CHAPTER IV

BOUNDARY LIMIT OF F(X,y)

4.1 Introduction

We recall that, for each $p > 0$, we define the class of G^p to consists of all those systems of conjugate harmonic functions $F(X,y) = (u(X,y), v_1(X,y),\ldots, v_n(X,y))$ defined on $\mathbb{R}^+_{n+1}$ satisfying

$$\int_{\mathbb{R}_n} \{X^2+(y+1)^2\}^{-\frac{1}{2}(n+1)} |F(X,y)|^p dX \leq A < \infty \qquad (4.1.1)$$

for $y > 0$.

We shall exploit (4.1.1) and theorem 2.2.1, to obtain boundary values of $F(X,y)$.

4.2 Boundary limit theorem

Lemma 4.2.1 Suppose μ is a positive measure on X, $1 \leq p < \infty$, $f \in L^p(\mu)$, $f_n \in L^p(\mu)$, $f_n(X) \to f(X)$ a.e., and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$, then $\|f-f_n\|_p \to 0$ as $n \to \infty$.

Proof By the inequality $|a-b|^p \leq \gamma_p(|a|^p+|b|^p)$, $a \geq 0$, $b \geq 0$, $0 < p < \infty$, where $\gamma_p = \max(1, 2^{p-1})$, we get

$$h_n = \gamma_n(|f|^p + |f_n|^p) - |f-f_n|^p \geq 0.$$

Since $|f|^p$, $|f_n|^p$ and $|f-f_n|^p$ are all integrable, h_n is μ-integrable.

By Fatou's lemma we get

$$2\gamma_p(\|f\|_p)^p = \int_X \lim_{n\to\infty} \inf [\gamma_p(|f|^p+|f_n|^p)-|f-f_n|^p]d\mu$$

$$\leq \lim_{n\to\infty} \inf \int_X [\gamma_p(|f|^p+|f_n|^p)-|f-f_n|^p]\, d\mu$$

$$= \lim_{n\to\infty} \int_X \gamma_p(|f|^p+|f_n|^p)d\mu - \lim_{n\to\infty} \sup \int_X |f-f_n|^p\, d\mu$$

$$= 2\gamma_p(\|f\|_p)^p - \lim_{n\to\infty} \sup \int_X |f-f_n|^p\, d\mu .$$

Since $2\gamma_p(\|f\|_p)^p$ is finite, we may substract it and obtain

$$\lim_{n\to\infty} \sup \int_X |f-f_n|^p\, d\mu \leq 0 . \tag{4.2.2}$$

But, if a sequence of nonnegative real numbers fails to converges to 0, then it upper limit is positive. Thus (4.2.2) implies that

$$\lim_{n\to\infty} \int_X |f-f_n|^p\, d\mu = 0 ,$$

i.e.,

$$\lim_{n\to\infty} (\|f-f_n\|_p)^p = 0 \quad \text{or}$$

$$\lim_{n\to\infty} (\|f-f_n\|_p) = 0 .$$

<u>Theorem 4.2.3</u>. Suppose $F(X,y)$ forms a system of conjugate harmonic functions and satisfies

$$\int_{\mathbb{R}_n} \frac{|F(X,y)|^p\, \partial X}{(X^2+(1+y)^2)^{\frac{1}{2}(n+1)}} \leq c < \infty \tag{4.2.4}$$

for $1 \leq p < \infty$, then $\lim_{y\to 0} F(X,y) = F(X,0)$ exists for almost all X in $\mathbb{R}_n$ and for $p > 1$, $\lim_{y\to 0} \int_{\mathbb{R}_n} \frac{|F(X,y)-F(X,0)|^p}{(X^2+(1+y)^2)^{\frac{n+1}{2}}} dX = 0$.

Proof Let $w(X,y)$ be one of the components $u(X,y)$, $v_1(X,y)$,... ..., $v_n(X,y)$ of $F(X,y)$. The inequality (4.2.4) implies that

$$\int_{\mathbb{R}_n} \frac{|w(X,y)|^p}{(X^2+(1+y)^2)^{\frac{1}{n}(n+1)}} dX \leq c < \infty .$$

If $p = 1$, by theorem 3.2.1, there is a harmonic majorant h_w of w such that

$$h_w(X,y) = ky + I_\nu(X,y)$$ where ν is a Radon measure on $\mathbb{R}_n$ such that

$$\int_{\mathbb{R}_n} (Z^2+1)^{-\frac{1}{2}(n+1)} d|\nu|(Z) \leq c$$ and k is the constant given in the theorem 3.2.1.

Since w is also a harmonic function in $\mathbb{R}^+_{n+1}$, then

$$w(X,y) = h_w(X,y) = ky + I_\nu(X,y) \qquad ((X,y) \in \mathbb{R}^+_{n+1}).$$

By the proposition 3.4.4, we have

$\lim_{y\to 0^+} w(X,y) = w(X,0)$ exists for almost all X in $\mathbb{R}_n$ and is equal to $D\nu(X)$. This proves that

$\lim_{y\to 0^+} F(X,y) = F(X,0)$ exists for almost all X in $\mathbb{R}_n$.

For the case $p > 1$, by theorem 3.2.2, there exists a harmonic majorant h_w of w such that

$$h_w(X,y) = I_{f_w}(X,y) \qquad ((X,y) \in \mathbb{R}^+_{n+1})$$

where $f_w \in K^p(\mathbb{R}_n)$ and for almost all X in $\mathbb{R}_n$ we have

$$\lim_{y\to 0} h_w(X,y) = f_w(X) .$$

Again, since w is a harmonic function, $\lim_{y\to 0} w(X,y) = w(X,0)$ exists for almost all X in $\mathbb{R}_n$ and is equal to $f_w(X)$.

This means

$$\lim_{y\to 0} F(X,y) = F(X,0) \text{ exists for almost every point } X \text{ in } \mathbb{R}_n .$$

To show the last part of theorem we note that, by theorem 3.4.6,

$$\lim_{y\to 0} \int_{\mathbb{R}_n} \frac{|w(X,y)|^p}{(X^2+(1+y)^2)^{\frac{1}{2}(n+1)}}\, dX = \int_{\mathbb{R}_n} \frac{|f_w(X)|^p}{(1+X^2)^{\frac{1}{2}(n+1)}}\, dX .$$

By lemma 4.2.1, we get

$$\lim_{y\to 0} \int_{\mathbb{R}_n} \left| \frac{w(X,y)}{(X^2+(1+y)^2)^{\frac{1}{2p}(n+1)}} - \frac{f_w(X)}{(1+X^2)^{\frac{1}{2p}(n+1)}} \right|^p dX = 0 .$$

Since $\lim_{y\to 0} (X^2+(1+y)^2)^{-\frac{1}{2p}(n+1)} = (X^2+1)^{-\frac{1}{2p}(n+1)}$, we get

$$\lim_{y\to 0} \int_{\mathbb{R}_n} \frac{|w(X,y)-f_w(X)|^p}{(X^2+(1+y)^2)^{\frac{1}{2}(n+1)}}\, dX = 0 . \qquad (4.2.5)$$

By the inequality $(a+b)^m \leqslant 2^m(a^m,b^m)$ $(a \geqslant 0,\ b \geqslant 0)$, together with (4.2.5) we get

$$\lim_{y\to 0} \int_{\mathbb{R}_n} \frac{|F(X,y)-F(X,0)|^p}{(X^2+(1+y)^2)^{\frac{1}{2}(n+1)}}\, dX = 0 ,$$

the theorem is now completely proved.