FIRST-PASSAGE TIME SECURITIES VALUATION UNDER JUMP-DIFFUSION MODEL USING PARTITIONING, EXPONENTIAL TWISTING, AND CONDITIONAL MONTE CARLO TECHNIQUE

Mr. Prachya Mongkolkul

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Financial Engineering
Department of Banking and Finance
Faculty of Commerce and Accountancy
Chulalongkorn University
Academic Year 2014

การคำนวณมูลค่าตราสารที่ขึ้นต่อเวลาผ่านครั้งแรก ภายใต้แบบจำลองการกระโดดดิฟฟิวชั่น โดยใช้การแยกส่วน การบิดแบบเอกซ์โพเนนเชียล และเทคนิคมอนติคาร์โลแบบมีเงื่อนไข

นายปรัชญา มงคลกุล

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาวิศวกรรมการเงิน ภาควิชาการธนาคารและการเงิน
คณะพาณิชยศาสตร์และการบัญชี จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2557

Thesis Title	FIRST-PASSAGE TIME SECURITIES VALUATION UNDER JUMP-DIFFUSION MODEL USING PARTITIONING, EXPONENTIAL TWISTING, AND CONDITIONAL MONTE CARLO TECHNIQUE
By	Mr. Prachya Mongkolkul
Field of Study	Financial Engineering
Thesis Advisor	Sira Suchintabandid, Ph.D.

Accepted by the Faculty of Commerce and Accountancy, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

................................Dean of the Faculty of Commerce and Accountancy
(Associate Professor Pasu Decharin, Ph.D.)

THESIS COMMITTEE

....Chairman
(Anant Chiarawongse, Ph.D.)

....Thesis Advisor
(Sira Suchintabandid, Ph.D.)

....Examiner
(Thaisiri Watewai, Ph.D.)

....External Examiner
(Kridsda Nimmanunta, Ph.D.)

ปรัชญา มงคลกุล : การคำนวณมูลค่าตราสารที่ขึ้นต่อเวลาผ่านครั้งแรก ภายใต้แบบจำลองการกระโดดดิฟฟิวชั่น โดยใช้การแยกส่วน การบิดแบบเอกซ์โพเนนเชียล และเทคนิคมอนติคาร์โลแบบมีเงื่อนไข (FIRST-PASSAGE TIME SECURITIES VALUATION UNDER JUMP-DIFFUSION MODEL USING PARTITIONING, EXPONENTIAL TWISTING, AND CONDITIONAL MONTE CARLO TECHNIQUE) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ดร.สิระ สุจินตะบัณฑิต, 39 หน้า.

งานวิจัยฉบับนี้พัฒนาวิธีการคำนวณมูลค่าตราสารที่มูลค่าเป็นฟังก์ชันของเวลาผ่านครั้งแรก และราคาของสินทรัพย์ทางการเงินอ้างอิง ณ เวลาผ่านครั้งแรก ภายใต้แบบจำลองการกระโดด และการแพร่กระจาย เนื่องจากการชำระหนี้ของตราสารสิทธิที่มีเงื่อนไขเวลาผ่านระดับนั้นมีการกระจายตัวสูง อีกทั้งเวลาผ่านระดับยังเกิดขึ้นได้ยาก การคำนวณมูลค่าตราสารดังกล่าวด้วยแบบจำลองมอนติคาร์โลจึงทำได้ยากตามไปด้วย และมักให้ผลลัพธ์ที่มีความแปรปรวนสูง เพื่อแก้ปัญหาดังกล่าว ผู้วิจัยได้พิสูจน์ และพัฒนาวิธีการคำนวณมูลค่า ประกอบด้วยเทคนิคการลดความแปรปรวน 3 ชนิด ได้แก่ การแยกส่วน การบิดแบบเอกซ์โพเนนเชียล และเทคนิคมอนติคาร์โลแบบมีเงื่อนไข นอกจากนั้น ยังวิเคราะห์ และพิสูจน์ค่าประมาณของโมเมนต์ที่สองของราคาคาดการณ์ ในการทดลองเชิงตัวเลข โดยพันธบัตรแปลงสภาพแบบมีเงื่อนไขจะถูกใช้เป็นตัวอย่าง เพื่อแสดงประสิทธิผลของวิธีการคำนวณมูลค่าที่นำเสนอ ร่วมกับพารามิเตอร์ที่ถูกเลือกจากวิธีที่ได้ในการวิเคราะห์ ซึ่งให้ผลลัพธ์ในการลดความแปรปรวนอย่างมีนัยสำคัญ

ภาควิชา การธนาคารและการเงิน ลายมือชื่อนิสิต ....

สาขาวิชา วิศวกรรมการเงิน ลายมือชื่อ อ.ที่ปรึกษาหลัก ....

ปีการศึกษา 2557

5582011226 : MAJOR FINANCIAL ENGINEERING
KEYWORDS: FIRST-PASSAGE TIME / DERIVATIVE PRICING / JUMP-DIFFUSION MODEL / MONTE CARLO SIMULATION / VARIANCE REDUCTION TECHNIQUE / CONTINGENT CONVERTIBLE BOND / KNOCK-IN OPTION

PRACHYA MONGKOLKUL: FIRST-PASSAGE TIME SECURITIES VALUATION UNDER JUMP-DIFFUSION MODEL USING PARTITIONING, EXPONENTIAL TWISTING, AND CONDITIONAL MONTE CARLO TECHNIQUE. ADVISOR: SIRA SUCHINTABANDID, Ph.D., 39 pp.

This paper develops an efficient simulation-based method to price derivative securities whose payoff depends on a first-passage time and the value of its underlying at that time, under the assumption that the underlying's process follows the jump-diffusion model. Because of the high variation of payoff and the rarity of the first-passage event, pricing such securities using Monte Carlo simulation is challenging, and usually results in a price estimate that has high variance. As a solution, we devise an improved method for pricing such securities by combining three techniques: partitioning, exponential twisting, and conditional Monte Carlo. We provide an analysis of the proposed method and derive an approximation for the second moment of the resulting price estimate. In our numerical experiments, we consider Contingent Convertible bonds as an example to demonstrate the effectiveness of the proposed method in reducing the variance of the price estimate. Numerical results show that the proposed method, with parameters selected through simple criteria laid out in the analysis, provides substantial variance reduction.

Department:	Banking and Finance	Student's Signature	....
Field of Study:	Financial Engineering	Advisor's Signature	....
Academic Year:	2014		

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor Mr.Sira Suchintabandid for his encouragement, tactical comments, and helped me in all the time of writing this thesis. He continually support and guide me in all the time.

I also would like to sincerely thank all other people who support me through this thesis.

CONTENTS

1. Introduction

This paper addresses the challenge in valuing derivative securities with “knock-in” feature whose payoff depends not only on the first time that the underlying breaches a barrier, but also on the value of the underlying at the that time. Contingent Convertible bonds (Coco), for example, is a hybrid security which will be automatically converted from debt into equity by a certain pre-defined trigger event which is the time τ that underlying cross the barrier. Its price is sensitive not only to the first-passage time τ, but also the share price S_τ at the first-passage time. Other examples of securities with such feature include knock-in American options, Lookback options, and American-style Asian option.

The valuation of these securities is relatively simple if we restrict ourselves to the case where the underlying process has continuous path. For example, if the underlying follows Geometric Brownian Motion (GBM), the closed-form valuation formula for such securities is available. However, the valuation will be more difficult if the underlying process does not have continuous path, because the overshoot’s distribution is often unknown (Kou, 2002). For example, if the underlying follows well-known Jump Diffusion Model (JDM), generally the closed-form valuation formula for such securities is not available. (Although the JDM can be approximated to be GBM by moment matching method (Bates, 1991), but many important features of JDM such as the leptokurtic and the ability to explain the volatility smile will be lost because the pure diffusion model with high volatility tends to cause many small jumps instead of instantaneous jumps which happen in the real market (Teneberg, 2012).) In this study, we assume that underlying follows the JDM with log-Normal jumps (Merton, 1976).

Because of the aforementioned issue with overshoot, some researchers considered the special case where the jump-size follows double-exponential distribution and used its Memory-Less property to obtain the distribution of the overshoot (Kou and Wang, 2003) so the closed-form valuation formulas for most of such securities are available (Kou and Wang, 2004). But for general case, Monte Carlo simulation remarks most convenient and universal.

Classical Monte Carlo simulation, while straightforward, produces estimates with high variance for the first-passage time problem (Primožič, 2011). In particular, the high variance of Monte Carlo estimate is due to a high variation of payoff (depending on the first-passage time τ) and the rare-event nature of the first-passage. Furthermore, using Monte Carlo simulation to generate price path inevitably requires time discretization, which, combined with the effort requires to deal with the high variance lead to unacceptably high computational effort. To solve this specific first-passage time problem, we enhance the traditional Monte Carlo simulation by combining three techniques: partitioning, exponential twisting, and conditional Monte Carlo.

In the partitioning technique, we divide the security into smaller sub-period. In each sub-period, we define a security that pays off the same as the original security if first passage occurs during that sub-period (effective period), and pays off zero otherwise. The newly defined security represents a "component" of the original security. Thus, the price of the original security can be expressed as the sum of the price of all components. Hence, our problem reduces to how to use Monte Carlo simulation to price each component. Because partitioning helps to limit the range of possible payoff of each component, the estimated price of the component given τ occurs in its effective period is expected to have small variance. Thus, if our proposed method can sample τ that falls in effective period, then we can reduce the variance of each component (and thus the variance of the full estimate) substantially. This technique is conceptually similar to stratified sampling or numerical integration over τ. Though, unlikely stratify sampling or numerical integration, our technique does not require explicit knowledge of the distribution of τ.

In order to sample τ that is likely to fall in a given period, we combine Monte Carlo simulation with importance sampling. Here, we use a class of importance sampling technique known as exponential twisting. In this technique, the component's price is expressed as expectation under a new probability measure that make the non-zero payoff event less rare. Changing probability measure by exponential twisting is convenient because under the new measure, the dynamic of the underlying process can be shown to be JDM still, but with parameters altered by a twist parameter. We choose the twist parameter so that, the underlying is unlikely to cross the barrier before the effective period, but at the same time, tends to breach barrier within the effective period. In this study, we also discuss the way to choose an appropriate twist parameter by minimizing an approximated second moment of the component's price estimator.

Under this partitioning technique with exponential twisting, the only relevant information about the path before effective period is whether or not the underlying share price crosses the barrier. We take advantage of this observation by applying the conditional Monte Carlo (Boyle et al., 1997) to simulate the price path before the effective period. In this technique, we sample the underlying's price just before and after each jump. Then rather than simulate the path between jumps to identify the first-passage, we adjust the value of the estimate by the probability that the underlying does not cross the barrier between two jumps. (This probability is available in closed-form because the fact that the process between two jumps is continuous.) This technique not only allow us to avoid discretize time (and discretization error) before the effective period, but also reduce the computational effort substantially.

All in all, this paper demonstrates the effectiveness of combination of these three techniques in reducing variance of the price estimate. We use Coco as a primary example. We also address and analyze important issues such as choice of partition and change of measure. In particular, we derive semi-analytical formula that approximates the second moment that depends on the twist parameters which could be used to choose appropriate twist parameters. The Numerical study shows significant variance reduction compare with the traditional Monte Carlo simulation and the stand-alone basic variance reduction technique. Our novel way of combining these three techniques adds to the body of literature on knock-in options by providing an efficient simulation-based method for computing the price.

Organization of this paper is as follows. Section 2 explicitly addresses the JDM and the Coco which will be used in this paper. Section 3 reviews three simulation techniques which will be applied together in the proposed method and also presents how to apply the proposed method on the sample security. Section 4 contains the result of numerical experiment and also discuss about a method to choose appropriate decision parameters. Section 5 concludes the paper, and an Appendix contains the proofs.

2. Model Setting

2.1. Dynamic of the Underlying

The assumption that the underlying share price of first-passage time dependent securities follows log-Normal JDM will be assumed. Let S_t is share price at time t. Given constant risk-free rate r, the process of S_t under the risk-neutral measure is given by (1) (Glasserman, 2003).

$$S_t = S_0 e^{\mu t + \sigma W(t)} \prod_{j=1}^{N(t)} Y_j \tag{1}$$

Where, σ is a constant, $W(t)$ is Wiener's process, Y_j is jump-size, $N(t)$ is a Poisson process with rate λ, and $\mu = r - \lambda \bar{Y} - \frac{1}{2}\sigma^2$, with $\bar{Y}$ being expected value of Y_j. (We define μ this way so that S_t is a martingale.) Assume that $Y_j, N(t)$ and $W(t)$ are independent to each other. In differential form:

$$\frac{dS_t}{S_{t^-}} = rdt + \sigma dW(t) + dJ_t - \lambda \bar{Y} dt \tag{2}$$

Where, J_t is a pure jump process which equals to $\sum_{j=1}^{N(t)} (Y_j - 1)$. Assume that $\log Y_j$ are i.i.d. $N(\bar{J}, \sigma_j^2)$ random variables. In this paper, S_0, r and $\bar{J}$ will be set to be 10, 5% and 0 respectively as a default except state differently.

2.2. Characteristics of Securities (Coco)

In order to find an appropriate simulation technique for securities of which the payoff is a function of the first-passage time τ, and underlying share price at the first-passage time S_τ, the Coco will be used as a representative. Coco is a hybrid security that is automatically converted from debt into equity when the trigger event, which is normally assumed to be τ as defines in equation (3), occurs.

$$\tau = \inf\{t > 0 | S_t \leq B\} \tag{3}$$

Where, B is implied trigger level or a barrier. At conversion, some parts of the notional principal N will be converted into pre-defined number of shares or the conversion rate C_r which could be calculated as in equation (4). The conversion fraction α is a parameter that determines the fraction of the Coco's face value that will be converted when the first-passage time occurs. If the conversion fraction equals to one, the Coco will be called a "Full" Coco which is recommended structure by the American Enterprise Institute (Spiegeleer, 2011). The Full Coco will be focused in this study.

$$C_r = \frac{\alpha N}{C_p} \tag{4}$$

Where, C_p is the conversion price. Note that in this paper, B and C_p will be set to be 2.5 and S_0 respectively except state differently.

The Coco's value can be expressed as in equation (5).

$$\text{Coco} = \mathbb{E}\left[\sum_{\delta=1}^{h} ce^{-rt_\delta}\mathbb{I}(\tau > t_\delta) + e^{-rT}N\mathbb{I}(\tau > T) + e^{-r\tau}C_r S_\tau \mathbb{I}(\tau \leq T)\right] \tag{5}$$

Where, $\mathbb{I}(\bullet)$ is an indicator function, c is coupon payment which will be set to be 8% compound quarterly except state differently and t_δ is time that δ^{th} coupon is paid where $\delta = 1, 2, \ldots, h$ and $t_h = T$. As present in (5), lower τ reduces the value of Coco. The maximum Coco's value is equal to the value of coupon bond. From (5), it could be concluded that when the first-passage time occurs, the Coco investors will suffer from the losses of conversion which is the cancellation of the coupons from τ to maturity and the conversion of N into a $C_r S_\tau$. These equation indicates that the value of Coco is really sensitive on two factors which are τ and S_τ.

2.3. Express Price as Expectation

The price of Coco can be computed from the expectation of Coco's value. Consider closely, the value of the Coco is equal to its maximum value minus by the conversion losses. Moreover, as mention before, the maximum value of the Coco is equal to the value of coupon bond and the value of the coupon bond can be computed easily by discounting its payoff. Therefore, the value of Coco can be expressed as in equation (6).

$$\text{Coco} = \text{Coupon bond} - \text{Conversion losses} \tag{6}$$

From equation (6), the value of Coco can be viewed as a combination of two securities which are the Coupon bond and the second security that has the same value as the Conversion loss. Let's refer to the second security (the conversion losses) as the Trigger Adjustment. The value of Trigger Adjustment X is defined as in equation (7). Because the value of the coupon bond can be obtained from closed-form formula so it is constant, this shows that the variance of Coco price comes only from the Trigger Adjustment part. This variance might seem to be low compare with the Coco's value. However, comparing with the Trigger Adjustment's value, this variance is unacceptably high. If the value of Trigger Adjustment is ignored, the Coco price will be the same as the coupon bond price. Therefore, in this research, in order to find a variance reduction technique for Coco, an efficiently method for evaluate Trigger Adjustment's value will be focus, which is the method to minimize the variance of estimator in equation (7).

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{\delta=1}^{h} ce^{-rt_\delta}\mathbb{I}(\tau \le t_\delta) + e^{-rT}N\mathbb{I}(\tau \le T) - e^{-r\tau}C_r S_\tau \mathbb{I}(\tau \le T)\right] \tag{7}$$

3. Suggested Method of Estimating the Expectation

To price the first-passage time securities, we propose a method that combines three techniques which are partitioning, exponential twisting and conditional Monte Carlo. The details of the methods can be shown as following.

3.1. Partitioning

The first-passage time dependent securities' payoff is a function of τ so the variation of payoff is high. Moreover, some events such as the early first-passage time have an extremely low probability of occurrence, but when these rare events occur, its extreme payoffs makes the estimator's variance high. These events can be regarded in the similar way as the tail-risk which cannot be ignored.

To solve these problems, we now describe our partitioning technique. Let's defined (8) as a component.

$$X^{[\alpha,\beta]} \triangleq X\mathbb{I}(\alpha < \tau \leq \beta) \tag{8}$$

The set of time $(\alpha, \beta]$ is an effective period. Let $\Gamma = \beta - \alpha$ denote the length of effective period, which will be set to be equal for every component. The value of the security can be viewed as a combination of h components, which have orderly effective period, $0 = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_{h-1} < \varepsilon_h = T$, as shown in equation (9).

$$\mathbb{E}[X] = \sum_{i=1}^{h} \mathbb{E}\left[X^{[\varepsilon_{i-1},\varepsilon_i]}\right] \tag{9}$$

Each component $X^{[\alpha,\beta]}$ will have the same payoff as the original security, only if the first-passage time occurs within the certain period, the effective period and zero payoff otherwise. Each component will be price by Monte Carlo simulation. If Γ is chosen to be small and given that the first-passage time occurs only within the effective period, the estimator's variance of that component is expected to be small. (In theory, if Γ is infinitesimally small and if the distribution of τ is known, then (9) can be seen to be equivalent to a numerical integration that yields a zero-variance estimate.) However, with too small Γ, event which the first-passage time occurs within the effective period becomes extremely rare event. An appropriate Γ will be suggested and discussed from the numerical result.

3.2. Exponential Twisting

After using partitioning technique to divide the security into many components, the simulation problem is now reduced to how to compute the value of each component $\mathbb{E}\left[X^{[\alpha,\beta]}\right]$ by Monte Carlo simulation efficiently. The only event that $X^{[\alpha,\beta]}$ is positive is when $\tau \in (\alpha,\beta]$ which is rare event.

We use importance sampling technique to deal with rare event problem effectively. The main idea of importance sampling is to simulate the sample from a new probability measure then adjust the value of each sample as shown in equation (10) (See (Večeř, 2011)).

$$\mathbb{E}[Y] = \widetilde{\mathbb{E}}[Y\mathcal{L}] \tag{10}$$

Where, Y is any random variable and $\mathcal{L}$ is the inverse of Radon-Nikodym derivative $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$ (or the likelihood ratio) which defines the new measure $\widetilde{\mathbb{P}}$ with respect to the original measure $\mathbb{P}$. Note that $\mathcal{L} > 0$ and $\widetilde{\mathbb{E}}[\mathcal{L}] = 1$.

In this work, we define the new measure using

$$\mathcal{L} = e^{-\int_0^\tau \theta_t d\ln(S_t) + \int_0^\tau \psi(\theta_t)dt} \tag{11}$$

Where, θ_t is a time dependent deterministic variable and $\psi(\theta_t) = \theta_t\mu + 0.5{\theta_t}^2\sigma^2 + \lambda\left(e^{\theta_t \bar{J} + 0.5{\theta_t}^2\sigma_J^2} - 1\right)$. It is easy to show that $\mathcal{L}$ satisfy $\widetilde{\mathbb{E}}[\mathcal{L}] = 1$. This setting is equivalent to use the special class of the importance sampling, the exponential twisting (See for example Glasserman, 2003) with parameter θ_t, on the increment $d\ln(S_t)$.

<u>Lemma 3.1</u> *Under the new measure defined by $\mathcal{L}$ in (11) the process of S_t follows the JDM as in equation (2), but its parameters are twisted from μ to μ_{θ_t} which equals to $\mu + \theta_t\sigma^2$, $N(t)$ to $N_{\theta_t}(t)$ which follows Poisson $\left(\lambda t e^{\theta_t \bar{J} + 0.5\theta_t^2\sigma_j^2}\right)$ and Y_j to $Y_j^{\theta_t}$ which follows $logNormal\left(\bar{J} + \theta_t\sigma_j^2, \sigma_j^2\right)$.*

The detail of Lemma 3.1 derivation is shown in Appendix 1. From Lemma 3.1, when θ_t is negative, the mean of the process will be shifted down, but the jump intensity will be increased. The alternation will increase the probability of the first-passage time.

In this work, for simplicity, we set

$$\theta_t = \begin{cases} \theta, & t \le \alpha \\ \vartheta, & t > \alpha \end{cases}$$

Having two differences values for before and within the effective period, θ and ϑ could be set to fit their role properly in each period. To reduce the variance efficiently, the new measure should be chosen to increases the probability of the importance event which is the event that has high pay off and high probability to occur in the original measure. For our components, this importance event is when $\tau \in (\alpha, \beta]$. Therefore, ϑ should be as negative as necessary to make the share price hit the barrier within the effective period. On the other hand, θ should be chosen in such a way that the share price is not likely to hit the barrier before α, but still close to the barrier at α which is to be slightly negative.

From those two twist parameters θ and ϑ, $\mathcal{L}$ can be written as in equation (12).

$$\mathcal{L} = \mathcal{L}_\alpha \mathcal{L}_\beta \tag{12}$$

Where,

$$\mathcal{L}_\alpha = exp\left(-\theta\, ln\left(\frac{S_\alpha}{S_0}\right) + \alpha\left(\theta\mu + 0.5\theta^2\sigma^2 + \lambda\left(e^{\theta\bar{J}+0.5\theta^2\sigma_J^2} - 1\right)\right)\right)$$

$$\mathcal{L}_\beta = exp\left(-\vartheta\, ln\left(\frac{S_\tau}{S_\alpha}\right) + (\tau-\alpha)\left(\vartheta\mu + 0.5\vartheta^2\sigma^2 + \lambda\left(e^{\vartheta\bar{J}+0.5\vartheta^2\sigma_J^2} - 1\right)\right)\right)$$

Even θ and ϑ should be chosen to be negative to shift the mean of the process down. With too negative twist parameters, the likelihood ratio will increase dramatically. On the other hand, with not enough negative value of θ and ϑ, the first-passage time might never occurs. Therefore, the value of these θ and ϑ are needed to be chosen carefully.

3.3. Conditional Monte Carlo

Crucial information for pricing $\mathbb{E}\left[X^{[\alpha,\beta]}\right]$ from before and within the effective period is difference. Because τ and S_τ is necessary information from the effective period, a traditional discretization simulation cannot be avoided. On the other hand, from before α, the only importance information is whether or not the underlying share price crosses the barrier so discretization is not the only option.

Metwally and Atiya (2002) proposed a simulation technique for barrier options pricing to observe whether or not the underlying share price breaches the barrier without discretization error (Metwally and Atiya, 2002). Adjusting the value of the sample by the probability that the share price does not breach the barrier between two jumps which is defined as in equation (13) (See (Karatzas and Shreve, 1991)), the technique allow us to simulate only the value of share price just before and after the jumps to observe the first-passage event. The concept of this method is generalized by technique called the conditional Monte Carlo (See Boyle et al., 1997).

$$\mathbb{P}\left(M_{(a,b]} \le B | S_a, S_b\right) = g(b-a, S_a, S_b) \quad (13)$$

Where, $M_{(a,b]} = \min(S_t, t \in (a,b])$

$$g(\Delta t, x, y) = \begin{cases} \exp\left(-\dfrac{2[\ln(B/x)][\ln(B/y)]}{\Delta t \sigma^2}\right) & , x > B, y > B \\ 1 & , \text{otherwise} \end{cases}$$

From the main idea of previous algorithms, the estimator of $X^{[\alpha,\beta]}$ can be formulated as shown below.

Twisting the probability measure then using the law of iterated expectation to given the jump time before the effective period.

$$\mathbb{E}\left[X^{[\alpha,\beta]}\right] = \widetilde{\mathbb{E}}\left[\mathbb{I}(\alpha < \tau \le \beta) X \mathcal{L}_\alpha \mathcal{L}_\beta\right]$$

$$= \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}_T\left[\mathbb{I}\left(M_{(0,\alpha]} > B\right)\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right) X \mathcal{L}_\alpha \mathcal{L}_\beta\right]\right] \quad (14)$$

Where, $\widetilde{\mathbb{E}}_T[\bullet]$ is $\widetilde{\mathbb{E}}\left(\bullet \,\middle|\, T_j, j = 1,2,\dots,N(\alpha)\right)$ and T_j is jump time of j^{th} jump, $0 = T_0 < T_1 < \dots < T_{N(\alpha)}$.

The conditional expectation in (14) can be transformed and simplify by using the law of iterated expectation. After that, we consider the Markov property to view the inner conditional expectation as a function of S_α to be able to take expectation as following. Let $\{\mathcal{F}_t\}_{t\ge0}$ denote the filtration generated by the path of $\{S_t\}_{t\ge0}$.

$$\widetilde{\mathbb{E}}_T\left[\widetilde{\mathbb{E}}_T\left[\mathbb{I}\left(M_{(0,\alpha]} > B\right)\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right) X \mathcal{L}_\alpha \mathcal{L}_\beta \middle| \mathcal{F}_\alpha\right]\right] = \widetilde{\mathbb{E}}_T\left[\mathbb{I}\left(M_{(0,\alpha]} > B\right)\mathcal{L}_\alpha f(S_\alpha)\right]$$

$$= \widetilde{\mathbb{E}}_T\left[\widetilde{\mathbb{E}}_T\left[\mathbb{I}\left(M_{(0,\alpha]} > B\right)\mathcal{L}_\alpha f(S_\alpha)\middle| S_{T_j^\pm}, j = 1,2,\dots,N(\alpha)+1\right]\right]$$

$$= \widetilde{\mathbb{E}}_T\left[\mathcal{L}_\alpha f(S_\alpha)\widetilde{\mathbb{E}}_T\left[\mathbb{I}\left(M_{(0,\alpha]} > B\right)\middle| S_{T_j^\pm}, j = 1,2,\dots,N(\alpha)+1\right]\right]$$

$$= \widetilde{\mathbb{E}}_T\left[\mathcal{L}_\alpha f(S_\alpha)\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)\right]$$

Where, we define $\Delta T_j \coloneqq T_j - T_{j-1}$, $T_{N(\alpha)+1} \coloneqq \alpha$, , $S_{T_j^{\pm}} = \lim_{h\to 0^{\pm}} S_{T_j+h}$, $f(S_\alpha) = \widetilde{\mathbb{E}}_T\left[\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right)X\mathcal{L}_\beta \middle| \mathcal{F}_\alpha\right]$ and $g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)$ is the probability given in (13).

The last expectation above, after substituting $f(S_\alpha)$ and using the law of iterated expectation, simplifies to

$$\widetilde{\mathbb{E}}_T\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right)X\mathcal{L}_\alpha\mathcal{L}_\beta\right]$$

The event that the underlying might cross the barrier between two jumps is taking into account by $\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)$ term which is the probability that the share price path does not hit the barrier between two jumps given the value of underlying just before and after jumps. Aside from reducing the variance and eliminating the discretization error, this method also has another great benefit which is the significant reduction of the computation effort.

3.4. Putting it together: Explicit expression of Estimator

To simulate the first-passage time securities' price efficiently, the proposed method combine three techniques in section 3.1-3.3. First, we use partitioning to divides the securities into many components to reduce the variation of payoff. Then exponential twisting is suitable to deal with the rare positive payoff property of the component. At last, conditional Monte Carlo can help reducing the computational effort and eliminate the discretization error before the effective period. The estimator of the component can be defined as in equation (15).

$$\mathbb{E}\left[X^{[\alpha,\beta]}\right] = \widetilde{\mathbb{E}}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right)X\mathcal{L}_\alpha\mathcal{L}_\beta\right] \qquad (15)$$

The optimum θ and ϑ are the set that minimize the variance of the estimator. Since, the value of the estimator will always be the same regardless the choice of twist parameter, minimizing the variance of the estimator is the same as minimizing the second moment $M^{(2)}$ which is defined as in equation (16).

$$M^{(2)} = \widetilde{\mathbb{E}}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)^2 \mathbb{I}\left(M_{(\alpha,\beta]} \leq B\right) X^2 \mathcal{L}_\alpha^2 \mathcal{L}_\beta^2\right] \tag{16}$$

Choosing appropriate value for θ and ϑ is an interesting problem. The more negative value of θ will reduce the value of $\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)^2$. On the other hand, it will increase the value of $\mathcal{L}_\alpha$. Likewise, similar tradeoff also occurs with ϑ. These tradeoff must be considered to choose the appropriate value of θ and ϑ.

Choosing θ and ϑ by minimizing equation (16) directly might not be possible because the expectation in equation (16) is too complicated to get a closed-form formula. In this study, θ and ϑ will be chosen by optimizing an approximated second moment $\widehat{M}^{(2)}$ instead. Using the fact that when Γ is small, we can approximate X by its upper bound X_m which is maximum value of $X^{[\alpha,\beta]}$ and $\tau - \alpha$ by $\frac{1}{2}(\beta - \alpha)$ so the approximation can be shown as following. After making these substitutions, the expectation in (16) can be shown (see Appendix 2) to simplify to

$$M^{(2)} \approx e^{(\beta+\alpha)\gamma(\vartheta)} {S_0}^{2\theta} B^{-2\vartheta} X_m^2 \widetilde{\mathbb{P}}\left(M_{(0,\alpha]} > B\right)$$

$$\times \widetilde{\mathbb{E}}^{M_\alpha}\left[{S_\alpha}^{2(\vartheta-\theta)}\, \widetilde{\mathbb{P}}^{M_\alpha}\left(M_{(\alpha,\beta]} \leq B \middle| S_\alpha\right) \widetilde{\mathbb{E}}^{M_\alpha}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right) \middle| S_\alpha\right]\right]$$

Where, $\gamma(x) = x\mu + 0.5x^2\sigma^2 + \lambda\left(e^{x\bar{J}+0.5x^2\sigma_J^2} - 1\right)$, $\mathbb{E}^{M_\alpha}[\bullet]$ is $\mathbb{E}\left[\bullet \middle| M_{(0,\alpha]} > B\right]$ and $\mathbb{P}^{M_\alpha}[\bullet]$ is $\mathbb{P}\left[\bullet \middle| M_{(0,\alpha]} > B\right]$. The detail of this approximation is shown in Appendix 2.

Under JDM, some terms such as $\widetilde{\mathbb{P}}\left(M_{(0,\alpha]} > B\right)$, $\widetilde{\mathbb{P}}\left(M_{(\alpha,\beta]} \leq B \middle| S_\alpha\right)$, and the expectation terms do not have closed-form formulas so we approximated the JDM process of the underlying with an adjusted GBM by using moment matching method (Bates, 1991). The process of the adjusted GBM can be showed as in equation (17). In (17), $\ln(dS_t)$ and $\ln\left(dS_t^{'}\right)$ have the same first and second moment.

$$S_t \approx S_t' = S_0 e^{\mu_\theta' t + \sigma_\theta' W(t)} \tag{17}$$

Where,

$$\mu_\theta' = \mu_\theta + \lambda_\theta \bar{J}_\theta$$

$${\sigma_\theta'}^2 = \sigma^2 + \lambda_\theta\left(\bar{J}_\theta^2 + \sigma_j^2\right)$$

Note that the exponential twist did not affect the value of σ and σ_j, but the value of σ_θ' is a function of λ_θ and $\bar{J}_\theta$ so the value of σ_θ' is a function of exponential twist parameter. Therefore, $\widehat{M}^{(2)}$ is defined as in equation (18).

$$\begin{aligned}\widehat{M}^{(2)} = {} & e^{2\alpha\gamma(\theta)+(\beta-\alpha)\gamma(\vartheta)} {S_0}^{2\theta} B^{-2\vartheta} X_m^2 \widetilde{\mathbb{P}}_G\left(M_{(0,\alpha]} > B\right) \\ & \times \widetilde{\mathbb{E}}_G^{M_\alpha}\left[{S_\alpha}^{2(\vartheta-\theta)} \widetilde{\mathbb{P}}_G^{M_\alpha}\left(M_{(\alpha,\beta]} \leq B \middle| S_\alpha\right) g(\alpha, S_\alpha, S_0)\right]\end{aligned} \tag{18}$$

Where, $\mathbb{E}_G[\bullet]$ and $\widetilde{\mathbb{P}}_G[\bullet]$ are expectation and probability while approximate that $S_t \approx S_t'$. Detail of the approximation will be showed in Appendix 3. The appropriate value of θ and ϑ will be chosen by minimize $\widehat{M}^{(2)}$ in equation (18). Detail of the approximation will be showed in Appendix 3.

Before discussing about the algorithm of the proposed method, first let's briefly review the traditional discretization simulation. The basic of the traditional discretization method is to discretize time into many tiny time steps then simulates the share price value at each time steps S_t to observe τ and S_τ. Then compute the value of $X^{[\alpha,\beta]}$. Average the value from N scenarios to obtain the value of the estimator. This method not only produces the estimate with high variance, but also requires long computational time because of the discretization. The proposed method can be applied to solve these problems as the following.

1. Optimizing $\widehat{M}^{(2)}$ in equation (18) to select appropriate twist parameters.
2. Simulate number of Jump up to α ($N_\theta(\alpha) \sim$ Poisson$(\lambda_\theta \alpha)$) and Jump timeT_j.
3. Simulate share price path before α $S_{T_j^{\pm}}, j = 1,2,\dots,N_\theta(\alpha)+1$ from (19) and (20)

$$S_{T_j^-} = S_{T_{j-1}^+} e^{\mu_\theta \Delta T_j + \sigma\sqrt{\Delta T_j} z_j} \tag{19}$$

$$S_{T_j^+} = S_{T_j^-} e^{(\sigma_j y_j + \bar{J}_\theta)} \tag{20}$$

   Where, z_j follows $N(0,1)$ and y_j follows $N(0,1)$. Note that $T_{N_\theta(\alpha)+1} = \alpha$, $y_{N_\theta(\alpha)+1} = 0$

4. Compute $\mathcal{L}_\alpha$ and $g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)$.
5. Using the traditional discretization simulation method as shown in the previous algorithm to simulate share price path within the effective period.
6. Obtain τ and S_τ from share price path from step 4th if the share price never hit barrier, $\tau = \infty$.
7. Compute $\mathcal{L}_\beta$ and $X^{[\alpha,\beta]}$
8. Continue doing step 1st – 7th for N replications then compute the value of the first and second moment from (15) and (16) respectively

4. Numerical Experiment

The effectiveness of the proposed method depends on how well the two main importance choices, which are i) set of twist parameters and ii) length of effective period, are made. The following sections investigate, through numerical experiment, these choices and provide recommendations of how to select them.

4.1. Effect of Twist parameters

As mentioned in section 3.2, the good set of twist parameters is the one that increase the probability of importance event, which is when the first-passage time occurs within the effective period. The twist parameter within the effective period ϑ should be as negative as necessary to make the underlying share price hits the barrier within this period. The twist parameter before the effective period θ should also be negative to help driving the underlying share price to move close to the barrier before the effective period, but not so negative as to make the first-passage time tends to occur before the effective period. These were confirmed as shown in figure 1.

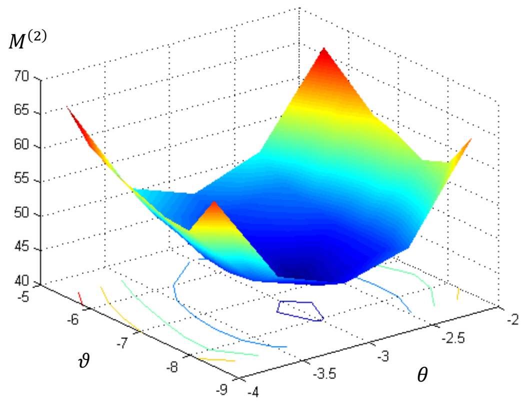

Figure 1 Numerical example of $M^{(2)}$ when vary twist parameters. The underlying has parameters as follows: $\sigma = 0.25, \lambda = 0.25$ and $\sigma_j = 0.25$.

From figure 1, the proposed method will present great variance reduction ratio, if the set of twist parameters is chosen appropriately.

Appropriate twist parameters may be chosen by optimizing $\widehat{M}^{(2)}$ which was shown in section 3.4. Note that $\widehat{M}^{(2)}$ successfully captures the variation between $M^{(2)}$ and the twist parameters as shown in figure 2.

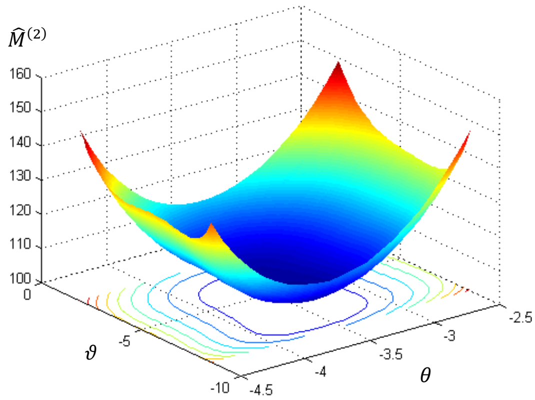

Figure 2 Numerical example of $\widehat{\mathbf{M}}^{(2)}$ when vary twist parameters. The underlying has parameters as follows: $\boldsymbol{\sigma} = \mathbf{0.25}, \boldsymbol{\lambda} = \mathbf{0.25}$ and $\boldsymbol{\sigma_j} = \mathbf{0.25}$.

The θ and ϑ which are obtained by minimizing $\widehat{M}^{(2)}$ gave not much difference variance reduction ratio compare with the optimum one as shown in table 1 so we would suggest to used this method to select appropriate twist parameters.

Table 1 Numerical example of variance reduction ratio of components when use optimum θ&ϑ compare with appropriated θ&ϑ from the suggested method when $(\alpha, \beta] = (5, 5.25]$.

Parameters			Variance Reduction Ratio	
σ	λ	σ_j	Optimum $\theta\&\vartheta$	$\theta\&\vartheta$ From $\widehat{M}^{(2)}$
0.2	0.2	0.2	99.22%	99.16%
0.4	0.2	0.4	86.65%	84.63%
0.2	0.1	0.1	99.20%	99.18%
0.3	0.2	0.1	95.18%	95.13%
0.3	0.1	0.2	95.47%	94.93%
0.15	0.2	0.3	98.98%	92.21%

The results of $X^{[\alpha,\beta]}$ simulation of sample Trigger Adjustment show significant variance reduction for every component. For example when set $\sigma \in [0.1,0.4]$, $\lambda \in [0,0.2]$, and $\sigma_j \in [0.1,0.4]$, the results show variance reduction ratio range from 84.08% to 99.99%. Another advantage of the proposed method is the discretization does not need before the effective period. This helps reduce the computational effort drastically.

4.2. Effect of Length of Effective Period

This section discusses the appropriate choice of Γ, the length of effective period. The variance of the full estimator $\mathbb{E}[X]$ can be written as a combination of h components as equation (21).

$$\sum_{i=1}^{h} \mathrm{Var}\left[\hat{X}^{[\varepsilon_{i-1},\varepsilon_i]}\right] \tag{21}$$

As mentioned in section 3.1, too small length Γ makes the first-passage time becomes too rare. On the other hand, too large Γ will increase the variation of the possible payoff. This section seeks to determine an appropriate Γ to help reducing the variance of the complete security efficiently.

To demonstrate this point, consider the following numerical experiment. Consider the following five choice of length $\Gamma = 1, 0.5, 0.25, 0.125$ and 0.0625 (i.e., annually, semi-annually, quarterly, and so on.). For a given set of underlying's parameters, we find that extreme value of Γ ($\Gamma = 1$ and 0.0625) often give high variance for estimator. The choice of Γ that give lowest variance is shown in table 2.

Table 2 Numerical example of the best choice of Γ for difference set of underlying's parameters.

σ	0.2	0.3	0.3	0.2	0.25	0.3	0.15	0.15	0.1
λ	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.15	0.2
σ_j	0.1	0.2	0.2	0.2	0.2	0.3	0.2	0.15	0.2
Best Γ	0.125	0.125	0.125	0.25	0.25	0.25	0.5	0.5	0.5

For example, for parameters $\sigma = 0.25$, $\lambda = 0.2$ and $\sigma_j = 0.2$, if Γ is chosen to be 0.125 or 0.5, the total variance of the estimate price will increase by 5% and 20% respectively compare with when $\Gamma = 0.25$. This implies the existence of an optimal value of Γ.

However, in practice, it is difficult to pinpoint the optimum value for Γ. Nevertheless, we find an approximate linear relationship between Γ and logarithm of average variance of h components. If we defined the average variance as $\bar{V} = \frac{1}{h}\sum_{i=1}^{h} \mathrm{Var}\left[X^{[\varepsilon_{i-1},\varepsilon_i]}\right]$, the following plot shows a linear trend between $\ln(\bar{V})$ and Γ.

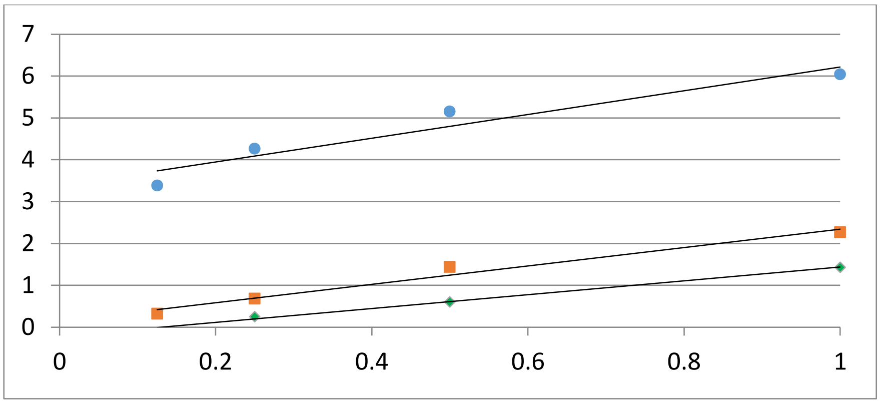

Figure 3 Numerical example of the relation of $\ln(\bar{V})$ and Γ.

Figure 3 implies an approximate relationship of the form

$$\bar{V} = Ke^{\Gamma/\Gamma^*}$$

Where, K and Γ^* are some constants. It follows that

$$\frac{\partial}{\partial \Gamma}\left[\frac{T}{\Gamma}Ke^{\Gamma/\Gamma^*}\right] = 0$$

$$\Gamma = \Gamma^*$$

Γ can be estimated by test run and fit the linear regression between $\ln(\bar{V})$ and Γ. The slope is mainly affected by source of variation of the underlying process. When the variation mainly comes from the diffusion process, the slope will be steep, and vice versa. This is because when the variation mainly comes from the jump process, with too small Γ, the jump is not likely to occur so an appropriate value of Γ^* should not be too small. Alternatively, we also found that $\Gamma^* \approx 1 - \frac{\mathbb{P}_\sigma(\tau \leq T)}{\mathbb{P}_G(\tau \leq T)}$ is a good guideline for choosing appropriate Γ for this setting, in particular, for range of length parameters used in experiment above.

Finally, The proposed method was test on many samples by varying the volatile parameters as follow: $\sigma \in [0.1,0.4]$, $\lambda \in [0,0.2]$, and $\sigma_j \in [0.1,0.4]$ and choosing θ & ϑ by minimizing $\widehat{M}^{(2)}$ and Γ as suggestion above. The number of scenario for each simulation method is computed in such a way that each method have approximated same number of underlying share price value that need to be simulated. The simulation results of the proposed method on the sample Trigger Adjustments show significantly variance reduction ratio range from 91.12% to 99.99% when compare with the basic simulation. Figure 4 shows the relation between the adjusted volatility (from the adjusted GBM) and the variance reduction ratio. We also test an effectiveness of the proposed method by varying the barrier level B while set $\sigma = 0.2, \lambda = 0.2$ and $\sigma_j = 0.2$. The results show 99.76%, 92.77%, 91.11% and 90.76% variance reduction ratio for $B = 1.5, 2.5, 4$ and 5 respectively. As expected, with low adjusted volatility or low barrier level which indicates the rarity of the first-passage event, the variance reduction ratio will be high and vice versa. The simulation time of the proposed method over the traditional method is range from 0.09 to 0.11. Please note that these ratios might vary depended on many variables such as performance of PC that used to run, coding, underlying parameters etc.

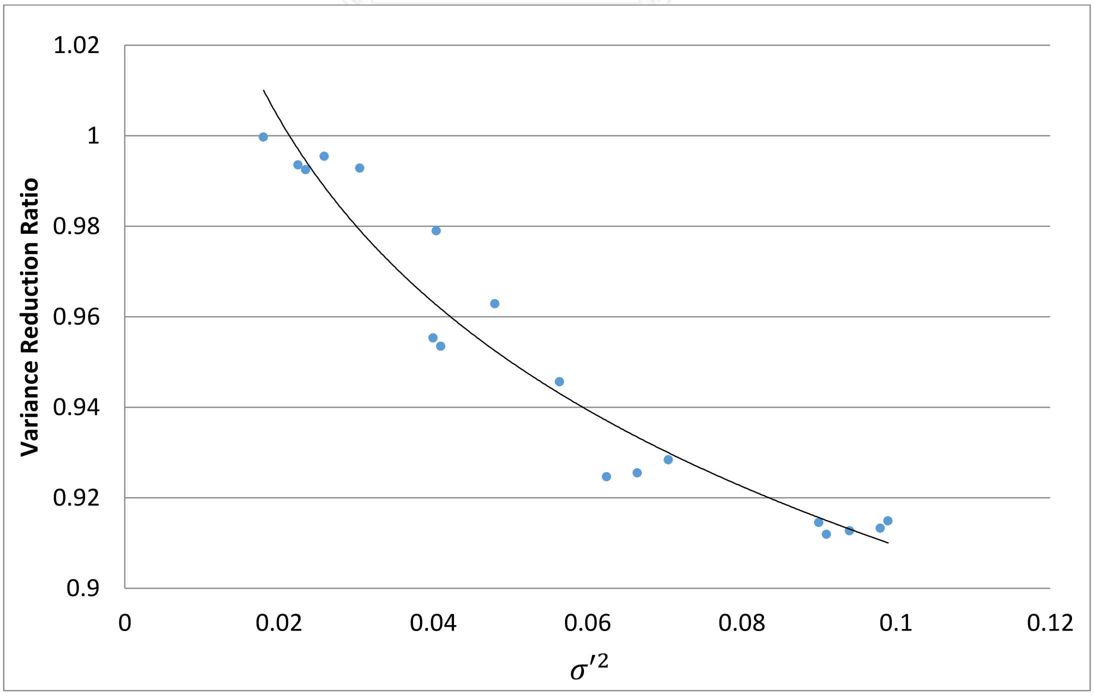

Figure 4 Numerical example of relation of variance reduction ratio and adjusted volatility

Comparing the proposed method with the exponential twist without partitioning (whose twist parameter is chosen by the same algorithm as a component with $(\alpha, \beta] = (0, T])$, the proposed method gave a better variance reduction ratio for every sample. The variance reduction ratio of the proposed method compare with the result from exponential twisted method is 67.35% on average. Furthermore, we also observe how the effectiveness of variance reduction depends on the volatile parameters of the underlying. For both proposed method and well-known exponential twisted technique, when the volatile parameters decrease, the efficiency of both variance reduction methods will be increase. This is because when the volatile parameters increase, this problem becomes less rare event problem so the methods that were designed to increase simulation efficiency of rare event problem becomes less necessary. On the other hands, when the problem becomes extremely rare (ex. = 0.15, $\lambda = 0.1$, $\sigma_j = 0.1$), the efficiency of both methods are very high. Especially for the proposed method, even when compare with the exponential twisted method, the variance reduction ratio is still 97.98%.

5. Conclusion

We have developed a simulation method for pricing the first-passage time dependent securities under the assumption that its underlying process follows the JDM. The method combines three variance reduction techniques which are partitioning, exponential twisting and conditional Monte Carlo. Partitioning is used to limit the variation of the payoff that is conditional on the first-passage time taking value within certain period. We apply a version of exponential twisting to deal with the rare non-zero payoff event and has convenient of preserving the JDM dynamic of underlying in this simulation pricing. Lastly, taking advantage of the fact that whether or not the underlying breaches the barrier is the only relevant information about the underlying path before the effective period, the conditional Monte Carlo helps avoiding discretization error and reducing the computational effort. Two main decision parameters which need to be chosen carefully to apply this method effectively are the twist parameters and the length of partition. We derived approximated second moment which is function of twist parameters in semi-closed-form which can be minimized to choose appropriated twist parameters. We also discuss about the method to select appropriate length of effective period from the numerical experiment. Finally, we applied this method to compute the price of sample securities and illustrated its effectiveness through numerical results.

Appendix 1

Lemma 3.1

When the underlying share price follows the log-Normal JDM, the increment of logarithm of the underlying share price could be defined as in equation (22).

$$\Delta \ln(S_n) = \mu\Delta t + \sigma\sqrt{\Delta t}z_n + \sum_{j=N((n-1)\Delta t)+1}^{N(n\Delta t)} \ln(Y_j) \tag{22}$$

Where, $\Delta \ln(S_n) = \ln(S_{n\Delta t}) - \ln\left(S_{(n-1)\Delta t}\right)$, z_n standard normal random variable. Let $\psi(\theta)$ be the logarithm of moment generating function of $\Delta \ln(S_n)$, which can be shown as the following

$$\begin{aligned}
\psi(\theta) &= \ln \mathbb{E}[\exp(\theta\Delta \ln(S_n))] \\
&= \ln \mathbb{E}\left[\exp\left(\theta\left(\mu\Delta t + \sigma\sqrt{\Delta t}z_n + \textstyle\sum_{j=N((n-1)\Delta t)+1}^{N(n\Delta t)} \ln(Y_j)\right)\right)\right] \\
&= \ln\left(\exp(\theta\mu\Delta t)\,\mathbb{E}\left[\exp\left(\theta\left(\sigma\sqrt{\Delta t}z_n + \textstyle\sum_{j=N((n-1)\Delta t)+1}^{N(n\Delta t)} \ln(Y_j)\right)\right)\right]\right) \\
&= \ln\left(\exp(\theta\mu\Delta t + 0.5\theta^2\sigma^2\Delta t)\,\mathbb{E}\left[\exp\left(\theta\left(\textstyle\sum_{j=N((n-1)\Delta t)+1}^{N(n\Delta t)} \ln(Y_j)\right)\right)\right]\right) \\
&= \ln\left(\begin{array}{l}\exp(\theta\mu\Delta t + 0.5\theta^2\sigma^2\Delta t) \\ \times\, \mathbb{E}\left[\mathbb{E}\left[\exp\left(\theta\left(\sum_{j=N((n-1)\Delta t)+1}^{N(n\Delta t)} \ln(Y_j)\right)\right)\middle| N\left((n-1)\Delta t\right), N(n\Delta t)\right]\right]\end{array}\right) \\
&= \ln\left(\exp(\theta\mu\Delta t + 0.5\theta^2\sigma^2\Delta t)\textstyle\sum_{m=0}^{\infty}\frac{e^{-\lambda\Delta t}(\lambda\Delta t)^m}{m!}\,\mathbb{E}\left[\exp\left(\theta\left(\sum_{j=1}^{m}\ln(Y_j)\right)\right)\right]\right) \\
&= \ln\left(\exp(\theta\mu\Delta t + 0.5\theta^2\sigma^2\Delta t)\textstyle\sum_{m=0}^{\infty}\frac{e^{-\lambda\Delta t}(\lambda\Delta t)^m}{m!}\exp\left(m\theta\bar{J} + 0.5\theta^2\sigma_j^2 m\right)\right) \\
&= \ln\left(\exp(\theta\mu\Delta t + 0.5\theta^2\sigma^2\Delta t - \lambda\Delta t)\textstyle\sum_{m=0}^{\infty}\frac{\left(\lambda\Delta t e^{\theta\bar{J}+0.5\theta^2\sigma_j^2}\right)^m}{m!}\right) \\
&= \ln\left(\exp\left(\theta\mu\Delta t + 0.5\theta^2\sigma^2\Delta t + \lambda\Delta t\left(e^{\theta\bar{J}+0.5\theta^2\sigma_j^2} - 1\right)\right)\right) \\
&= \theta\mu\Delta t + 0.5\theta^2\sigma^2\Delta t + \lambda\Delta t\left(e^{\theta\bar{J}+0.5\theta^2\sigma_j^2} - 1\right)
\end{aligned}$$

Therefore, the likelihood ratio can be written as in equation (23)

$$e^{-\theta\Delta\ln(S_n)+\psi(\theta)\Delta t} = \exp\begin{pmatrix} -\theta\ln\left(\frac{S_n}{S_{n-1}}\right) + \theta\mu\Delta t + 0.5\theta^2\sigma^2\Delta t \\ +\lambda\Delta t\left(e^{\theta\bar{J}+0.5\theta^2\sigma_J^2} - 1\right) \end{pmatrix} \quad (23)$$

To obtain the process of the underlying after the twist, the moment generating function of $\Delta\ln(S_n)$ is defined as in equation (24).

$$\mathbb{E}\left[e^{\rho\Delta\ln(S_n)}\right] = \exp\left(\rho\mu\Delta t + 0.5\rho^2\sigma^2\Delta t + \lambda\Delta t\left(e^{\rho\bar{J}+0.5\rho^2\sigma_J^2} - 1\right)\right) \quad (24)$$

Under the new measure defined by (11), the moment generating function of $\Delta\ln(S_n)$ is

$$\begin{aligned}\tilde{\mathbb{E}}\left[e^{\rho\Delta\ln(S_n)}\right] &= \exp\left(-\theta\mu\Delta t - 0.5\theta^2\sigma^2\Delta t - \lambda\Delta t\left(e^{\theta\bar{J}+0.5\theta^2\sigma_J^2} - 1\right)\right) \\ &\quad\times\exp\left((\rho+\theta)\mu\Delta t + 0.5(\rho+\theta)^2\sigma^2\Delta t + \lambda\Delta t\left(e^{(\rho+\theta)\bar{J}+0.5(\rho+\theta)^2\sigma_J^2} - 1\right)\right) \\ &= \exp\begin{pmatrix}\rho\mu\Delta t + 0.5(\rho^2+2\rho\theta)\sigma^2\Delta t \\ +\lambda\Delta t e^{\theta\bar{J}+0.5\theta^2\sigma_J^2}\left(e^{\rho\left(\bar{J}+\theta\sigma_J^2\right)+0.5\rho^2\sigma_J^2} - 1\right)\end{pmatrix} \\ &= \exp\begin{pmatrix}(\mu\Delta t + \theta\sigma^2\Delta t)\rho + 0.5\rho^2\sigma^2\Delta t \\ +\lambda\Delta t e^{\theta\bar{J}+0.5\theta^2\sigma_J^2}\left(e^{\rho\left(\bar{J}+\theta\sigma_J^2\right)+0.5\rho^2\sigma_J^2} - 1\right)\end{pmatrix} \\ &= \exp\left(\mu_\theta\Delta t\rho + 0.5\rho^2\sigma^2\Delta t + \lambda_\theta\Delta t\left(e^{\rho(\bar{J}_\theta)+0.5\rho^2\sigma_J^2} - 1\right)\right)\end{aligned}$$

$$\tilde{\mathbb{E}}\left[e^{\rho\Delta\ln(S_n)}\right] = \exp\left(\rho\mu_\theta\Delta t + 0.5\rho^2\sigma^2\Delta t + \lambda_\theta\Delta t\left(e^{\rho\bar{J}_\theta+0.5\rho^2\sigma_J^2} - 1\right)\right) \quad (25)$$

Where, $\mu_\theta = \mu + \theta\sigma^2$, $\lambda_\theta = \lambda t e^{\theta\bar{J}+0.5\theta^2\sigma_J^2}$, $N_\theta(t)$ is Poisson process with rate λ_θ, and $\bar{J}_\theta = \bar{J} + \theta\sigma^2$. As present in equation (24) and (25), the moment generating function in both equations have same form, but with difference parameters. Therefore, after the exponential twisting the dynamic of the underlying process is still JDM.

Appendix 2

Approximate the second moment of the component price estimator

Let

$\mathbb{E}^{M_\alpha}[\bullet]$: $\mathbb{E}\left[\bullet\,\middle|M_{(0,\alpha]} > B\right]$

$\mathbb{P}^{M_\alpha}[\bullet]$: $\mathbb{P}\left[\bullet\,\middle|M_{(0,\alpha]} > B\right]$

$\gamma(x)$: $x\mu + 0.5x^2\sigma^2 + \lambda\left(e^{x\bar{J}+0.5x^2\sigma_J^2} - 1\right)$

$$M^{(2)} = \widetilde{\mathbb{E}}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)^2 \mathbb{I}\left(M_{(\alpha,\beta]} \le B\right)X^2\mathcal{L}_\alpha^2\mathcal{L}_\beta^2\right]$$

$$= \widetilde{\mathbb{E}}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)\mathbb{I}\left(M_{(0,\alpha]} > B\right)\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right)X^2\mathcal{L}_\alpha^2\mathcal{L}_\beta^2\right]$$

$$= e^{2\alpha\gamma(\theta)}{S_0}^{2\theta}\widetilde{\mathbb{P}}\left(M_{(0,\alpha]} > B\right)$$

$$\times\, \widetilde{\mathbb{E}}^{M_\alpha}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right)X^2{S_\alpha}^{2(\vartheta-\theta)}{S_\tau}^{-2\vartheta}e^{2(\tau-\alpha)\gamma(\vartheta)}\right]$$

Where, the last equality follows from substituting $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ from (12).

Using the fact that when Γ is small, we can approximate X by its upper bound X_m which is maximum value of $X^{[\alpha,\beta]}$ and $\tau - \alpha$ by $\frac{1}{2}(\beta - \alpha)$

$$\approx e^{2\alpha\gamma(\theta)}{S_0}^{2\theta}B^{-2\vartheta}X_m^2\widetilde{\mathbb{P}}\left(M_{(0,\alpha]} > B\right)e^{(\beta-\alpha)\gamma(\vartheta)}$$

$$\times\, \widetilde{\mathbb{E}}^{M_\alpha}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right){S_\alpha}^{2(\vartheta-\theta)}\right]$$

$$= {S_0}^{2\theta}B^{-2\vartheta}X_m^2\widetilde{\mathbb{P}}\left(M_{(0,\alpha]} > B\right)e^{(\beta+\alpha)\gamma(\theta)}$$

$$\times\, \widetilde{\mathbb{E}}^{M_\alpha}\left[\widetilde{\mathbb{E}}^{M_\alpha}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right)\mathbb{I}\left(M_{(\alpha,\beta]} \le B\right){S_\alpha}^{2(\vartheta-\theta)}\middle|\mathcal{F}_\alpha\right]\right]$$

$$= {S_0}^{2\theta} B^{-2\vartheta} X_m^2 \widetilde{\mathbb{P}}\left(M_{(0,\alpha]} > B\right) e^{(\beta+\alpha)\gamma(\theta)}$$

$$\times \widetilde{\mathbb{E}}^{M_\alpha}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right) {S_\alpha}^{2(\vartheta-\theta)} \widetilde{\mathbb{P}}^{M_\alpha}\left(M_{(\alpha,\beta]} \leq B \middle| S_\alpha\right)\right]$$

$$= e^{(\beta+\alpha)\gamma(\theta)} {S_0}^{2\theta} B^{-2\vartheta} X_m^2 \widetilde{\mathbb{P}}\left(M_{(0,\alpha]} > B\right)$$

$$\times \widetilde{\mathbb{E}}^{M_\alpha}\left[{S_\alpha}^{2(\vartheta-\theta)} \widetilde{\mathbb{P}}^{M_\alpha}\left(M_{(\alpha,\beta]} \leq B \middle| S_\alpha\right) \widetilde{\mathbb{E}}^{M_\alpha}\left[\prod_{j=1}^{N_\theta(\alpha)+1} g\left(\Delta T_j, S_{T_{j-1}^+}, S_{T_j^-}\right) \middle| S_\alpha\right]\right]$$

Which yields (18).

Appendix 3

Approximate JDM by adjusted GBM

We approximated the JDM process of the underlying with an adjusted GBM by using a matching moment method [2]. Let set S_t' to follow the GBM whose process has the same first and second moment as process of S_t. Then the parameter of S_t' process can be obtained as the following

$$\mathbb{E}[\ln(dS_t')] = \mathbb{E}[\ln(dS_t)]$$

$$\mu_\theta' dt = \mu_\theta dt + \lambda_\theta \bar{J}_\theta dt$$

$$\therefore \mu_\theta' = \mu_\theta + \lambda_\theta \bar{J}_\theta$$

$$\mathbb{E}[\ln(dS_t')^2] = \mathbb{E}[\ln(dS_t)^2]$$

$$\sigma_\theta'^2 dt = \sigma^2 dt + \lambda_\theta\left(\bar{J}_\theta^2 + \sigma_j^2\right)dt$$

$$\therefore \sigma_\theta'^2 = \sigma^2 + \lambda_\theta\left(\bar{J}_\theta^2 + \sigma_j^2\right)$$

From this approximation, the closed-form of some terms in the approximated second moment which do not have closed-form under JDM is now available (See Karatzas and Shreve, 1991) as the following.

$$\widetilde{\mathbb{P}}\left(M_{(0,\alpha]} > B\right) \approx \widetilde{\mathbb{P}}_G\left(M_{(0,\alpha]} > B\right) = 1 - \left(N(d_\alpha^-) + \left(\frac{B}{S_0}\right)^{\left(\frac{2\mu_\theta'}{\sigma_\theta'^2}\right)} N(d_\alpha^+) \right)$$

$$\widetilde{\mathbb{P}}\left(M_{(\alpha,\beta]} \le B \middle| S_\alpha\right) \approx \widetilde{\mathbb{P}}_G\left(M_{(\alpha,\beta]} \le B \middle| S_\alpha\right) = N\left(d_\beta^-\right) + \left(\frac{B}{S_0}\right)^{\left(\frac{2\mu_\vartheta'}{\sigma_\vartheta'^2}\right)} N\left(d_\beta^+\right)$$

Where, $d_\alpha^\pm = \frac{\ln\left(\frac{B}{S_0}\right) \pm \mu_\theta' \alpha}{\sigma_\theta' \sqrt{\alpha}}$ and $d_\beta^\pm = \frac{\ln\left(\frac{B}{S_\alpha}\right) \pm \mu_\vartheta'(\beta - \alpha)}{\sigma_\vartheta' \sqrt{(\beta - \alpha)}}$.

For $\widetilde{\mathbb{E}}^{M_\alpha}\left[\prod_{j=1}^{N_\theta(\alpha)+1} \widetilde{\mathbb{P}}_T\left(M_{\Delta T_j} > B \middle| S_{T_{j-1}^+}, S_{T_j^-}\right)\middle|S_\alpha\right]$ terms, because we approximate the JDM process with the adjusted GBM process so $N(\alpha)$ should be zero.

$$\widetilde{\mathbb{E}}^{M_\alpha}\left[\prod_{j=1}^{N_\theta(\alpha)+1} \widetilde{\mathbb{P}}_T\left(M_{\Delta T_j} > B \middle| S_{T_{j-1}^+}, S_{T_j^-}\right)\middle|S_\alpha\right] \quad \approx \mathrm{g}(\alpha, \mathrm{S}_0, S_\alpha)$$

$$= 1 - exp\left(-\frac{2[\ln(B/S_0)][\ln(B/S_\alpha)]}{\alpha\sigma_\theta'^2}\right)$$

Lastly, because $\widetilde{\mathbb{E}}_G^{M_\alpha}\left[S_\alpha{}^{2(\vartheta-\theta)}\widetilde{\mathbb{P}}_G^{M_\alpha}\left(M_{(\alpha,\beta]} \le B\middle|S_\alpha\right)\mathrm{g}(\alpha, \mathrm{S}_0, S_\alpha)\right]$ is function of S_α and the probability density function of S_α given $M_\alpha > B$ is available (See Karatzas and Shreve, 1991), the conditional expectation can be computed by numerical integration.

REFERENCES

1. Bates, D.S. The Crash of'87: Was It Expected? The Evidence from Option Market. The Journal of Finance 46:3, 1009-1044, July 1991
2. Boyle, P., Broadie, M., and Glasserman, P. Monte Carlo methods for security pricing. Journal of Economic Dynamics and Control 21:8, 1267-1321, 1997
3. Glasserman, P. Monte Carlo Methods in Financial Engineering. New York: Springer, 2003
4. Karatzas, I. and Shreve, S.E. Brownian Motion and Stochastic Calculus. New York: Springer, 1991
5. Kou, S.G. A jump-diffusion model for option pricing. Management Sci. 48:8, 1086-1101, 2002
6. Kou, S.G. and Wang, H. First Passage Times of a jump diffusion process. Adv. Appl. Prob. 504-531, 2003
7. Kou, S.G. and Wang, H. Option Pricing Under a Double Exponential Jump Diffusion Model. Management science 50:9, 1178-1192, 2004
8. Merton, R.C. Option Pricing when Underlying Stock Returns are Discontinuous. Journal of Financial Economics 3:1, 125-144, 1976
9. Metwally, S.A. and Atiya, A.F. Using Brownian bridge for fast simulation of jump-diffusion processes and barrier options. Journal of Derivatives (fall) 10:1, 43-54, 2002
10. Primožič, T. Estimating expected first passage times using multilevel Monte Carlo algorithm. Master's Thesis, University of Oxford, 2011
11. Spiegeleer, J.D. Pricing contingent convertibles: A derivatives approach. Master's Thesis, Department of Mathematics,, Katholieke University Louven, 2011
12. Teneberg, H. Pricing Contingent Convertibles using an Equity Derivatives Jump Diusion Approach. Master's Thesis, SAS Institute, 2012
13. Večeř, J. Stochastic Finance: A Numeraire Approach. CRC Press, 2011

VITA

Name –Surname Prachya Mongkolkul

Date of Birth November 15, 1990

Address 25 Soi Chimphli 27 Chimphli Rd. Chimphli Tlingchan Bangkok 10170

Work place FWD Life Insurance

Position Supervisor - Corporate Actuarial (Business Partner and Planning)

Education Background

2012 Industrial Engineering Chulalongkorn University, Thailand