)$, we have, by (3.3.6)

$$\lim_{j\to+\infty} \int_{\mathbb{R}_n} s(Z,b_j)\chi_F(Z)d\nu(Z) = \int_{\mathbb{R}_n} f(Z)\chi_F(Z)d\nu(Z)$$

$$= \int_F f(Z)(Z^2+1)^{-\frac{1}{2}(n+1)} dZ$$

$$\geq 0 .$$

This contradicts the supposition that $f < 0$ on F and F has positive Lebesque measure.

3.4 Futher properties of Poisson integral

Let $K^p(\mathbb{R}_n)$, $p \geq 1$ denote the class of all real valued measurable functions defined on $\mathbb{R}_n$ such that

$$\int_{\mathbb{R}_n} (Z^2+1)^{-\frac{1}{2}(n+1)} |f(Z)|^p dZ \leq c < \infty,$$

i.e, $K^p(\mathbb{R}_n) = L^p(\nu) \supseteq L^p(\mathbb{R}_n)$.

For any function f in $K^p(\mathbb{R}_n)$, $p \geq 1$, we define , for each (X,y) in $\mathbb{R}^+_{n+1}$, the Poisson integral of f to be

$$I_f(X,y) = \int_{\mathbb{R}_n} f(Z)P(X-Z,y)dZ = \int_{\mathbb{R}_n} f(X-Z)P(Z,y)dZ.$$

And, if μ is a signed Radon measure in $\mathbb{R}_n$ such that

$$\int_{\mathbb{R}_n} (Z^2+1)^{-\frac{1}{2}(n+1)} \, d|\mu|(z) < \infty ,$$

we define, for each (X,y) in $\mathbb{R}^+_{n+1}$, the Poisson integral of μ to be

$$I_\mu(X,y) = \int_{\mathbb{R}_n} P(X-Z,y)d\mu(Z) .$$

The Poisson integral has the following properties.

3.4.1 $I_f(X,y)$; the Poisson integral of a function f in $K^p(\mathbb{R}_n)$, and $I_\mu(X,y)$; the Poisson integral of a Radon measure such that $\int_{\mathbb{R}_n} (Z^2+1)^{-\frac{1}{2}(n+1} \, d|\mu|(Z) < \infty$, are harmonic in $\mathbb{R}^+_{n+1}$.

3.4.2 $\lim_{y\to 0^+} I_f(X,y) = f(X)$ a.e. X in $\mathbb{R}_n$.

3.4.3 If we define $D\mu(X) = \lim_{r\to 0} \frac{\mu B(X,r)}{\nu_n(r)}$ whenever the limit exists, where

$B(X,r)$ = the ball in $\mathbb{R}_n$ with center at X, radius r.

$\nu_n(r)$ = the volume of the ball with radius r, in $\mathbb{R}_n$,

and define $D_\mu(X)$ arbitrarily at point X where the limit does not exist. It is known that $\lim_{r\to 0} \frac{\mu B(X,r)}{\nu_n(r)}$ exists for almost all X in $\mathbb{R}_n$ and $D\mu \in L'(\mathbb{R}_n)$ (see, Rudin [8], theorem 8.6).

With this definition, we get the following proposition .

Proposition 3.4.4. Let $I_\mu(X,y)$ be the Poisson integral of a Radon measure satisfying $\int_{\mathbb{R}_n} (Z^2+1)^{-\frac{1}{2}(n+1)} d|\mu|(Z) < \infty$ in $\mathbb{R}_n$, then $\lim_{y\to 0^+} I_\mu(X,y)$ exists for almost all X in $\mathbb{R}_n$ and is equal to $D\mu(X)$.

To prove this we first show that we can assume $D\mu(X) = 0$.

If $D\mu(X) = a \neq 0$, we will consider the signed measure $\mu - a\nu$ where ν is the measure on $\mathbb{R}_n$ defined by

$$\nu(E) = \int_E dZ .$$

Now $\quad D(\mu - a\nu)(X) = D\mu(X) - aD\nu(X)$.

Since $\quad \nu(B(X,r)) = \int_{B(X,r)} dZ = \nu_n(r)$.

Then $D(\mu - a\nu)(X) = 0$ and

$$\int_{R_n} P(X-Z,y)d(\mu - a\nu)(Z) = \int_{\mathbb{R}_n} P(X-Z,y)d\mu(Z) - a \int_{\mathbb{R}_n} p(X-Z,y)d\nu(Z)$$

$$= h(X,y) - a .$$

(Since $\nu(E) = \int_E dZ$, $\int_{\mathbb{R}_n} p(X-Z,y)d\nu(Z) = \int_{\mathbb{R}_n} P(X-Z,y)dZ$ and the right side equals to 1).

If we can show that $\lim_{y\to 0^+} (I_\mu(X,y) - a) = 0$, then we have $\lim_{y\to 0^+} I_\mu(X,y) = a$. Therefore by replacing μ by $\mu - a\nu$ we can assume $a = 0$, and we will show that $\lim_{y\to 0^+} I_\mu(X,y) = 0$.

Since $\lim_{r\to 0} \frac{\mu B(X,r)}{\nu_n(r)} = 0$, given $\varepsilon > 0$, $\exists\, \delta > 0$ such that for all r with $r < 2\delta$ we have $|\mu B(X,r)| < \varepsilon\, \nu_n(r)$.

By Lebesque's dominated convergence theorem we get

$$\lim_{y\to 0^+} \int_{\mathbb{R}_n - B(X,\delta)} P(X-Z,y)\,d\mu(Z) = 0 .$$

It remains to prove $\int_{B(X,\delta)} P(X-Z,y)\,d\mu(Z) \to 0$ as $y \to 0^+$

Let $$J(y) = \int_{B(X,\delta)} \frac{y}{(|X-Z|^2+y^2)^{\frac{1}{2}(n+1)}}\, d\mu(Z) ,$$

and $$F(r) = \int_{B(X,r)} d\mu(Z).$$

Then $$J(y) = \int_0^\delta \frac{y}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dF(r).$$

By integrating by parts

$$J(y) = \left[y\, \frac{F(r)}{(r^2+y^2)^{\frac{1}{2}(n+1)}} \right]_{r=0}^{\delta} + (n+1)y \int_0^\delta \frac{r\,F(r)}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr$$

$$= y\, \frac{F(\delta)}{(\delta^2+y^2)^{\frac{1}{2}(n+1)}} - y\, \frac{F(0)}{y^{n+1}} + (n+1)y \int_0^\delta \frac{r\,F(r)}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr$$

since $|F(r)| = |\mu\, B(X,r)| < \varepsilon\, \nu_n(r)$. (3.4.5)

Therefore $y \frac{F(\delta)}{(\delta^2+y^2)^{\frac{1}{2}(n+1)}} \longrightarrow 0$ as $y \to 0$.

And since $F(0) = \mu B(X,0) = \mu(\{X\}) = 0$, $y \frac{F(0)}{y^{n+1}} = 0$.

Then by (3.4.5) we get

$$\left| n+1 \int_0^{\delta} \frac{y\, r\, F(r)}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr \right| < \varepsilon(n+1) \int_0^{\delta} \frac{y\, r\, \nu_n(r)}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr = I(y) \quad \text{say.}$$

By integrating by parts

$$I(y) = \varepsilon y \left[\frac{-\nu_n(r)}{(r^2+y^2)^{\frac{1}{2}(n+1)}} \right]_{r=0}^{\delta} + \varepsilon y \left[\int_0^{\delta} \frac{\nu_n}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr^n \right]$$

where $\nu_n = \nu_n(1) = \frac{1}{r^n}\, \nu_n(r)$. ([4])

$$= I_1 + I_2 \quad \text{say.}$$

We see that $I_1(y) \to 0$ as $y \to 0$.

Consider

$$I_2(y) = \varepsilon y \int_0^{\delta} \frac{\nu_n}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr^n$$

$$= \varepsilon n y \nu_n \int_0^{\delta} \frac{r^{n-1}}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr .$$

Since

$$\int_0^{\delta} \frac{r^{n-1}}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr < \int_0^{\infty} \frac{r^{n-1}}{(r^2+y^2)^{\frac{1}{2}(n+1)}}\, dr$$

$$= \frac{c_n}{\sigma_n} \qquad \text{(see 3.1.2)}$$

and $0 \leq I_2(y) < \varepsilon n y \nu_n \dfrac{c_n}{\sigma_n}$.

We can clearly see that $I_2(y) \longrightarrow 0$ as $y \longrightarrow 0$, that is $I(y) \longrightarrow 0$ as $y \longrightarrow 0$ and so we get

$$|J(y)| \longrightarrow 0 \quad \text{as} \quad y \longrightarrow 0.$$

And, then the proof is complete.

<u>Theorem 3.4.6.</u> If $s \geqslant 0$ and $s \in S^p$ where $p > 1$, then the limit of $K(s^p,y)$ as $y \to 0^+$ exists and is given by

$$\lim_{y \to 0^+} K(s^p,y) = \int_{\mathbb{R}_n} (Z^2+1)^{-\frac{1}{2}(n+1)} |f(Z)|^p \, dZ$$

where f is the non-negative measurable function of theorem 3.2.2. (See in Nualtaranee, [6] for the proof.)