Credit contagion and portfolio choice

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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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งานวิจัยฉบับนี้ศึกษาเกี่ยวกับการแพร่ระบาดด้านเครดิตและการเลือกกลุ่มลงทุน งานวิจัยมุ่งเน้นที่ปัจจัยที่กระทบต่อการกระจุกตัวด้านเครดิต และ ความเสี่ยงจากการแพร่ระบาดด้านเครดิต นอกจากนี้การศึกษาได้ตรวจสอบผลของปัจจัยเหล่านี้ต่อการตัดสินใจการของธนาคารเกี่ยวกับกลุ่มลงทุนด้านสินเชื่อ วิทยานิพนธ์ประกอบด้วยสามบทความที่มีเนื้อหาครบถ้วนในตัว

บทความแรกเสนอกรอบการประเมินความเสี่ยงกระจุกตัวของกลุ่มลงทุนด้านสินเชื่อโดยประเมินเงินกองทุนทางเศรษฐศาสตร์ตลอดวัฐจักรของเศรษฐกิจ ผลการศึกษาเชิงประจักษ์โดยใช้ข้อมูลการผิดนัดชำระหนี้จากธนาคารแห่งประเทศไทยแสดงให้เห็นว่าความเสี่ยงกระจุกตัวด้านเครดิตเป็นสาเหตุหลักของเหตุการณ์การสูญเสียอย่างรุนแรงของมูลค่ากลุ่มลงทุนการให้กู้ยืมด้านสินเชื่อของธนาคารพาณิชย์ การศึกษาแสดงให้เห็นว่าปัจจัยความเสี่ยงกระจุกตัวเพิ่มความรุนแรงที่มีผลเชิงลบต่อการสูญเสียของกลุ่มลงทุนอันเนื่องมาจากตัวแปรทางเศรษฐกิจ อันได้แก่ เงินเฟ้อ ดัชนีผลผลิตอุตสาหกรรม และ การว่างงาน

บทความที่สองเสนอกรอบการวิเคราะห์สำหรับการศึกษาการแพร่ระบาดด้านเครดิตของกลุ่มลงทุนด้านสินเชื่อในเชิงการบริหารความเสี่ยงที่มีผลต่อการประเมินเงินกองทุน การประมาณค่าพารามิเตอร์ของแบบจำลองสามารถหาได้จากข้อมูลอนุกรมเวลาของประวัติการผิดนัดชำระหนี้ด้วยวิธีของโมเมนต์ แบบจำลองไม่ได้ระบุสมมติฐานที่ชัดเจนของความสัมพันธ์ทางธุรกิจหรือโครงสร้างเงินทุนของบริษัท

บทความที่สามเสนอแบบจำลองการตัดสินใจงบดุลของธนาคารเพื่อศึกษาผลกระทบของความเสี่ยงด้านการแพร่ระบาดด้านเครดิต และ ตลาดสินเชื่อรองต่อการจัดสรรกลุ่มลงทุนด้านสินเชื่อ การศึกษานี้แนะนำว่าความเสี่ยงด้านการแพร่ระบาดด้านเครดิตสร้างโอกาสให้กับธนาคารที่มีฐานเงินทุนที่แข็งแกร่งในการสร้างผลกำไรภายใต้ภาวะความบีบคั้นทางเศรษฐกิจ และตลาดสินเชื่อรองเพิ่มมูลค่าของผู้ถือหุ้น

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This research studies credit contagion and portfolio choice of loan portfolio for banking institutions. The research focuses on the factors affecting the credit concentration and contagion risk of loan portfolios and examines the effect of those risks on the bank's loan portfolio decision. The thesis comprises three self-contained essays. The first essay proposes a framework to analyze the concentration risk of loan portfolios by quantifying the amount of economic capital throughout the economic cycle. The study using the Bank of Thailand’s default data demonstrates that credit concentration risk is a major cause of extreme credit portfolio losses faced by commercial banks. It shows that the concentration risk factor intensifies the adverse effect on the portfolios’ losses from the three prominent macroeconomic variables, which are inflation, manufacturing production index and unemployment.

The second essay proposes a framework to analyze the credit contagion of loan portfolios with a risk management perspective concerning risk capital quantification. The model parameters can be estimated from the time-series of the portfolio default rate with the simple method of moments. The model does not impose any explicit assumption about the business relationship between counterparties or capital structure of the firm.

The third essay proposes a bank's balance sheet decision model to study the impact of contagion risk and the impact of loan sale market on the loan portfolio allocation. This study suggests that contagion risk creates an opportunity for the banks with the high quality of capital to increase the profit during the stress economic regime and the loan sale market enhances the shareholders' value creation.

Department:	Banking and Finance	Student's Signature	....
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CHAPTER 1: CREDIT CONCENTRATION AND CYCLICAL ECONOMIC CAPITAL OF LOAN PORTFOLIOS

1. INTRODUCTION

Basel Committee on Banking Supervision (BCBS, 2009) defines a broad definition of risk concentration as any single exposure or group of similar exposures to the same borrower, industry or risk factor with the potential to produce losses large enough to threaten a bank's creditworthiness or ability to maintain its core operations. In line with the broad definition, Basel Committee on Banking Supervision (BCBS, 2008) describes credit concentration risk as a concentrated risk exposure within a specific risk category such as single name or sector. The portfolio with exposure concentration to a single name or sector has each obligor exposes to the common risk source. Concentration risk at the aggregate financial system can be vulnerable to the economic stability as the distress event can simultaneously affect all the borrowers. Both bankers and regulators can better mitigate the risk of large loss with the better accurate level of concentration risk quantification and its impending effect in the financial system.

Macroeconomic variables partly describe the state of economy, and specify the level of credit risk at each point of the economic cycle. The fully described risk factors improve the accuracy of risk quantification and help better manage the surprise large credit losses. Duffie, Eckner, Horel, and Saita (2009) finds that the unobserved risk factor enhances a large default loss from the level determined by the observable risk factors. The latent factor is not captured by the current macroeconomic variables definition; therefore, ignoring the adverse effect of macro factors on assessing the concentration risk to a latent common factor makes the risk quantification become more inaccurate. Underestimation of risk capital can happen during the economic downturn as the concentration risk factor may move into the adverse direction. The reverse is true during the economic upturn and the overestimation may reduce credit creation to support the economic growth. This study helps identify the concentration risk factor after taking into account the effect of economic cycle on the vulnerability of the economy shared by each industry loan portfolio. The result allows for an accurate assessment of risk concentration and capital throughout the economic cycle so that banks can plan for an efficient capital allocation, a quality capital accumulation, and a long-term lending decision. Košak, Li, Lončarski, and Marinč (2015) shows that banks with high quality capital have competitive strength during the crisis period. The policy makers benefit from an accurate concentration risk quantification by being allowed for the accurate determination of the additional capital reserve requirements when they anticipates the increase in the underlying risk of the economy.

Allen, Carletti, and Marquez (2011) argues that the level of capital that can prevent the system from the crisis must be accountable for the accurate risk level of the system, and may not align with the regulatory capital level. A weak level of capital in the financial system is one of the reasons of credit crunch that deteriorates the economic wealth (Ivashina & Scharfstein, 2010). The cyclical portfolio's contribution to the riskiness of the economy due to the portfolio's exposure to the concentration risk factor should determine the amount of risk capital influenced by the concentration risk factor. If this level of riskiness directly relates to the large loss of the portfolio, which in turn contributes to the large loss of the economy, it should be able to confirm that the portfolio's risk contribution to the overall concentration risk of the economy is vulnerable to the economic sustainability.

This study measures the level of concentration risk after taking into account the effect of economic cycle on the vulnerability of the economy, examines the influences of the common risk factor on the risk capital quantification in each industry loan portfolio of the Thai loan market, and examines the explanatory of the concentration risk on the extreme loss from default across economic cycle.

In this study, I first define the concentration risk factor of the Thai loan market as the factor to which the total country portfolio composing of the equal-weight industry sub-portfolios has the highest net exposure considering the effect of macroeconomic variables on the economic fragility. The study measures the level of concentration risk of the Thai loan market using the Herfindahl index of the weighted net exposure to each of the common risk factor of the total country portfolio. The result suggests that the concentration risk factor would lead to a potential large loss during the unfavourable economic scenario. The degree of economic vulnerability due to the risk concentration varies by the economic cycle. The risk concentration would not make a portfolio at risk during strong growth periods as high as during the downturn periods. I quantify the time-varying risk capital of the loan portfolio due to the concentration risk factor and examine whether the factor explains the large loss of the portfolio. The result suggests that this level of concentration risk is vulnerable to the large loss of the total Thai loan market.

This article makes three contributions to the literature. First, to the best of my knowledge, it is the first study that provides a multi-factor macro-economic linked default rate model to quantify the capital that incorporates the effect of concentration risk and considers the stage of economy described by the macro factors. The proposed macro-linked multi-factor macro model framework is equipped with the parameter estimation framework that overcomes the unavailability of the default timing data, and the dynamic of asset value of loan portfolios. The approximation of the asset value dynamic is to use the equity value, which is also not available for the non-listed firm. The framework allows a closed-form expression to estimate the default rate model at the portfolio level linking with macro-economic variables. The asset value is required under a structural based default rate model such as Merton (1974), Vasicek (1991), Pykhtin (2004), Gordy and Lütkebohmert (2013). Most practitioners and researchers infer the asset value from the market value of equity and use it for parameter calibration, such as in Crosbie and Bohn (2003) and Bharath and Shumway (2008). The default timing at the obligor level is the required input data for parameter estimation by the intensity based model framework including Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), Lando (1998), Carling et al. (2007), Das, Duffie, Kapadia, and Saita (2007), and Bharath and Shumway (2008).

Second, it assesses the influence of the concentration risk on the cyclical risk capital quantification. It finds that the concentration risk factor alters the amount of risk capital to the different direction throughout the economic cycle. It increases the risk capital during the distress regime and decreases it during the good economic regime. The result shows that the risk capital based on the concentration risk factor does better to safeguard the portfolio loss during the stress period. It is less conservative during the low risk period; therefore, it helps banks to unleash their risk taking capacities during the favorable economy.

Third, it assesses how the concentration risk explains the large default loss at each industry loan portfolio that may lead to a total large loss in the Thai loan market. The study finds that the important macroeconomic variables strengthen the adverse effect of the concentration risk factor on the portfolio loss.

In what follows, I discuss the model framework and the parameter estimation approach. I use the historical data of default rates from the Bank of Thailand reported by commercial banks operating in Thailand. Based on the estimated model, I identify the concentration risk factor, measure the level of concentration, and evaluate if this concentration risk causes a large default loss. Later, I show that the risk concentration alters the risk capital and the concentration risk factor affects the demanded capital throughout the economic cycle.

2. METHODOLOGY

2.1. The model

Due to its broad implications to portfolio management including capital allocation, portfolio risk management, and the pricing of credit portfolio instruments as well as its nice features to describe the default mechanism, the structural model family (first developed in a single factor framework by Merton (1974) and further by (Vasicek, 2002)) has been adopted widely by financial institutions. Several extended versions into multifactor frameworks include well-known Credit Portfolio Managers from Moody's, CreditMetrics from JPMorgan, and many others as discussed in Crouhy, Galai, and Mark (2000) and Altman and Saunders (1998). Both assume that the borrower defaults if the value of the borrower's asset at maturity falls below the obligation value. Though its clear default mechanism is beneficial and simplified, the structural model posits a challenging implementation issue. This involves the estimation of the model parameters including the asset return correlation and the default boundary. It is generally known that the asset value and default boundary are not observable because the market data are not available. Therefore, most practitioners and researchers infer asset value from the market value of equity and use them for parameter calibration, such as in Crosbie and Bohn (2003) and Bharath and Shumway (2008). The requirement of a marketable value of equity is prohibitive for the use of this approach in estimating the portfolio model where the obligors are non-listed firms. Nonetheless, equity inferred asset correlation estimation is not straightforward, and it requires a numerical procedure.

My approach adapts the framework from Vasicek (2002) and Jakubik and Schmieder (2008) to construct the model and addresses the above-mentioned calibration issues. As a result, the calibrated models from my methodology serve my study objective for concentration risk analysis.

Let $\tilde{L}_j$ denote the portfolio loss of a homogeneous portfolio of industry j, where the portfolio has $\hat{K}_j$ number of homogeneous loans of equal notional amount in the portfolio. Assume the notional amount of each loan is $1/\widehat{K}_j$. Assuming the loss given default is equal to one, $\tilde{L}_j$ is given by:

$$\tilde{L}_j = \sum_{i=1}^{\widehat{K}_j} \tilde{L}_{ij},$$

where $\tilde{L}_{ij}$ is $1/\widehat{K}_j$ if loan i in industry j defaults or 0 if loan i in industry j does not default. At time T, the i^{th} borrower in portfolio j defaults with probability

$$p_{ij} = P\left[A_{ij}(T) < B_j(T|x)\right]. \tag{1}$$

where $A_{ij}(T)$ is the i^{th} borrower's asset value in the portfolio of industry j, which reflects his ability to pay at time T and $B_j(T|x)$ is the default barrier, which reflects the obligation of the representative borrower to pay and his willingness to pay in portfolio of industry j at time T. The barrier depends on the vector of economic factors x. The economic factors are an adjustment of the representative borrower of portfolio j throughout the economic cycle, which may reflect the potential change in the financial obligation to other financial liabilities besides those indicated by the loan contract. The difference between the default barrier and the wealth level determines the likelihood to default.

The asset level of the i^{th} borrower in portfolio of industry j represents his wealth level. The log asset value at time T is assumed to be:

$$logA_{ij}(T) = logA_{jo} + \mu_j T - \frac{1}{2}\sigma_j^2 T + \sigma_j\sqrt{T}U_{ij}, \tag{2}$$

where U_{ij} is a standard normal random variable for loan i in industry j, μ_j is drift, and σ_j is volatility of asset return.

Let us specify $B_j(T|x)$ as follows:

$$B_j(T|x) = \alpha_j e^{\sigma_j\sqrt{T}\beta_j{}'x}, \tag{3}$$

where α_j is a constant parameter representing the debt obligation of any borrower i in portfolio of the industry j, and β_j is an m-dimensional vector of coefficients representing the sensitivities of the default barrier to the change in each of the macroeconomic variable. The term $e^{\sigma_j\sqrt{T}\beta_j{}'x}$ is to allow the default barrier to vary by economic cycle. Note that, without loss of generality, I write $\sigma_j\sqrt{T}$ and $\beta_j{}'$ separately to segregate the sensitivity of the macro variables from the asset volatility.

By (2) and (3), I obtain the following default probability:

$$p_{ij}(x) = P\left[A_{ij}(T) < B_j(T|x)\right] = P\left[U_{ij} < \frac{log\alpha_j - logA_{jo} - \mu_j T + \frac{1}{2}\sigma_j^2 T}{\sigma_j\sqrt{T}} + \beta_j{}'x\right] \quad (4)$$

$$= \Phi\left(c_j + \beta_j{}'x\right), \quad (5)$$

where $\Phi(\cdot)$ is the cumulative probability distribution function of a standard normal random variable and c_j is the static component of the default boundary and is given by:

$$c_j = \frac{log\alpha_j - logA_{jo} - \mu_j T + \frac{1}{2}\sigma_{ij}^2 T}{\sigma_j\sqrt{T}}. \quad (6)$$

Observe that $p_{ij}(x)$ does not depend on i because all the loans in the same industry are identical in their default characteristics.

To model default dependence between loans, I assume that the standard normal random variables U_{ij} are jointly standard normal with equal pairwise correlation $\sqrt{\rho_j}$ with the common risk factor W_j of industry j and $\sqrt{1-\rho_j}$ with the idiosyncratic risk factor Z_{ij}, where Z_{ij} and W_j are independent. The random variable U_{ij} is given by:

$$U_{ij} = W_j\sqrt{\rho_j} + Z_{ij}\sqrt{1-\rho_j}. \quad (7)$$

To allow the correlation of asset returns between industries, I assume that the common risk factor W_j of industry j is a composite factor of the independent common risk factors $Y = (Y_1, Y_2, \ldots Y_J)'$ with the following relationship:

$$W_j = \sum_{k=1}^{J} c_{kj} Y_k.$$

where $\sum_{k=1}^{J} c_{kj} = 1$. Therefore, the model assumes that there are J systematic risk factors commonly shared by J industry loan portfolios and (7) can be represented by:

$$U_{ij} = \sum_{k=1}^{J} v_{kj} Y_k + Z_{ij}\sqrt{1-\rho_j}, \quad (8)$$

where v_{kj} is $c_{kj}\sqrt{\rho_j}$. Conditional on W_j, the probability of default of i in j is:

$$P\left[\tilde{L}_{ij} = 1 \middle| W_j, x\right] = \Phi\left(\frac{c_j + \beta_j{}'x - W_j\sqrt{\rho_j}}{\sqrt{1-\rho_j}}\right), \quad (9)$$

The number of defaults of portfolio j follows the binomial distribution with parameters $P[\tilde{L}_{ij} = 1|W_j, x]$ and $\widehat{K}_j$. By the law of large numbers and with the loss given default equal to one, the portfolio loss amount approaches probability of default (9). This assumption is held for large homogeneous loan portfolios. It is clear that the risk factors determining the time-varying portfolio loss rates are macroeconomic variables (x) and unobservable (latent) risk factors (Y_j). Equation (9) is similar to equation (4) in Jakubik and Schmieder (2008) and equation (3) in Vasicek (2002) with the modification of the additional term $\beta_j{}'x$.

By construction the common risk factor $W = (W_1, W_2 \ldots . W_J)'$ are jointly normal and they are independent of Z_{ij} for any loan i and industry j. I represent $Y = (Y_1, Y_2 \ldots . Y_J)'$ by J mutually independent common risk factors $\hat{Y} = (\hat{Y_1}, \hat{Y_2}, \ldots \hat{Y_J})'$, which are a principal component of W. Each industry loan portfolio shares the vector of common risk factors $\hat{Y}$. Therefore, each industry common risk factor W_j is decomposable into J principal factors by the principal component decomposition method as follows:

The vector $W = (W_1, W_2 \ldots . W_J)'$ satisfies the following system of equation.

$$W = \tilde{C}\hat{Y},$$

where $\hat{Y} = (\hat{Y_1}, \hat{Y_2}, \ldots \hat{Y_J})'$ is the vector of principal components of W and $\tilde{C}$ is the J by J matrix of principal component coefficients, which is the matrix of column eigen vector of the covariance matrix of W. Let the first principal component $\hat{Y_1}$ represents the component that explains the highest variation of industry risk factors $W_1, W_2 \ldots . W_J$, and the second principal component $\hat{Y_2}$ represents the component that explains the second highest variation of those factors, and so on.

Therefore, (8) can be written as:

$$U_{ij} = \sqrt{\rho_j}\tilde{C}_j\hat{Y} + Z_{ij}\sqrt{1 - \rho_j}, \quad (10)$$

where $\tilde{C}_j$ is the norm vector taken from the j^{th} row of $\tilde{C}$. Let $V_j = \sqrt{\rho_j}\tilde{C}_j$ and $\tilde{v}_{kj}$ be the k^{th} element of V_j , specifying the risk exposure of the portfolio of industry j to risk factor k, $\hat{Y_k}$. The relationship (10) can be rewritten as:

$$U_{ij} = \sum_{k=1}^{J} \tilde{v}_{kj}\hat{Y}_k + Z_{ij}\sqrt{1 - \rho_j}, \quad (11)$$

The value of $\sqrt{\sum_{k=1}^{J} \tilde{v}^2{}_{kj}}$ determines the asset correlation between each obligor in the portfolio j and is equivalent to $\sqrt{\rho_j}$. In addition, the value $\sqrt{\sum_{j=1}^{J} w_j^2 \sum_{k=1}^{J} \tilde{v}^2{}_{kj}}$, where $\sum_{j=1}^{J} w_j = 1$, and w_j is the weight of industry j in the total country portfolio, determines the total standard deviation of asset return of the total portfolio described by all the *K* factors.

2.2. The concentration risk factor and the level of concentration

Assuming equally weighted portfolio of industry loan portfolios, the net exposure of the portfolio to each common latent risk factor k represents the total exposure to the k^{th} common risk factor of total country portfolio, which is $\sum_{j=1}^{J} \tilde{v}_{kj}$. The concentration risk factor is determined as the common risk factor with the highest net exposure deemed as the highest contributor to the total portfolio default correlation. Since each common risk factor is standard normal random variable, the highest level of the net exposure indicates the factor that contributes the highest variance to the asset return.

I use the Herfindahl index, which is the sum square of weighted net exposure of the equally weighted portfolio to each of the common risk factors to measure the level of concentration risk of the Thai loan market. The square net exposure to each risk factor represents the variance of the total portfolio asset return caused by that risk factor. The Herfindahl index indicates the degree of default dependence between each industry in the total country portfolio because the asset return of each industry portfolio are correlated through the common risk factor. As a result, the high value of Herfindahl index represents the high potential of large loss due to the high level of concentration risk.

2.3. The capital quantification

The cumulative loss distribution function $F(\cdot) = P(\tilde{L}_j \leq l|x)$ of the portfolio of industry j and its density function $f(\cdot)$ are given by:

$$F(l; c_j, \rho_j, \beta_j | x) = \Phi\left(\frac{\sqrt{1-\rho_j}\Phi^{-1}(l) - \Phi^{-1}(c_j) - \beta_j{}'x}{\sqrt{\rho_j}}\right), \quad (12)$$

$$f\left(l;c_j,\rho_j,\beta_j|x\right)=\sqrt{\frac{1-\rho_j}{\rho_j}}\exp\left(-\frac{1}{2\rho_j}\left(\sqrt{1-\rho_j}\Phi^{-1}(l)-\Phi^{-1}(c_j)-\beta_j{}'x\right)^2+\frac{1}{2}\left(\Phi^{-1}(l)\right)^2\right), \tag{13}$$

where l is the portfolio loss rate. The loss at q-quantile of the loss distribution function is given by:

$$L_{qj}|x=\Phi\left(\frac{\sqrt{\rho_j}\Phi^{-1}(q)+\Phi^{-1}(c_j)+\beta_j{}'x}{\sqrt{1-\rho_j}}\right). \tag{14}$$

See the proof of (12) through (14) in Appendix A.

Conditional on x and $\widehat{Y_k}$, the cumulative loss distribution function $\bar{F}(\cdot)=P(\tilde{L}_j\leq l|\widehat{Y_k},x)$ is given by:

$$\bar{F}\left(L_q;c_j,\rho_j,\beta_j,V_j\middle|\widehat{Y_k},x\right)=\Phi\left(\frac{\sqrt{1-\rho_j}\Phi^{-1}(l)-\Phi^{-1}(c_j)-\beta_j'x+\tilde{v}_{kj}\widehat{Y_k}}{\parallel V_{kj}\parallel}\right). \tag{15}$$

where $V_j=(\tilde{v}_{1j},..\tilde{v}_{J,j})$, $V_{kj}=(\tilde{v}_{1j},..,\tilde{v}_{k-1,j},\tilde{v}_{k+1,j},\dots,\tilde{v}_{J,j})$ and $\parallel V_{kj}\parallel$ be the norm of V_{kj}. The proof of (15) is similar to the proof of (12) in Appendix A. Equation (15) allows to solve the conditional loss quantile using the formulation in (16) in order to study the implication of the exposure to the common risk factor ($\tilde{v}_{kj}$) on high loss quantile. The conditional loss quantile of portfolio j on the k^{th} principal component and economic variables x is given by:

$$\overline{L_{qJk}}|\widehat{Y_k},x=\Phi\left(\frac{\parallel V_{kj}\parallel\Phi^{-1}(q)+\Phi^{-1}(c_j)+\beta_j'x-\tilde{v}_{kj}\widehat{Y_k}}{\sqrt{1-\rho_j}}\right). \tag{16}$$

See proof in Appendix B. The portfolio loss quantile conditional on the k^{th} principal component, $\overline{L_{qJk}}|\widehat{Y_k},x$, can be interpreted as an economic capital at a q confident level by industry j under a particular economic regime defined by a state vector $[\widehat{Y_k},x]$.

2.4. The parameter estimation

To estimate the parameters of each portfolio (β_j,c_j,ρ_j). I maximize the log-likelihood function of the form:

$$\max_{\beta_j,c_j,\rho_j}\sum_{n=1}^{N}(\log(f\left(l_{jn};c_j,\rho_j,\beta_j\middle|x_n\right)\cdot g(x_n|x_{n-1},\dots x_0)), \tag{17}$$

where $g(x_n|x_{n-1},\dots x_0)$ is the probability density function of state vector x_n conditional on the entire historical value $x_{n-1},\dots,x_0$ and N is the number of observations, and the likelihood function $f(l_{jn};c_j,\rho_j,\beta_j|x_n)$ is given by (13). It is assumed that the conditional loss distribution l_n on the macro factors x_n is independent across time, and the dynamic of x_n does not depend on l_n.

Since the first order condition of (17) does not involve $g(x_n|x_{n-1},\dots,x_0)$, (17) can be reduced to[1]

$$\max_{\beta_j,c_j,\rho_j}\sum_{n=1}^{N}\log(f(l_{jn};c_j,\rho_j,\beta_j|x_n). \tag{18}$$

Note that Jakubik and Schmieder (2008) maximized the likelihood of binomial distribution of the number of defaults with probability following (9). This approach requires the knowledge of the number of credits in each period as well as the need for numerical integration to integrate out the latent variable in (9). My proposed approach does not require numerical integration and the default data at the loan level; therefore, the implementation is more economical and is suitable for the large loan portfolio default rate model.

3. EMPIRICAL ANALYSIS, RESULTS, AND DISCUSSION

3.1. The data

The historical default data, calculated as the new non-performing loan of the current period as a percentage of loan outstanding of the industry of the same period, of five industrial portfolios were obtained from the Bank of Thailand for parameter estimation and empirical analysis. These include agriculture, manufacturing, commerce, real estate, and personal consumption. The first four of the five portfolios are loans to corporations, while the last one is a retail loan portfolio. The data covers the period from the third quarter of 2001 to the fourth quarter of 2014.

The economic variables are chosen as follows: 1) real gross domestic product (GDP) growth rate representing the direction of economy, 2) stock market price index return representing the financial market condition, 3) seasonally adjusted unemployment rate reflecting the health of the economy, 4) seasonally adjusted

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[1] See (Hayashi, 2000) for conditional maximum likelihood estimation.

manufacturing production index (MPI) as an indicator of the cyclical growth of an economy, 5) real effective exchange rate index determining competitive advantage of country's export, 6) headline inflation as an indicator of the stage of economic cycle, and 7) weighted average deposit interest rate indicating the financial cost to corporate investments. The weighted average deposit interest rate directly relates to the lending interest rate as the deposit interest rate is the cost of fund borne by banks.

Seasonally adjusted unemployment rate (UMP), weighted average deposit interest rate of all commercial banks (WIR), real effective foreign exchange rate index (FX), and seasonally adjusted manufacturing production index (MPI) are from Bank of Thailand. Headline Inflation (INF) and GDP index are taken from Thomson Reuters. SET price index is taken from Bloomberg. The data cover the period from the 3rd Quarter of 2001 to the 4th Quarter of 2014. The monthly data series are available for MPI, WIR, and SET, and the quarterly data series are available for UMP, WIR, GDP, and INF. Since the default rate data is available on a quarterly basis, I take the last month of the quarter to form the quarterly macro factors of MPI, WIR, and SET data.

Because nominal income incorporates inflation, the information regarding the level of inflation and real income (through GDP measure) captures the nominal income completely. Assuming the role of credit on the growth of property price, the theoretical model of McQuinn and O'Reilly (2008) captures the important roles of the interest rate and disposable income indicative of the lending capacity as the key determinants in the boom and bust in property price. Therefore, levels of interest rate, inflation, and GDP are inferable to the level of property price index. Nominal income and the housing price index are excluded in this analysis for the above-mentioned reason. The GDP and unemployment are regarded as fundamental to the default rate in the literature (Figlewski, Frydman, & Liang, 2012; Jakubik & Schmieder, 2008). It is well known that stock price is a leading indicator of economic status, and it has been taken as a macro-economic covariate in default risk modelling (Bharath & Shumway, 2008; Duffie et al., 2009; Lando & Nielsen, 2010). The pairwise correlations of economic variables are shown in Table1. There is no unit of measure of GDP, SET, FX and MPI as they are index data. The unit of measure of inflation and interest rate is percentage. The symbols GDP, SET, FX, and MPI represent the change versus 1 year ago in the logarithm of the value. The transformation to log return of the index data of GDP, SET, FX, and MPI is taken to standardize the variables, which are comparable to the percentage measure; therefore, the pair-wise correlation can be interpreted as the correlation of the percentage movement between two variables.

Table1: Correlations between economic variables.

	GDP	SET	UMP	MPI	FX	INF	WIR
GDP	1	0.59	0.09	0.83	0.44	-0.12	-0.04
SET		1	-0.02	0.55	0.34	-0.31	-0.33
UMP			1	0.27	-0.07	-0.40	0.04
MPI				1	0.31	-0.27	-0.18
FX					1	0.05	0.23
WIR						1	0.36

The correlations between the default rate and the economics variables are reported in Table 2. The data cover the period from the 3rd Quarter of 2001 to the 4th Quarter of 2014, which is the entire data series available.

The change in GDP and MPI positively correlate with the default rate, while inflation negatively correlates with the default rate. This may seem counterintuitive; however, these economic variables and the default rates are cyclical and auto correlated, which requires some lead time for the variables to take the effect (i.e., the drop in MPI may recede when the default rate is turning to peak). Since the policy maker targets the inflation, moderate inflation is a sign of healthy economics, and this explains the negative relationship with the default rate.

Table 2: Time-series correlation between portfolio default rate and economic variables.

Industry	GDP	SET	UMP	MPI	FX	INF	WIR
Agriculture	0.10	-0.11	0.67	0.21	0.07	-0.25	0.32
Manufacturing	0.10	-0.19	0.67	0.28	-0.10	-0.31	0.14
Commerce	0.12	0.01	0.78	0.26	-0.03	-0.35	0.23
Real Estate	0.14	-0.05	0.56	0.21	-0.02	-0.17	0.10
Personal Consumption	0.03	-0.06	0.72	0.15	-0.01	-0.30	0.22

3.2. The estimated parameters

3.2.1. The lag selection

Before I estimate the model, I identify the potential economic variables and their lags to form problem (18). To do that, the univariate test of statistical significance using the likelihood ratio is conducted by comparing the likelihood of problem (18) obtained from the standard Vasicek's model, which is (18) without any macro factor, and from the model with one economic factor. To identify the lag variable, the likelihood of each variable is compared against that of its own lag variable (the previous period data) and only the one with the highest value is chosen. The most significant lag of each economic variable from the univariate test enters the multivariate model. If there is at least one variable that is not significant in the multivariate test, the second most significant lag of that economic variable is replaced and the multivariate model is re-estimated. The multivariate model is re-estimated by changing the lag until all the variables are significant. If none of the lag is significant in the multivariate model, that variable is dropped. If there are more than one variable that is not significant in the multivariable model, the economic variable that has the highest number of significant lags in the univariate test is chosen for a new lag selection. The new significant lag is chosen or dropped before the new lag selection of the next economic variable that has the second highest number of significant lags can start, and so on.

I consider the number of lag from 0 to 4 lags for the SET index, GDP, manufacturing production index, foreign exchange, inflation and real interest rate, and the number of lag from 0 to 2 for the unemployment rate due to the limited historical data, which is available from the 1st Quarter of 2001. Instead of moving the data period to start in the 1st Quarter of 2002 in order to have four lags data for the unemployment rate, I choose to reduce the number of lags to two in order to start the analysis from the 3rd quarter of 2001, consistent with the default rate data. This is to make it closest possible to the Thailand banking crisis in 1998.

Table 3 shows the result of the univariate test of each economic variable and each lag period. Table 4 shows the estimation result of the model with the set of variables whose lag value corresponds to the most significant lag for each economic variable from the univariate test. Most industry portfolios give the significant result of all variables at 1% significance level except the personal consumption industry portfolio in which the FX and MPI are insignificant, and inflation in manufacturing portfolio is significant at 10%. Therefore, the second most significant FX lag variable for personal consumption portfolio is chosen to the model, which is lag 2; however, this lag variable is not significant either. Then, the least significant FX lag, which is lag 4 is chosen but this lag variable is still not significant. Therefore, I remove FX from the model. The MPI is still insignificant in the model without FX. Therefore, the MPI is removed since there is only one lag that is significant in the univariate test. The final model excludes FX and MPI, and only UMP, INF and WIR enter the model with the most significant lag from the univariate test.

Table 3: The univariate estimated parameters of industrial loan portfolios.

Variable	Lag	Parameter	Agriculture	Manufacturing	Commerce	Real Estate	Personal Consumption
GDP	0	ρ (x10^{-2})	6.5900	5.7300	5.2500	8.7400	2.4400
		C (x10^{-3})	1.5000	1.3000	1.0500	1.3000	0.9300
		β_{GDP}	0.0078	0.0050	0.0069	0.0089	0.0016
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	1	ρ (x10^{-2})	6.5700	5.7500	5.2600	8.7700	2.4400
		C (x10^{-3})	1.4900	1.3100	1.0500	1.3100	0.9400
		$\beta_{GDP(-1)}$	0.0085	0.0038	0.0067	0.0078	0.0004
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	2	ρ (x10^{-2})	6.4400	5.6600	5.2200	8.7400	2.4300
		C (x10^{-3})	1.4300	1.2500	1.0300	1.3000	0.9200
		$\beta_{GDP(-2)}$	0.0122	0.0081	0.0083	0.0088	0.0025
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	3	ρ (x10^{-2})	6.4700	5.6800	5.2500	8.5800	2.4400
		C (x10^{-3})	1.4400	1.2600	1.0400	1.2300	0.9300
		$\beta_{GDP(-3)}$	0.0116	0.0075	0.0072	0.0132	0.0017
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	4	ρ (x10^{-2})	6.5800	5.6700	5.2700	8.6900	2.4400
		C (x10^{-3})	1.4900	1.2500	1.0500	1.2700	0.9500
		$\beta_{GDP(-4)}$	0.0084	0.0078	0.0064	0.0106	-0.0003
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
SET	0	ρ (x10^{-2})	6.6100	5.5400	5.3300	8.7900	2.4200
		C (x10^{-3})	1.6800	1.4400	1.1400	1.4700	0.9600
		β_{SET}	-0.0027	-0.0045	-0.0008	-0.0027	-0.0014
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	1	ρ (x10^{-2})	6.6300	5.5400	5.2900	8.8700	2.3300
		C (x10^{-3})	1.6800	1.4400	1.1600	1.4400	0.9800
		$\beta_{SET(-1)}$	-0.0024	-0.0045	-0.0019	-0.0006	-0.0031
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	2	ρ (x10^{-2})	6.7000	5.7000	5.3300	8.8000	2.4100
		C (x10^{-3})	1.6200	1.4100	1.1400	1.3800	0.9700
		$\beta_{SET(-2)}$	0.0001	-0.0024	-0.0005	0.0024	-0.0017
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	3	ρ (x10^{-2})	6.6200	5.7700	5.3000	8.6200	2.4300
		C (x10^{-3})	1.5700	1.3600	1.1000	1.3200	0.9300
		$\beta_{SET(-3)}$	0.0025	0.0002	0.0017	0.0046	0.0008
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	4	ρ (x10^{-2})	6.5800	5.7700	5.2100	8.6400	2.3800
		C (x10^{-3})	1.5500	1.3500	1.0700	1.3200	0.9100
		$\beta_{SET(-4)}$	0.0031	0.0006	0.0032	0.0044	0.0022
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
UMP	0	ρ (x10^{-2})	3.1300	3.2400	1.8500	5.5200	1.0200

Variable	Lag	Parameter	Agriculture	Manufacturing	Commerce	Real Estate	Personal Consumption
		c (x10^{-3})	0.4900	0.5000	0.3300	0.4300	0.4500
		β_{UMP}	0.2797	0.2345	0.2746	0.2745	0.1728
		P-Value	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
	1	ρ (x10^{-2})	3.2300	2.6500	1.5400	3.9400	0.7900
		c (x10^{-3})	0.5200	0.4500	0.3200	0.3300	0.4300
		$\beta_{UMP(-1)}$	0.2586	0.2444	0.2689	0.3123	0.1747
		P-Value	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
	2	ρ (x10^{-2})	3.8400	3.6300	1.7200	4.6500	1.0800
		c (x10^{-3})	0.6400	0.6100	0.3700	0.4200	0.4900
		$\beta_{UMP(-2)}$	0.2036	0.1757	0.2274	0.2505	0.1376
		P-Value	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
MPI	0	ρ (x10^{-2})	6.1900	5.1800	4.8500	8.3800	2.3600
		c (x10^{-3})	1.5000	1.2400	1.0400	1.3100	0.9100
		β_{MPI}	0.0156	0.0168	0.0151	0.0154	0.0063
		P-Value	0.0326**	0.0127**	0.0200**	0.0676*	1.0000
	1	ρ (x10^{-2})	6.2000	5.2400	4.9200	8.2700	2.3700
		c (x10^{-3})	1.5000	1.2500	1.0400	1.2900	0.9200
		$\beta_{MPI(-1)}$	0.0154	0.0159	0.0140	0.0171	0.0058
		P-Value	0.0352**	0.01867**	0.0323**	0.0425**	1.0000
	2	ρ (x10^{-2})	6.0500	5.0900	4.7700	8.2200	2.3300
		c (x10^{-3})	1.4700	1.2200	1.0200	1.2800	0.9100
		$\beta_{MPI(-2)}$	0.0177	0.0181	0.0164	0.0179	0.0072
		P-Value	0.0156**	0.0072***	0.0115**	0.0343**	1.0000
	3	ρ (x10^{-2})	5.9600	5.1900	4.8200	8.0800	2.3000
		c (x10^{-3})	1.4500	1.2300	1.0200	1.2600	0.9000
		$\beta_{MPI(-3)}$	0.0191	0.0169	0.0160	0.0200	0.0081
		P-Value	0.0094***	0.0135**	0.0156**	0.01927**	0.0736*
	4	ρ (x10^{-2})	6.1300	5.1200	4.8700	7.8600	2.3900
		c (x10^{-3})	1.4600	1.2100	1.0200	1.2200	0.9200
		$\beta_{MPI(-4)}$	0.0171	0.0182	0.0154	0.0230	0.0051
		P-Value	0.02396**	0.0090***	0.0220**	0.0076***	1.0000
FX	0	ρ (x10^{-2})	6.6700	5.7400	5.3400	8.8700	2.4400
		c (x10^{-3})	1.6000	1.3900	1.1300	1.4300	0.9500
		β_{FX}	0.0078	-0.0087	0.0001	-0.0018	-0.0005
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	1	ρ (x10^{-2})	6.6200	5.7500	5.3000	8.8700	2.3700
		c (x10^{-3})	1.5900	1.3800	1.1400	1.4300	0.9600
		$\beta_{FX(-1)}$	0.0142	-0.0071	-0.0095	-0.0022	-0.0131
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	2	ρ (x10^{-2})	6.6700	5.7700	5.2100	8.8700	2.2600
		c (x10^{-3})	1.6100	1.3600	1.1500	1.4300	0.9700
		$\beta_{FX(-2)}$	0.0075	0.0002	-0.0171	-0.0010	-0.0202
		P-Value	1.0000	1.0000	1.0000	1.0000	0.0400**

Variable	Lag	Parameter	Agriculture	Manufacturing	Commerce	Real Estate	Personal Consumption
	3	ρ (x10^{-2})	6.7000	5.7600	5.1400	8.8500	2.2200
		c (x10^{-3})	1.6300	1.3600	1.1600	1.4400	0.9700
		$\beta_{FX(-3)}$	0.0010	0.0046	-0.0211	-0.0078	-0.0222
		P-Value	1.0000	1.0000	1.0000	1.0000	0.0220**
	4	ρ (x10^{-2})	6.6500	5.7700	5.1500	8.8300	2.2900
		c (x10^{-3})	1.6500	1.3600	1.1600	1.4500	0.9700
		$\beta_{FX(-4)}$	-0.0110	0.0022	-0.0212	-0.0102	-0.0189
		P-Value	1.0000	1.0000	1.0000	1.0000	0.0584*
INF	0	ρ (x10^{-2})	6.0300	5.3800	4.6600	8.7100	2.1700
		c (x10^{-3})	2.1300	1.6900	1.4900	1.6400	1.1400
		β_{INF}	-0.0920	-0.0706	-0.0918	-0.0451	-0.0574
		P-Value	0.0140**	0.0436**	0.0057***	1.0000	0.0107**
	1	ρ (x10^{-2})	6.3400	5.5500	5.0100	8.7800	2.3400
		c (x10^{-3})	1.9900	1.6100	1.3800	1.5900	1.0600
		$\beta_{INF(-1)}$	-0.0668	-0.0528	-0.0638	-0.0340	-0.0354
		P-Value	0.0756*	1.0000	0.0568*	1.0000	1.0000
	2	ρ (x10^{-2})	6.4800	5.5800	5.1500	8.6800	2.3900
		c (x10^{-3})	1.9100	1.5900	1.3200	1.6600	1.0200
		$\beta_{INF(-2)}$	-0.0521	-0.0485	-0.0484	-0.0494	-0.0241
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	3	ρ (x10^{-2})	6.6800	5.6000	5.2000	8.5400	2.4200
		c (x10^{-3})	1.7100	1.5700	1.2800	1.7400	0.9900
		$\beta_{INF(-3)}$	-0.0161	-0.0460	-0.0406	-0.0659	-0.0146
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
	4	ρ (x10^{-2})	6.7000	5.6700	5.2500	8.6600	2.4400
		c (x10^{-3})	1.6300	1.5300	1.2600	1.6800	0.9700
		$\beta_{INF(-4)}$	-0.0002	-0.0352	-0.0331	-0.0517	-0.0070
		P-Value	1.0000	1.0000	1.0000	1.0000	1.0000
WIR	0	ρ (x10^{-2})	6.3700	5.7100	5.1500	8.6100	2.3400
		c (x10^{-3})	1.0200	1.1200	0.7800	0.9300	0.7200
		β_{RIR}	0.1085	0.0473	0.0832	0.0982	0.0603
		P-Value	0.0918*	1.0000	1.0000	1.0000	1.0000
	1	ρ (x10^{-2})	6.0400	5.6000	4.9300	8.5700	2.2300
		c (x10^{-3})	0.8300	0.9700	0.6500	0.9000	0.6400
		$\beta_{RIR(-1)}$	0.1520	0.0784	0.1192	0.1044	0.0854
		P-Value	0.0150**	1.0000	0.0337**	1.0000	0.0240**
	2	ρ (x10^{-2})	5.7700	5.4900	4.7500	8.4900	2.0900
		c (x10^{-3})	0.7300	0.8900	0.5900	0.8500	0.5700
		$\beta_{RIR(-2)}$	0.1770	0.0969	0.1403	0.1147	0.1064
		P-Value	0.0034***	0.0916*	0.0099***	1.0000	0.0035***
	3	ρ (x10^{-2})	5.4800	5.4000	4.5600	8.3500	1.9500
		c (x10^{-3})	0.6600	0.8400	0.5400	0.7900	0.5300
		$\beta_{RIR(-3)}$	0.1950	0.1077	0.1545	0.1289	0.1210

Variable	Lag	Parameter	Agriculture	Manufacturing	Commerce	Real Estate	Personal Consumption
		P-Value	0.0007***	0.0505*	0.0029***	0.0595*	0.0005***
	4	ρ (x10^{-2})	5.3800	5.3200	4.4100	8.2300	1.8600
		c (x10^{-3})	0.6500	0.8000	0.5100	0.7500	0.5100
		$\beta_{RIR(-4)}$	0.1946	0.1137	0.1623	0.1369	0.1259
		P-Value	0.0004***	0.0302**	0.0010***	0.0357**	0.0001***

**** Significant at 1%, ** Significant at 5%, * Significant at 10%.*

Table 4: The multivariate estimated parameters of industrial loan portfolios.

Industry	ρ (x10^{-2})	c (x10^{-3})	UMP	MPI (x10^{-3})	FX (x10^{-3})	INF (x10^{-3})	WIR
Agriculture	2.69***	3.33***	0.24***	6.75***		-36.00***	0.13***
Std.Error	0.00	0.58	2.50x10^{-03}	1.19		3.28	1.37x10^{-03}
lag			*0*	*3*		*0*	*4*
Manufacturing	3.04***	4.06***	0.21***	9.20***		-3.58*	0.05***
Std.Error	0.00	0.43	2.25x10^{-03}	1.04		2.41	1.27
lag			*1*	*2*		*0*	*4*
Commerce	1.58***	2.46***	0.24***	5.89***		-30.32***	0.10***
Std.Error	0.00	0.37	2.57x10^{-03}	0.96		3.15	1.27x10^{-03}
lag			*1*	*2*		*0*	*4*
Real Estate	5.22***	2.73***	0.26***	7.26***			0.10***
Std.Error	0.00	0.52	2.43x10^{-03}	1.62			2.67x10^{-03}
lag			*1*	*4*			*4*
Personal Consumption	0.91***	3.74***	0.16***	0.15	-0.32	-22.65***	0.07***
Std.Error	0.00	0.21	2.37x10^{-03}	0.76	2.23	2.17	0.86x10^{-03}
lag			*1*	*3*	*3*	*0*	*4*

*Following asymptotic normality of extremum estimators proposition 7.8 of (Hayashi, 2000), the asset correlation **ρ** and **c** of each portfolio are tested for statistical significance from 10^{-6} and from its average default rate respectively, and the factor loading coefficients are tested for their statistical significance from zero. The result in the table reports the factor loadings obtained by jointly estimating all significant factors according to (18). Note that the function (18) is undefined at **ρ** equal zero and **c** equal zero or one; therefore, the significant test is conducted at near zero, which is 10^{-6} for **ρ** and at the portfolio time-series average default rate for **c**. *** Significant at 1%, ** Significant at 5%, * Significant at 10%.*

3.2.2. The estimation result

The estimated parameters are provided in Table 5. The data cover the period from the 3rd Quarter of 2001 to the 4th Quarter of 2014, which is the entire data series available.

The estimated parameters are obtained from maximizing the log likelihood function (18). Table 5 reports the final model. Almost all portfolios give the significant result of UMP, MPI, INF, and WIR variables at 1% significance level except the personal consumption industry portfolio (in which only UMP, INF and WIR are significant at 1%), real estate (in which only UMP,MPI and WIR are significant at 1%), and inflation in the manufacturing portfolio is significant at 10%.

Table 5: The multivariate models of industrial loan portfolios.

Industry		ρ (x10^{-2})	c (x10^{-3})	UMP	MPI (x10^{-3})	INF (x10^{-3})	WIR
Agriculture		2.69^{***}	3.33^{***}	0.24^{***}	6.75^{***}	-36.00^{***}	0.13^{***}
	Std.Error	0.00	0.58	2.50x10^{-03}	1.19	3.28	1.37x10^{-03}
	lag			*0*	*3*	*0*	*4*
Manufacturing		3.04^{***}	4.06^{***}	0.21^{***}	9.20^{***}	-3.58*	0.05^{***}
	Std.Error	1.42x10^{-08}	0.43	2.25x10^{-03}	1.04	2.41	1.27
	lag			*1*	*2*	*0*	*4*
Commerce		1.58^{***}	2.46^{***}	0.24^{***}	5.89^{***}	-30.32^{***}	0.10^{***}
	Std.Error	1.05x10^{-08}	0.37	2.57x10^{-03}	0.96	3.15	1.27x10^{-03}
	lag			*1*	*2*	*0*	*4*
Real Estate		5.22^{***}	2.73^{***}	0.26^{***}	7.26^{***}		0.10^{***}
	Std.Error	1.43x10^{-08}	0.52	2.43x10^{-03}	1.62		2.67x10^{-03}
	lag			*1*	*4*		*4*
Personal Consumption		0.91^{***}	3.80^{***}	0.16^{***}		-22.27^{***}	0.07^{***}
	Std.Error	0.00	0.21	2.97x10^{-03}		3.44	0.77x10^{-03}
	lag			*1*		*0*	*4*

*The estimated parameters of multivariate factors after removing the insignificant variables from prior multivariate estimations. The unreported result shows that GDP, SET and FX variables are insignificant in any portfolio. Almost all portfolios give the significant result of UMP, MPI, INF, and RIR variables at 1% significance level except the personal consumption industry portfolio (in which only UMP, INF and RIR are significant at 1%), the real estate portfolio (in which only UMP,MPI and RIR are significant at 1%), and inflation in the manufacturing portfolio is significant at 10%. *** Significant at 1%, ** Significant at 5%, * Significant at 10%.*

3.3. The empirical analysis

3.3.1. Identifying the concentration risk factor

The portfolio latent factor W_j is filtered out using (9) by plugging in the historical default rate on the left-hand side of (9) and estimated parameters indicated in Table 5 and economic variables on the right-hand side. The time series W_j of each portfolio are transformed to obtain the time-series vectors of principal components and the value of exposure $\tilde{v}_{kj}$, which is the exposure to risk factor k of portfolio j represented by the k^{th} principal component.

Table 6 shows the values of the portfolio exposure ($\tilde{v}_{kj}$) to each principal component. The result suggests that the five portfolios are highly concentrated to the first principal risk component with positive exposure in all portfolios. The absolute net exposure (ANE) to each principal component is the absolute value of the sum of the exposure to that principal component across all industries. The absolute weighted net exposure to each principal component is the equally weighted average of ANE. I define the normalized weighted exposure (NWE) as the absolute weighted net exposure normalized by the sum of the absolute weighted net exposure from all factors. The NWE of the total portfolio to the first principal component is the highest; therefore, the first principal component is the concentration risk factor. The Herfindahl index of the total country portfolio is calculated as the sum square of NWE of each factor, and is equal to 0.2525 by which the first three principal risk factors highly contribute to the Herfindahl index. The Herfindahl index value is between zero and one.

Table 6: Exposure of industry loan portfolio to principal components.

Industry	1st Component	2nd Component	3rd Component	4th Component	5th Component
Agriculture	0.06	-0.07	-0.04	0.13	0.01
Manufacturing	0.04	-0.07	0.02	-0.06	0.14
Commerce	0.06	-0.06	0.00	-0.06	-0.07
Real Estate	0.01	-0.01	0.22	0.05	-0.03
Personal Consumption	0.07	0.06	0.00	0.00	0.01
ANE	0.24	0.15	0.20	0.06	0.06
NWE	0.34	0.21	0.28	0.08	0.08
Herfindahl Index			0.2525		

Panel a) of Table 7 depicts the correlation between the latent factors W_j and the economic variables. The symbols GDP, SET, FX, and MPI represent the change versus 1 year ago in the logarithm of the value. The data cover the period from 3rd Quarter of 2001 to 4th Quarter of 2014. The percentages of the total variance explained by each component are as follows: the first component (65.92%), the second component (27.73%), the third component (3.78%), the fourth component (1.47%), and the fifth component (1.09%).

The GDP and SET are not significant in the univariate tests. The FX is not significant in the multivariate model. The low absolute correlation of these macro factors with the latent factors W_j of any portfolios, which range from -0.02 to 0.15, confirms the insignificance of these macro factors. In contrast, unemployment is the most significant variable that correlates highly with each latent factor W_j with the level ranging from 0.38 to 0.7 in absolute value. Panel a) and Panel b) of Table 7 show that GDP and SET index exhibit low correlation with every latent factors and every principal component. In contrast, Panel b) of Table 7 shows that unemployment is highly negatively correlated with the first principal component, while inflation is positively correlated. The percentages of the total variance explained by each component are as follows: the first component (65.92%), the second component (27.73%), the third component (3.78%), the fourth component (1.47%), and the fifth component (1.09%).

Table 7: Time-series correlation between common risk factors and economics variables.

Latent Factor	GDP	SET	UMP	MPI	FX	INF	RIR
$W_{Agriculture}$	-0.15	-0.11	-0.69	-0.34	-0.02	0.79	0.06
$W_{Manufacturing}$	-0.11	-0.03	-0.67	-0.37	0.08	0.76	0.14
$W_{Commerce}$	-0.14	-0.16	-0.73	-0.35	0.03	0.83	0.10
$W_{Real\ Estate}$	-0.13	0.11	-0.38	-0.19	-0.07	-0.11	0.03
$W_{Personal\ Consumption}$	-0.07	-0.20	-0.50	-0.16	0.05	0.31	0.16

Panel A) The correlation between the latent factors and the economic variables.

Principal Components	GDP	SET	UMP	MPI	FX	INF	RIR
1st Component	-0.13	-0.20	-0.72	-0.30	0.04	0.64	0.15
2nd Component	0.08	-0.06	0.28	0.21	0.01	-0.53	0.04
3rd Component	-0.09	0.15	-0.19	-0.11	-0.04	-0.31	0.03
4th Component	-0.08	0.04	-0.08	-0.02	-0.18	-0.13	-0.10
5th Component	0.07	0.22	0.04	-0.08	0.16	-0.03	0.16

Panel B) The correlation between the principal components and the economic variables.

The sign of the exposure of the portfolio to each principal component determines how the wealth level of the portfolio that is influenced by that principal component. From equations (2) and (8), we have: $logA_{ij}(T) \propto \sum_{k=1}^{J} v_{kj}Y_k$. The portfolio with the positive exposure to the principal component will be adversely affected by the decrease in the value of that principal component. On the other hand, the sign of the factor loading of the portfolio to each of the economic factor determines how the default behavior is influenced by that economic factor. From (3), we have $\log\left(B_j(T|x)\right) \propto \sum_{k=1}^{m} \beta_{kj}x_k$. In addition, how the portfolio's exposure to the principal component (v_{kj}) and to the economic variable (β_{kj}) impact the risk of the portfolio depends on the correlation between the economic factor and the principal component. The effect of the positive exposure to the first principal component of the portfolios seems to be offset by their positive exposures to the interest rate since the first principal risk component positively correlates with interest rate. For example, an increase in the interest rate, which results in a higher default barrier, is likely to happen together with an increase in the first principal component, which results in a higher asset value. However, the opposite is true for unemployment, MPI, and inflation. The factor loadings of unemployment and MPI of all the portfolios are positive, while they are negative for inflation; however, the correlation between MPI and the first principal component, and unemployment and the first principal component are negative, and the correlation between inflation and the first principal component is positive. These three economic factors amplify the adverse effect of the concentration risk factor on the downside of portfolio loss during the distress regime. It can be seen that the first principal risk component tends to increase the risk to the portfolio loss when the unemployment, MPI, and inflation increase the risk to the industries. This demonstrates that these three economic factors amplify the adverse effect of the concentration risk factor on the downside of portfolio loss during the distress regime. Therefore, it is conjectured that the highest exposure to the first principal risk component could be a major source of large loss faced by each industry loan portfolios.

The next section examines how the concentration risk factor influences the risk capital quantification and its implication to the risk measurement and cyclical economic capital.

3.3.2. The influence of concentration risk factor on the risk capital

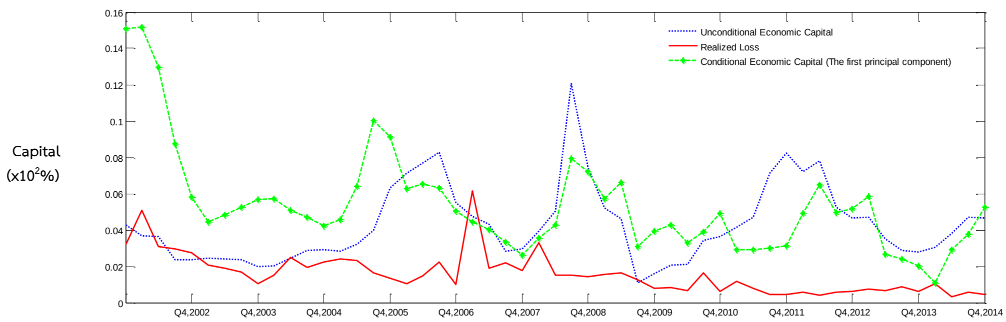
Let me introduce the definition of the risk capital as follows:

Definition A: *The T-horizon* unconditional *Capital at Risk at q confidence level is the amount of capital, or unconditional economic capital (EC), required to absorb the loss of loan portfolio over a target horizon T within a given confidence level q. The q-unconditional capital at risk or q-unconditional economic capital of portfolio* ***j*** *is measured by equation (14) using the loss and economic data of current period to estimate the unconditional EC of the next horizon.*

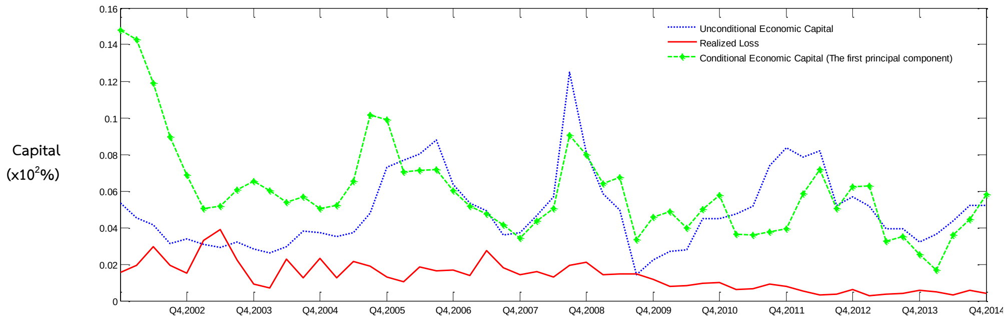
Definition B: *The T-horizon* K^{th} *component Conditional Capital at Risk at q confidence level is the amount of capital, or* K^{th} *Component conditional economic capital (EC), required to absorb the loss of loan portfolio over a target horizon T within a given confidence level q considering the realization of the* K^{th} *principal component . The q-conditional on* K^{th} *component capital at risk or economic capital of portfolio* ***j*** *is* *measured by equation (16) using the loss and economic data of current period to estimate the conditional EC of the next horizon.*

Employing conditional risk capital as a loss cushion is more effective than using the unconditional capital because it does better in loss cushion and not overestimate risk. Figure 1 shows the estimated risk capitals of the portfolio loss quantile at 99.99% for the unconditional capital (dotted line), the first component conditional capital (dashed line), and the realized loss (solid). The estimated economic capital of each quarter takes the loss and economic data from the current period to compute the capital for the loss in the next quarter. The realized loss is the loss of the current period and the EC is estimated using the loss and economic data from the previous quarter. The capital requirement measured in Figure 1 is time-varying throughout the economic cycle. The number of periods in which the realized loss breaches the unconditional portfolio EC is as follows: agriculture (6), manufacturing (3), commerce (13), real estate (0), and personal consumption (15). The number of periods the realized loss breaches the conditional portfolio EC are as follows: agriculture (1), manufacturing (0), commerce (0), real estate (0), and personal consumption (8). Therefore, this suggests that the conditional EC safeguards against a portfolio shortfall during high risk better than the unconditional EC does. Conversely, the conditional EC is less conservative during tranquil periods. For example, as shown in Figure 1 the unconditional EC of personal consumption portfolio during the period from Q4-2013 to Q4-2014, which is the period of low realized loss, demands higher amount of capital than the conditional EC.

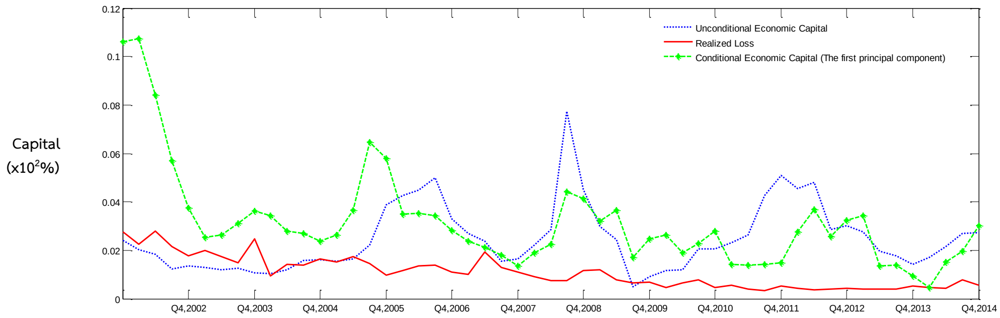
Figure 1: Portfolio Economic Capital (EC) and realized loss.



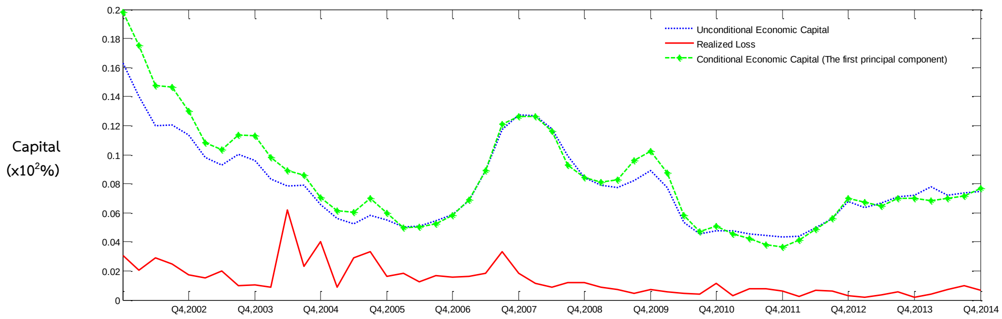
Agriculture



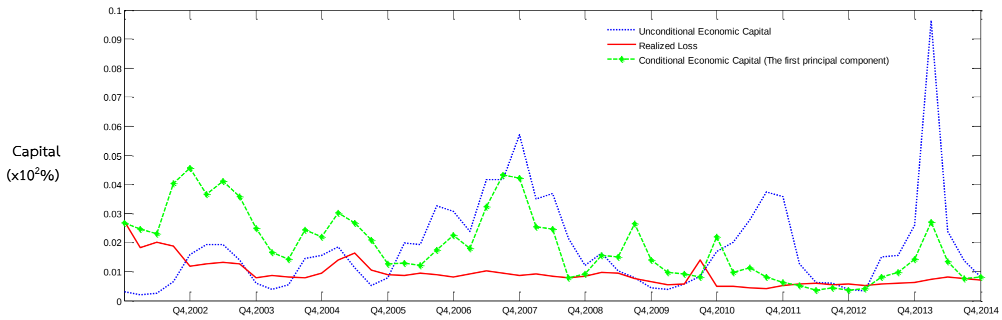
Manufacturing



Commerce



Real Estate

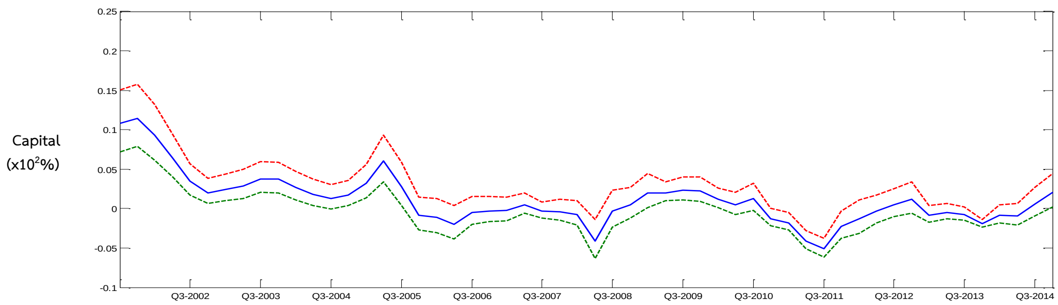


Personal Consumption

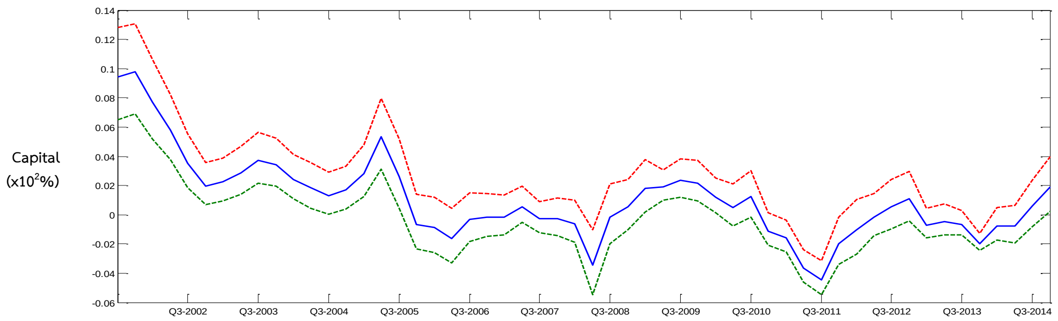
The estimated risk capital of the portfolio loss quantile at 99.99%. The unconditional capital (dashed line), the first component conditional capital (dotted line), and the realized loss (solid line).

Considering the filtered principal component and the set of economic variables realized in the same period, I vary the concentration risk factor for three standard deviations of its return distribution from the realized value and evaluate the amount of conditional EC in excess of the unconditional one. Figure 2 shows the extra EC, the conditional EC in excess of the unconditional one, to the change in the concentration risk factor. The dotted lines show the extra EC evaluated at the plus and minus three standard deviations from the realized concentration risk factor value. As expected, the extra amount of the capital of the real estate portfolio is less sensitive to the variation of the concentration risk factor as the range of excess capital (with the standard deviation of the range at 0.0028) compared to those of other portfolios, whose values are more than 0.01. This is because the real estate portfolio's exposure to the concentration risk factor is the lowest of all those of the others.

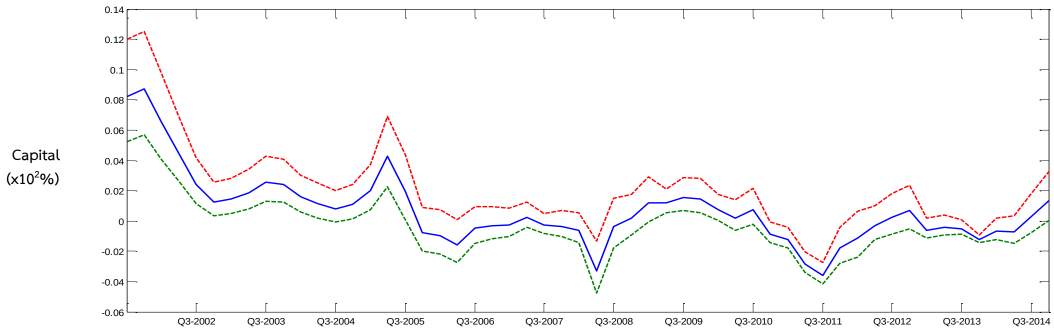
Figure 2: Factor sensitivity of EC to the change in the first principal risk component.



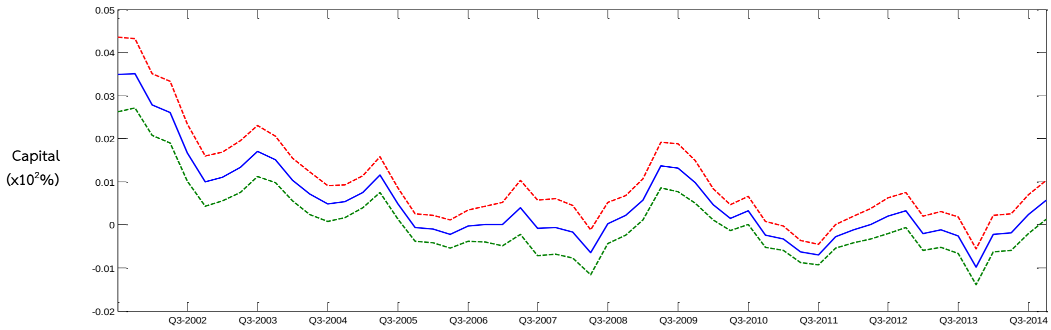
Agriculture



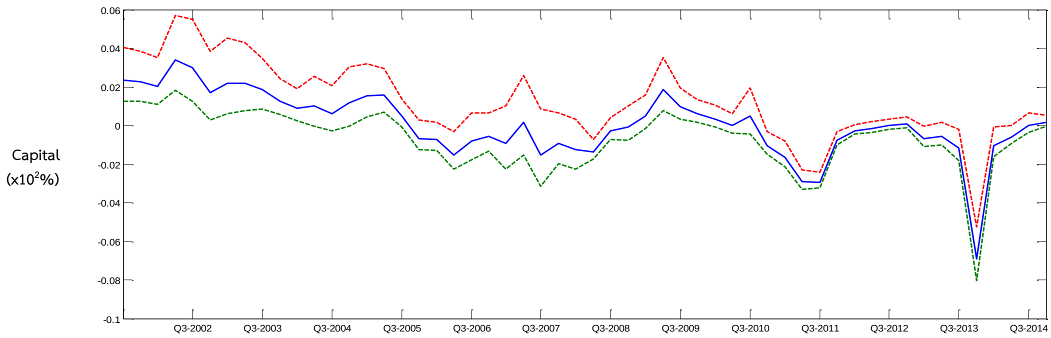
Manufacturing



Commerce



Real Estate



Personal Consumption

The standard deviations of the range of the excess capital of each portfolio are as follows: agriculture (0.0139), manufacturing (0.0102), commerce (0.0126), real estate (0.0028), and personal consumption (0.0105).

The difference of the amount of risk capital conditional on the concentration risk factor and that of the unconditional capital determines the portfolio risk level due to the concentration risk factor throughout the economic cycle. Since each industry portfolio is a part of the economy, the amount of the different capital of each industry portfolio determines the risk level contributed to loss of the economy due to the exposure to the concentration risk factor of that industry. The portfolio's risk contribution to the overall concentration risk of the economy could be substantial if the risk level measured by the different amount of capital explains the large loss of the portfolio, which in turn contributes to the large loss of the economy.

The next section examines whether the determined risk level by the concentration risk factor explains the large portfolio loss.

3.3.3. The explanatory power of the concentration risk factor to the large default loss

From the previous section, it shows that all the portfolios have the positive exposure to the concentration risk factor and this factor correlates with many economic variables, which are inflation, MPI and unemployment, in the direction that influences the default risk of each industry portfolio at the same time. However, this factor correlates with the interest rate in the direction that cancels the contribution to the default risk. In addition, the diversification due to the other common risk factor should reduce the event of simultaneous extreme losses of these industry portfolios. Therefore, the test to understand if the concentration risk factor is a major source of risk that creates large default losses of those industrial portfolios is performed.

I conduct the test whether the extreme loss event, the event that the amount of loss exceeds the specified quantile $\tilde{q}$ of the loss distribution conditional on the economic regime, can be determined by the excess capital demanded by each principal risk factor (the conditional economic capital measured by (16) over the unconditional economic capital measured by (14)).

The probit model of each portfolio takes the following form:

$$P[l_{jt} > L_{\tilde{q},j}|x_t] = \Phi\left(\hat{b}_{0k} + \hat{b}_k(\overline{L_{qjk}}|\hat{Y}_{k,t-1}, x_{t-1} - L_{q,j}|x_{t-1})\right), \quad (19)$$

where $\overline{L_{qjk}}|\hat{Y}_{k,t-1}, x_{t-1}$ is the capital conditional on the k^{th} principal component, $\hat{Y}_k$, and $\hat{b}_{0k}$ and $\hat{b}_k$ are the parameter-pair to be estimated. In addition, $\overline{L_{qjk}}|\hat{Y}_{k,t-1}, x_{t-1}$ and $L_{qj}|\, x_{t-1}$ are defined in (14) and (16) at q = 0.9999.

The analysis is done for the loss threshold $\tilde{q}$ at 0.5, 0.8, 0.9 and 0.999. The result shows that only the concentration risk factor explains the loss exceeding the median and 80$^{\text{th}}$ quantile of the loss distribution conditional on the economic regime of every five industries portfolios. In addition, only this concentration risk factor explains the loss beyond 99.9$^{\text{th}}$ quantile of agriculture, manufacturing, commerce, and personal consumption. Since all these four industries have relatively large exposures to the first principal component and the real estate has a very low exposure, which is 0.01, the first principal component is a risk factor that drives simultaneous loss at high quantile of agriculture, manufacturing, commerce, and personal consumption but not the real estate. This can be concluded that the concentration risk factor explains the large default loss in four out of five industry portfolios. That means if the state of economy with the concentration risk factor adversely affect the default probability, those four portfolios are risk for simultaneous default leading to large loss, in which each portfolio can exceed the 99.9^{th} quantile of loss distribution. Let us observe that the third principal component drives the loss above 99.9^{th} quantile of the real estate portfolio. This is because the real estate portfolio has a high exposure to the third principal component. Therefore, the large loss in the real estate portfolio happens due to a risk factor different from the factor that drives the large loss in other industries.

Table 8: The principal component as a predictive variable to excess default rate.

Principal Component	Industry	Agriculture	Manufacturing	Commerce	Real Estate	Personal Consumption
1^{st} PC	$\hat{b}_0$	0.14	-0.51**	0.35*	0.41**	0.68***
	$\hat{b}_1$	37.75***	42.12***	49.22***	45.50*	65.25***
2^{nd} PC	$\hat{b}_0$	0.64***	-0.02	0.63***	0.59***	0.84***
	$\hat{b}_1$	42.09***	25.47***	45.19**	-28.07	60.70***
3^{rd} PC	$\hat{b}_0$	0.38**	-0.08	0.41**	1.64***	0.58***
	$\hat{b}_1$	41.71	93.47	-515.33	17.51**	13456.24**
4^{th} PC	$\hat{b}_0$	1.16***	0.02	0.78**	0.57***	0.51***
	$\hat{b}_1$	55.83**	28.05	96.73	1.02	270.54
5^{th} PC	$\hat{b}_0$	0.34*	0.74	0.68**	0.57***	0.87***
	$\hat{b}_1$	237.81	38.96*	64.56	-23.55	3300.04**

*Panel a) The principal component as the factor determining a predictive variable to predict one quarter ahead the excess default rate, which is measured by the realized loss exceeding the loss at 50 quantile of the conditional loss distribution $L_{\tilde{q}j}|x$ at $\tilde{q}$ equal to 0.5. *** Significant at 1%, ** Significant at 5%, * Significant at 10%.*

Principal Component	Industry	Agriculture	Manufacturing	Commerce	Real Estate	Personal Consumption
1^{st} PC	$\hat{b}_0$	-0.19	-0.95***	0.13	-0.53**	0.46*
	$\hat{b}_1$	33.1***	47.47***	80.66***	41.82**	123.31***
2^{nd} PC	$\hat{b}_0$	0.29	-0.31*	0.55***	-0.31*	0.63***
	$\hat{b}_1$	38.14***	14.51*	67.71***	-22.88	87.91***
3^{rd} PC	$\hat{b}_0$	0.11	-0.34*	0.26	0.51	0.29
	$\hat{b}_1$	30.55	75.03	-365.36	14.65**	7125.14
4^{th} PC	$\hat{b}_0$	0.94**	-0.06	0.89***	-0.29*	0.29
	$\hat{b}_1$	58.61**	63.33	169.84**	26.57	-861.45
5^{th} PC	$\hat{b}_0$	0.06	0.84	0.65**	-0.29	0.53**
	$\hat{b}_1$	-145.36	55.71**	97.01*	106.24	2947.1**

*Panel b) The principal component as the factor determining a predictive variable to predict one quarter ahead the excess default rate, which is measured by the realized loss exceeding the loss at 50 quantile of the conditional loss distribution $L_{\tilde{q}j}|x$ at $\tilde{q}$ equal to 0.8. *** Significant at 1%, ** Significant at 5%, * Significant at 10%.*

Principal Component	Industry	Agriculture	Manufacturing	Commerce	Real Estate	Personal Consumption
1st PC	$\hat{b}_0$	-0.45**	-1.2***	-0.07	-0.61***	0.33
	$\hat{b}_1$	40.84***	47.71***	43.28***	-3.96	109.84***
2nd PC	$\hat{b}_0$	0.12	-0.47**	0.39*	-0.64***	0.54**
	$\hat{b}_1$	39.69***	14.71*	70.59***	12.57	79.52***
3rd PC	$\hat{b}_0$	-0.02	-0.51***	0.12	1.37**	0.23
	$\hat{b}_1$	40.68	16.83	-201.52	40.21***	5896.96
4th PC	$\hat{b}_0$	1.24***	-0.22	0.75**	-0.63***	0.24
	$\hat{b}_1$	91.76***	62.71	174.91**	-0.2	-879.42
5th PC	$\hat{b}_0$	-0.09	0.93*	0.49*	-0.62***	0.47**
	$\hat{b}_1$	-159.51	69.01***	96.2*	78.31	2823.47**

Panel c) The principal component as the factor determining a predictive variable to predict one quarter ahead the excess default rate, which is measured by the realized loss exceeding the loss at 50 quantile of the conditional loss distribution $L_{\tilde{q}j}|x$ *at* $\tilde{q}$ *equal to 0.9. *** Significant at 1%, ** Significant at 5%, * Significant at 10%.*

Principal Component	Industry	Agriculture	Manufacturing	Commerce	Real Estate	Personal Consumption
1st PC	$\hat{b}_0$	-1.27***	-1.87***	-0.8***	-1.86***	-0.35
	$\hat{b}_1$	34.86***	24.24***	28.77***	12.79	150.25***
2nd PC	$\hat{b}_0$	-0.64***	-1.29***	-0.42**	-1.83***	0
	$\hat{b}_1$	11.49	6.82	51.51**	-39.75	29.76
3rd PC	$\hat{b}_0$	-0.65***	-1.33***	-0.52***	-1.09	-0.11
	$\hat{b}_1$	30.32	91.14	1985.87	14.22	4931.19
4th PC	$\hat{b}_0$	-0.23	-1.35***	-0.05	-1.78***	-0.1
	$\hat{b}_1$	33.7	-7.68	142.07*	23.86	-990.77
5th PC	$\hat{b}_0$	-0.72***	-1.2*	0.09	-1.79***	-0.02
	$\hat{b}_1$	-286.05	5.39	184.3**	75.75	1149.1

Panel d) The principal component as the factor determining a predictive variable to predict one quarter ahead the excess default rate, which is measured by the realized loss exceeding the loss at 50 quantile of the conditional loss distribution $L_{\tilde{q}j}|x$ *at* $\tilde{q}$ *equal to 0.999. *** Significant at 1%, ** Significant at 5%, * Significant at 10%.*

The results show that the concentration risk factor vastly influences the portfolio's risk contribution to the overall concentration risk of the economy. The different amount of capital due to this concentration risk factor explains the large loss of the portfolio, which is a part of the loss of the economy.

4. CONCLUSION

I identify the concentration risk factor of the aggregated country loan portfolio using the Thailand data and quantify the level of the concentration risk. I show that the concentration risk factor explains the large default loss of four out of five industry beyond the 99.9th quantile of the conditional loss distribution and the concentration risk factor tends to increase the adverse effect on the portfolios' losses from the three prominent macroeconomic variables, which are inflation, manufacturing production index and unemployment. I provide the link between the concentration risk factor and the macro variables in quantifying the economic capital that is time-varying by the stage of the economy. In addition, I quantify the regime-dependent economic capital due to the concentration risk factor. The study shows that the conditional economic capital based on the concentration risk factor safeguards against the portfolio loss better than the unconditional economic capital. In addition, the conditional economic capital based on the concentration risk factor is less conservative during the low default risk period. The framework proposed in this study enables banks to estimate the multi-factor macro-linked default rate model that can be applied to assess the level of concentration risk of any other total loan portfolio as well as to evaluate the degree of impact from the concentration risk to the large loss of loan portfolios.

APPENDIX

Appendix A.

- *The proof of (12):*

Denote $P[L_{ij} = 1 | W_j, x]$ by $p(W_j|x)$. Following the proof in Vasicek (1991), the cumulative probability conditional on x that the percentage loss on a portfolio does not exceed l is given by:

$$\bar{\bar{F}}(l; c_j, \rho_j, \beta_j, \hat{K}_j | x_T) = \sum_{i=0}^{[l\hat{K}_j]} \binom{\hat{K}_j}{i} \int_0^1 p(W_j|x_T)^i (1 - p(W_j|x_T))^{\hat{K}_j - i} dP[p(W_j|x_T)], \qquad \text{(A.1)}$$

where $[l\hat{K}_j]$ is the floor function of $l\hat{K}_j$.

By the law of large numbers, when $\hat{K}_j$ is infinitely large and l is significantly greater than zero, it is satisfied by:

$$\lim_{\hat{K}_j \to \infty} \sum_{i=1}^{[l\hat{K}_j]} \binom{\hat{K}_j}{i} p(W_j|x)^i \left(1 - p(W_j|x)\right)^{\hat{K}_j - i} = \begin{matrix} 0 \; if \; \hat{K}_j \leq \; p(W_j|x) \\ 1 \; if \; \hat{K}_j > \; p(W_j|x) \end{matrix}. \qquad \text{(A.2)}$$

Therefore, a large loan portfolio cumulative loss distribution function (A.1) is given by:

$$\lim_{\hat{K}_j \to \infty} \bar{\bar{F}}(l; c_j, \rho_j, \beta_j, \hat{K}_j \, | x_T) = F(l; c_j, \rho_j, \beta_j | x_T) = P[p(W_j|x_T) \leq l], \qquad \text{(A.3)}$$

From (9), we get:

$$F\left(l; c_j, \rho_j, \beta_j \middle| x_T\right) = \Phi\left(\frac{\sqrt{1-\rho_j}\Phi^{-1}(l) - \Phi^{-1}(c_j) - \beta_j{}'x}{\sqrt{\rho_j}}\right), \quad \text{(A.4)}$$

which is given by (12). Q.E.D.

- ***The proof of (13):***

By definition, I get:

$$f\left(l; c_j, \rho_j, \beta_j \middle| x\right) = \frac{dF\left(l; c_j, \rho_j, \beta_j \middle| x\right)}{dl} = \frac{dF\left(l; c_j, \rho_j, \beta_j \middle| x\right)}{d\Phi^{-1}(l)} * \frac{d\Phi^{-1}(l)}{dl}. \quad \text{(A.5)}$$

It is clear that:

$$\frac{dF\left(l; c_j, \rho_j, \beta_j \middle| x_T\right)}{d\Phi^{-1}(l)} = \sqrt{\frac{1-\rho_j}{\rho_j}}\phi\left(l; c_j, \rho_j, \beta_j \middle| x_T\right), \quad \text{(A.6)}$$

where ϕ is the standard normal density function, Φ and the derivative of its inverse are given by:

$$\phi\left(l; c_j, \rho_j, \beta_j \middle| x_T\right) = \phi\left(\frac{\sqrt{1-\rho_j}\Phi^{-1}(l) - \Phi^{-1}(c_j) - \beta_j{}'x}{\sqrt{\rho_j}}\right), \quad \text{(A.7)}$$

$$\frac{d\Phi^{-1}(l)}{dl} = \frac{1}{\frac{dl}{d\Phi^{-1}(l)}}. \quad \text{(A.8)}$$

Since $l = \Phi(\Phi^{-1}(l))$, l can be written as:

$$l = \int_{-\infty}^{\Phi^{-1}(l)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \quad \text{(A.9)}$$

$$\frac{dl}{d\Phi^{-1}(l)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{[\Phi^{-1}(l)]^2}{2}}. \quad \text{(A.10)}$$

By plugging (A.8) – (A.10) into (A.5), I obtain (13).

- ***The proof of (14):***

By replacing l with L_q in (10), I obtain:

$$F\left(L_q; c_j, \rho_j, \beta_j \middle| x\right) = \Phi\left(\frac{\sqrt{1-\rho_j}\Phi^{-1}(L_q) - \Phi^{-1}(c_j) - \beta_j{}'x}{\sqrt{\rho_j}}\right) = q. \quad \text{(A.13)}$$

By taking an inverse of $F^{-1}(q; c_j, \rho_j | x)$, I obtain:

$$L_{qj}|x = \Phi\left(\frac{\sqrt{\rho_j}\Phi^{-1}(q) + \Phi^{-1}(c_j) + \beta_j{}'x}{\sqrt{1-\rho_j}}\right), \tag{A.14}$$

which is given by (14). Q.E.D.

Appendix B.

From (15)

$$\bar{F}\left(L_q; c_j, \rho_j, \beta_j, V_j \middle| \widehat{Y_k}, x\right) = \Phi\left(\frac{\sqrt{1-\rho_j}\Phi^{-1}(L_q) - \Phi^{-1}(c_j) - \beta_j' x + \tilde{v}_{kj}\widehat{Y_k}}{\|V_{kj}\|}\right) = q, \tag{B.1}$$

$$\overline{L_{qjk}} \middle| \widehat{Y_k}, x = \Phi\left(\frac{\| V_{kj} \| \Phi^{-1}(q) + \Phi^{-1}(c_j) + \beta_j' x - \tilde{v}_{kj}\widehat{Y_k}}{\sqrt{1-\rho_j}}\right). \tag{B.2}$$

Q.E.D.

CHAPTER 2: CREDIT CONTAGION MODEL OF LOAN PORTFOLIOS

1. INTRODUCTION

The default contagion is defined as the cascading defaults in payment obligation causing from the dependence between obligors through the common supply chain channel, trade linkage or the common creditor. For example, the bankruptcy of a large conglomerate firm could trigger default domino of firms in its supply network, leading to multiple large losses of portfolios with concentrated supply chain. This scenario could happen without a severe macroeconomic event. Default contagion creates large default loss because a default of one firm may lead to defaults of many other firms. Previous studies find that there are contagions in several financial crises such as Franklin and Douglas (2000), Longstaff (2010) and Giesecke and Kim (2011). Credit contagion, which is the spillover effect of credit worthiness deterioration through the counter-party network, generally covers default contagion and is a part of financial market distress.

The literatures mention two mechanics of default contagion. The first is the contagion triggered by the downturn of the economy caused by the shock to systematic risk factor, and the second is the contagion triggered by a shock to a specific risk factor of a particular firm or portfolio. After any particular firm defaults due to the systematic shock or specific shock, it increases the specific risk of the other firms in its private network, and induces a subsequence default and loss spiral. A contagion between firms can create a contagion between portfolios if the infected firms are belong to another portfolio. Without contagion, the defaults of two portfolios sharing the same common risk factors do not highly correlate during the specific distress event of one portfolio when the specific risk factor of that portfolio triggers the event. Examples of studies discussing the mechanics of default contagion include Azizpour, Giesecke, and Schwenkler (2014), and Errais, Giesecke, and Goldberg (2010). Errais et al. (2010) models the default contagion at the firm level, where the infected firms belong to the same portfolio. Their model specifies the impact of each default to the economy by the size of the debt outstanding of each default event. The factor driving the default contagion is the specific risk factor of the portfolio. The model of Azizpour et al. (2014) is different from that of Errais et al. (2010) as Azizpour et al. (2014)'s specifies the factors affecting the default contagion using the common economic factors, common unobservable factor, and specific risk factor of the portfolio. Azizpour et al. (2014)'s describes the richer sources of contagion as compared to Errais et al. (2010)'s.

Credit concentration and credit contagion can create a large loss although the source triggering the loss are different. A portfolio exposes to credit risk concentration if it has a large exposure to a single source of risk while it exposes to contagion risk if the others infect its constituents' default risks. With contagion, a portfolio without single name concentration can still be highly capital concentrated because the businesses of the obligors in that portfolio could be connected.

Estimation of portfolio risk models without the contagion component under the existence of the contagion risk poses the risk of in accurate capital quantification. The purpose of this study is to examine the implication of ignoring the contagion under the existence of contagion on the capital quantification of loan portfolio during crisis period of two major bank loan portfolios, which are corporate and retail loans. Providing an evidence showing the importance of the contagion modelling in risk capital quantification helps banks avoid the risk of undercapitalized during crisis period. In addition, this study evaluates the implication of ignoring the contagion risk in risk capital estimation error. Providing an understanding of the model risk of ignoring contagion helps banks make careful decision about their modelling choices.

The credit risk models can be classified into three groups. The first group known as structural-based models originates from the option pricing theory of Black-Scholes-Merton. Specifically, Merton (1974) defines that the firm defaults when its asset value falls below the firm's liability. Therefore, the default process is endogenous and the dynamic of the firm's asset value determines the likelihood to default. Examples are Asymptotic Single Risk Factor (ASRF), which is fundamental to the Internal-Risk-Based capital requirement (IRB) of the new Basel accord, Credit Portfolio Manager from Moody's and Credit Metrics from JPMorgan (Crouhy et al., 2000). Unlike Portfolio

Manager, ASRF assumes a single latent factor governs the default process. Credit portfolio models, which use a similar framework to Portfolio Manager, need an assumption on the liability structure and asset value dynamic at the firm level. Credit contagion portfolio models developed under this framework include Egloff, Leippold, and Vanini (2007) and Giesecke and Weber (2004). These models require expert judgment about business relationship between firms to determine the level of idiosyncratic shock dissipation. However, because of the normality assumption of the asset return distribution and continuous path of the Wiener process, the structural model cannot produce short-term defaults or jump risk that describes unanticipated large losses.

The models in the second group are the family of intensity-based models. In contrast to the structural-based models, the default processes are exogenous. Thus, the assumption about the firm's capital structure is not required. The earliest models in this group have intensity processes with constant hazard rates. Examples are Jarrow and Turnbull (1995), Jarrow et al. (1997) and Collin-Dufresne and Solnik (2001). To allow for interdependence of default times, the doubly stochastic Cox process extends the features of the constant hazard rate model. Example of this type of models are Lando (1998), Bharath and Shumway (2008), Shumway (2001) and many more. All of the aforementioned assume common risk factors drive the default correlation hence there is no characterization of the cascading default. However, many empirical tests including Jorion and Zhang (2007), Jorion and Zhang (2009), Das et al. (2007) and Lando and Nielsen (2010) confirm the existence of credit contagion. The Hawkes model goes one step further by allowing past default events to additionally explain the magnitude of the current intensity, for example, Giesecke and Kim (2011) and Yu (2007). More generally, affine jump-diffusion model has intensity process with the structure of drift, volatility, and jump intensity parameters affine on state variables. An example are Errais et al. (2010) and Gourieroux, Monfort, and Renne (2014), who assume a top-down version of a credit portfolio model with contagious jumps. In addition, another top-down single portfolio model of Azizpour et al. (2014) incorporates the Hawkes process and common observable factors to specify the source of default dependency.

Specifically, they define the contagion channel through the Hawkes process and the systematic risk through the common factors. All of these contagion models need credit securities price information or complete default history at a firm level to estimate the parameters. Hence, the calibration approaches associated with these models are not applicable for loan portfolios where only aggregate default frequency data are available. On the other hand, these approaches may require an intensive computation time and effort if the detail default data at loan level are available.

The third group of credit risk models is econometric-based. This group links the credit spread obtained from market data with the observable explanatory variables such as economic variables or accounting data of the firms. The functional forms of the relationships follow either a linear or a nonlinear model. For a comprehensive review of this group of model, see Duffie and Singleton (2003). However, the models do not specify the default correlation. There are models that link default probability with macro-economic variables such as Mckinsey's CreditPortfolioView for risk capital quantification. Nonetheless, these models do not feature default contagion.

The existing models in the literature lack features to separate the default correlation due to common factors and that due to cascading default or contagion of loan portfolios. The macroeconomic factors capture the default correlation due to the common exposure to the same risk factor while the contagion factors capture the default correlation due to the increase in the likelihood to default due to the past defaults. These two sources of default correlation model default correlation risk in different aspects and one cannot be used to replace the other. The existing models also fail to allow for parameter estimation when only limited detail of historical default data is available. The default data are usually available as the default rate time series, while the default timing data at the obligor level are quite limited. Therefore, it requires the development of the default contagion model that captures both sources of default correlation, and overcomes the data limitation. Although the model of Azizpour et al. (2014) captures both sources of default correlation, it requires a default timing at the obligor level.

Calibrating the model parameters of loan portfolio is challenging due to the data limitation both the length of historical default data, and the observability of default timing at obligor level. First, bank relies on relationship managers to review credit and determine default events on a monthly basis; therefore, only discrete time observation is available and immediate default timing is not attainable. Second, the number of credits can be so large, especially that of a retail loan portfolio. This makes it impractical for calibration using the data at the individual level. The above-mentioned literature exploits market information such as equity price, bond spread or credit-default swap spread (Bharath & Shumway, 2008; Fama & French, 1993; Friewald, Wagner, & Zechner, 2014) to calibrate the probability of default of corporations. However, it is common for loan portfolios that there is no securitization of the loan so the market price information of those loans are generally unavailable. A limited knowledge at the firm level default is a limitation of using the partial likelihood estimation to estimate the models in the literature. One of the approaches found in the literature is to employ two-step estimation. This approach first estimates the sensitivity to the common risk factor using the total default history with the add-on estimation of a contagion component in the second step. As a result, the common factor risk sensitivity may be overestimated in the first step because it may try to incorporate the correlation due to the contagion risk embedded in the data. In addition, the approach requires an assumption about the business relationship between the interconnected firms.

My approach is closet to Azizpour et al. (2014) that incorporates both sources of contagion, i.e. systematic and specific risk triggered contagion. The key is that my framework is an affine-jump diffusion, not Hawkes process. The advantage of having the affine model is to allow the derivation of the semi closed-form of moment conditions of default rates. The proposed framework features default process at the portfolio level similar to the top-down intensity based models. It has the contagion feature within the portfolio with common macroeconomic variables and common latent factor similar to Azizpour et al. (2014); however, it enhances to the literature by incorporating the contagion across portfolio and allows the semi closed-form moment conditions of default rates.

The proposed framework relies on the result in Pra, Runggaldier, Sartori, and Tolotti (2009), Giesecke, Spiliopoulos, Sowers, and Sirignano (2012) and Giesecke and Zhu (2013), which proves the law of large number of contagious portfolio default rate. That means asymptotic loss distribution of large portfolios with contagious jump intensities exists and the semi closed-form solution of moment conditions of default rates is available. With the macroeconomic variables, the model can quantify the portfolio loss quantile conditional on the economic regime.

This study makes three contributions to the literature. First, it provides the framework to estimate the default of loan portfolios that separates the quantification of default rates into two separate channels, through the channel that the loss is triggered from a systematic risk component and the channel that the loss is triggered from a specific risk component. Understanding of the channel of default correlation allows portfolio manager to allocate effectively a new investment and risk manager to prepare for a risk mitigation.

The framework overcomes the data limitation required by the general intensity based framework. This data limitation is the unavailability of the default timing at the obligor level. I develop a parameter estimation method based on the method of moments. The estimated process parameters provide the moments conditions that match the empirical moments of all portfolios simultaneously. Therefore, this approach only requires the time series of default rates at the portfolio level.

Second, it examines the impact of contagion to the capital at risk of the commercial and retail portfolios of total country U.S. banks during the crisis period. The result shows that ignoring the contagion risk during the crisis period results in an overestimation of the capital of the corporate portfolio when the stress is triggered from the systematic risk factor and an underestimation of the capital when the stress is triggered from the specific risk factors of either the corporate or retail portfolio. The overestimation of the capital in the corporate portfolio is because the default probability is overly sensitive to the systematic risk factor to compensate the missing contagion effect, which is significant for the corporate portfolio. Ignoring the contagion effect results in an underestimation of the capital when the distress is generated from the specific risk factors, as there is no loss spiral phenomena induced by the contagion mechanics.

Unlike the corporate portfolio, ignoring the contagion risk during the crisis period results in an underestimation of the capital of the retail portfolio when the stress is triggered from the systematic risk factor. The underestimation of the capital is a result of ignoring the high cross-contagion effect that intensifies the impact from systematic risk factor.

Lastly, it points out the importance of ignoring the contagion risk to capital quantification. Generally, the contagion component increases the model parameterization and makes it more flexible to match the tail of the empirical loss distribution. The result shows that ignoring the contagion effect increases the estimation error of the estimated Value at Risk capital.

In what follows, I explain the credit contagion model and the proposed framework on parameter estimation. Then, I analyze the results.

2. METHODOLOGY

2.1. The credit contagion model

I assume that there are I clusters of homogeneous firms in the economy, that the number of firms in each cluster is constant, and that the economy is continuously injecting new fund into the cluster to establish a new firm replacing defaulted firms. According to this assumption, the portfolio model is set up as follows.

The model is a top-down multi-factor default rate of loans at the portfolio level that allows credit contagion within and across portfolios. The top-down default rate models specify the default process at the portfolio level without characterization of the default rate process of each portfolio's constituent. I specify the contagion risk by allowing an increase in default intensity of other obligors after a default event of one obligor occurs. This infectious default could happen between the obligors within the same portfolio and/or across portfolios. The default of obligors within the same portfolio is named as self-exciting and the infectious default of the obligors across portfolio is named as cross-exciting.

Assume that only one credit event can happen in each given time t. Let $N_t = [N_{1,t}, \dots, N_{I,t}]'$ where $N_{i,t}$ denotes an observable counting process representing the number of defaults of loans within portfolio i occurred by time t whose intensity is $\Lambda_{i,t}$ for $i = 1, \dots, I$. Let $\Lambda_t = [\Lambda_{1,t}, \dots, \Lambda_{I,t}]'$. The intensity process Λ_t is driven by three types of risk factors; a set of J observable common risk factors $X_t = [X_{1,t}, \dots, X_{J,t}]'$, an unobservable common risk factor Y_t, and unobservable specific risk factors $\lambda_t = [\lambda_{1,t}, \dots, \lambda_{I,t}]'$. Specifically,

$$\Lambda_t = \mu^X X_t + \mu^Y Y_t + \lambda_t, \tag{1}$$

where $\mu^X \in \mathbb{R}^{I \times J}$ and $\mu^Y \in \mathbb{R}$ are factor exposures for portfolio i to observable risk factors X_t and an unobservable risk factor Y_t, respectively.

The common observable factors are defined and measurable, which can be macroeconomic variables, or the financial index variables that drive the default rates across obligors within and across portfolios. The common unobservable risk factor is latent variable, which cannot be represented by the contemporary measures; however, they drive the default risk of everyone in the economic system. The purpose of introducing the common unobservable risk factor is to capture the systematic default risk part that cannot be captured by the observable common factors. Das et al. (2007) and Duffie et al. (2009) show that the latent factor is an important systematic risk factor driving the large loss of corporates' defaults.

I assume that the observable common risk factors follow the mean-reverting process of the form

$$dX_t = \kappa(\theta - X_t)dt + \delta dW_t^X, \tag{2}$$

for $\kappa \in \mathbb{R}^{J \times J}$, $\theta \in \mathbb{R}^J$, and $\delta \in \mathbb{R}^{J \times J}$ where W^X is a standard J–dimensional Brownian motion. Similarly, the unobservable common risk factor follows the stochastic differential equation as follows:

$$dY_t = -\xi Y_t dt + dW_t^Y, \tag{3}$$

for $\xi \in \mathbb{R}$ where W^Y is a standard Brownian motion. I assume that X and Y are independent. That is Y represents a common factor not captured by X.

The unconditional distribution of common factors can be unbounded which can be described by the distribution of the random variable generated from the mean reverting process.

To capture default contagion within and across portfolios, I assume that the specific risk factor λ_t follows a mean-reverting process with mutually exciting jumps of the form:

$$d\lambda_t = \zeta(\phi - \lambda_t)dt + \sigma dW_t^{\lambda} + \eta dN_t, \tag{4}$$

for $\zeta \in \mathbb{R}^{I\times I}$, $\phi \in \mathbb{R}^{I}$, $\sigma \in \mathbb{R}^{I\times I}$, and $\eta \in \mathbb{R}^{I\times I}$ where W^{λ} is a standard I −dimensional Brownian motion. All W^{λ}, W^{X} and W^{Y} are independent. The last term in (4) represents the credit contagion effects happening within and across portfolios. Specifically, a default of a loan in portfolio j results in a jump of size $\eta_{i,j}$ in the specific risk factor $\lambda_{i,t}$ of portfolio i. The credit contagion within the portfolio i is captured by $\eta_{i,i}$ while the credit contagion across portfolios is captured by $\eta_{i,j}$ for $i \neq j$. I further assume that κ,ξ, ζ and σ are diagonal.

The linear combination of the common factors and specific risk factor determines the default intensity. Since the level of the common factor values can be negative as described by the mean reverting process, specifying the specific risk factor with the square root process with jump, which is the specification used in Errais et al. (2010), does not help avoid the negative default intensity of the portfolio. However, the simulation procedure puts the zero bound for the default intensity Λ_t.

Finally, assume that there is a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is a standard $L-$ dimensional Brownian motion $W = [W^{\lambda'}, W^{X'}, W^{Y}]'$ and an I −dimensional counting process N. Let $\mathcal{F}_t^O$ denote the filtration generated by X_t and N_t augmented by the null sets of $\mathbb{P}$, and $\mathcal{F}_t^U$ the $\mathbb{P}$ −augmented filtration generated by Y_t and λ_t. Let $\mathcal{F}_t = F_t^O \vee \mathcal{F}_t^U$, the smallest σ −field containing both $\mathcal{F}_t^O$ and $\mathcal{F}_t^U$. The filtration $\mathcal{F}_t^O$ represents the information set for observable processes, while the filtration $\mathcal{F}_t$ represents the extended information set for both observable and unobservable processes. Let $Z_t = [X_t', Y_t', \lambda_t']'$. It is clear that Z_t is a Markov process adapted to the filtration $\mathcal{F}_t$. Equations (2) - (4) can be rewritten in terms of Z_t as follows:

$$dZ_t = (Q_0 + Q_1 Z_t)dt + H^{1/2}dW_t + \Gamma dN_t, \tag{5}$$

where

$$Q_0 = \begin{bmatrix} \kappa\theta \\ 0 \\ \zeta\phi \end{bmatrix}, \quad Q_1 = \begin{bmatrix} -\kappa & 0 & 0 \\ 0 & -\xi & 0 \\ 0 & 0 & -\zeta \end{bmatrix}, \quad H = \begin{bmatrix} \delta\delta' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma\sigma' \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ 0 \\ \eta \end{bmatrix}.$$

The intensity process (1) can be rewritten as:

$$\Lambda_t = \mu Z_t, \tag{6}$$

where $\mu = [\mu^X, \mu^Y, I]$.

2.2. The parameter estimation

In this section, I describe the estimation method for the parameters of the model. The data are usually available as discrete observations, and most of the time with different frequency between the observable common risk factors X and the number of default events N. The data for the risk factors and the number of defaults is typically available at different frequency, i.e. daily, weekly, monthly or quarterly for market or economic data and quarterly or annually for default rate data. To utilize the richer availability of observable risk factor data, a two-step estimation method is applied in which the parameters associated with the dynamic of the risk factors (2) are estimated separately using a standard maximum likelihood estimation method and the estimated parameters are then used to estimate the parameters in (1), (3) and (4) using a method of moments. That is, I match the unconditional moments of N implied from the model to those implied from the number of default data. I obtain the standard maximum likelihood estimators of process parameters of X by transforming (2) into its equivalent econometric representation of autoregressive model and regressing the X_n on X_{n-1}.

The joint moments of X, Y and λ implied in the model moments justifies this two-step estimation approach. Specifically, given the known dynamic of X, the rest of the parameters within an admissible set that implicitly match the joint moments of X, Y, λ are chosen. The following subsections describe how to compute the unconditional moments of Z, and then the unconditional moments of N followed by an inference problem.

2.2.1. Unconditional Moments of Z

Given that the model defined by (5) - (6) is an affine model, the moment generating function is an affine function of Z. Precisely, for $0 \leq t < T$ and $u \in \mathbb{R}^L$ where $L = I + J + 1$, let

$$\Psi(u,t,T,Z_t) = \mathbb{E}\left[e^{u'Z_T} \mid Z_t\right] \tag{7}$$

denote the moment generating function of Z_T at u given Z_t. Following the analysis as in Duffie, Pan, and Singleton (2000), the result is as follows:

Proposition 1. Suppose that the expectation in (7) exists and is finite. Then it is given by:

$$\Psi(u,t,T,Z_t) = e^{\alpha(t)+\beta(t)'Z_t}, \tag{8}$$

where $\alpha(t) \in \mathbb{R}$ and $\beta(t) \in \mathbb{R}^L$ are the solution of the following system of ordinary differential equations (ODEs):

$$\dot{\alpha}(t) + Q_0{}'\beta(t) + \tfrac{1}{2}\beta(t)'H\beta(t) = 0, \tag{9}$$

$$\dot{\beta}(t) + Q_1{}'\beta(t) + \mu'\left(e^{\Gamma'\beta(t)} - \mathbf{1}\right) = 0, \tag{10}$$

with the terminal conditions $\alpha(T) = 0, \beta(T) = u$ where $\exp\left(\Gamma'\beta(t)\right)$ refers to an element-wise exponential function of $\Gamma'\beta(t)$, and $\mathbf{1}$ is a vector of ones.

Proof: This proposition is a special case of Duffie et al. (2000).

Observe that in general the solution $(\alpha(t), \beta(t))$ of the ODEs (9) - (10) depends on u through the terminal condition $\beta(T) = u$. To emphasize this, they are written as $\alpha(t,u)$ and $\beta(t,u)$ for $\alpha(t)$ and $\beta(t)$ with the terminal condition $\beta(T) = u$. Now the conditional moments of Z can be obtained from differentiating Ψ with respect to u at $u = 0$. For example, the first conditional moment of $Z_{i,T}$ given Z_t for $i = 1, \dots, L$ is as follows:

$$\begin{aligned} \mathbb{E}[Z_{i,T} \mid Z_t] &= \Psi_{u_i}(u,t,T,Z_t)|_{u=0} \\ &= \Psi(u,t,T,Z_t)\left(\alpha_{u_i}(t,u) + \beta_{u_i}(t,u)'Z_t\right)|_{u=0} \\ &= \alpha_{u_i}(t,0) + \beta_{u_i}(t,0)'Z_t, \end{aligned} \tag{11}$$

where $\Psi(0,t,T,Z_t) = 1$, and Ψ_{u_i}, α_{u_i} and β_{u_i} are the partial derivatives of Ψ, α and β with respect to u_i, respectively. To obtain the unconditional first moment of $Z_{i,T}$, the unconditional expectation are taken on both sides of (11), which yields:

$$\mathbb{E}[Z_{i,T}] = \alpha_{u_i}(t,0) + \beta_{u_i}(t,0)'\mathbb{E}[Z_t],$$

$i = 1, \dots, L$. Let assume that Z_t is stationary with finite expectation. I get $\mathbb{E}[Z_{i,t}] \equiv \mathbb{E}[Z_{i,0}]$ for all t. Thus, the values of $\mathbb{E}[Z_{i,0}], i = 1, \dots, L$ can be obtained from the following system of linear equations:

$$\mathbb{E}[Z_{i,0}] = \alpha_{u_i}(t,0) + \sum_{l=1}^{L} \frac{\partial \beta_l(t,0)}{\partial u_i} \mathbb{E}[Z_{l,0}], \tag{12}$$

$i = 1, \dots, L.$ Similarly, I obtain the second moments as follows:

$$\begin{aligned} \mathbb{E}[Z_{i,T} Z_{j,T} \mid Z_t] &= \Psi_{u_i,u_j}(u,t,T,Z_t)|_{u=0} \\ &= \alpha_{u_i,u_j}(t,0) + \beta_{u_i,u_j}(t,0)' Z_t \\ &\quad + (\alpha_{u_i}(t,0) + \beta_{u_i}(t,0)' Z_t)(\alpha_{u_j}(t,0) + \beta_{u_j}(t,0)' Z_t). \end{aligned} \tag{13}$$

Taking unconditional expectation on both sides of (13) and assuming the stationary property of Z give:

$$\begin{aligned} \mathbb{E}[Z_{i,0} Z_{j,0}] &= \alpha_{u_i,u_j}(t,0) + \alpha_{u_i}(t,0)\alpha_{u_j}(t,0) \\ &\quad + \left[\beta_{u_i,u_j}(t,0) + \alpha_{u_i}(t,0)\beta_{u_j}(t,0) + \alpha_{u_j}(t,0)\beta_{u_i}(t,0)\right]' \mathbb{E}[Z_0] \\ &\quad + \textstyle\sum_{1_1=1}^{L} \sum_{l_2=1}^{L} \left(\frac{\partial \beta_{l_1}(t,0)}{\partial u_i}\right)\left(\frac{\partial \beta_{l_2}(t,0)}{\partial u_j}\right) \mathbb{E}\left[Z_{l_1,0} Z_{l_2,0}\right], \end{aligned} \tag{14}$$

where $i, j = 1, \dots, L$. Given that $\mathbb{E}[Z_0]$ is known from the above calculation, the values of $\mathbb{E}[Z_{i,0} Z_{j,0}], i, j = 1, \dots, L$ can be obtained from solving the system of linear equations (14). The third moment of Z can be obtained similarly in sequence[2]. I now describe how to compute the unconditional moments of N from the unconditional moments of Z.

2.2.2. Unconditional Moments of N

For $0 \le t < T$ and $u \in \mathbb{R}^I$, let

$$\tilde{\Psi}(u,t,T,Z_t) = \mathbb{E}\left[e^{u' N_T} \mid Z_t\right] \tag{15}$$

denote the moment generating function of N_T at u given Z_t. Applying the analysis as in Duffie et al. (2000) I have the following result:

Proposition 2. Suppose that the expectation in (15) exists and is finite, and then it is given by:

$$\tilde{\Psi}(u,t,T,Z_t) = e^{\alpha(t) + \beta(t)' Z_t}, \tag{16}$$

---

[2]See Giesecke and Zhu (2013) for the condition for the existence of the moments of Z.

where $\alpha(t) \in \mathbb{R}$ and $\beta(t) \in \mathbb{R}^L$ are the solution of the following system of ordinary differential equations (ODEs):

$$\dot{\alpha}(t) + Q_0'\beta(t) + \frac{1}{2}\beta(t)'H\beta(t) = 0, \tag{17}$$

$$\dot{\beta}(t) + Q_1'\beta(t) + \mu'\left(e^{\Gamma'\beta(t)+U} - \mathbf{1}\right) = 0, \tag{18}$$

with the terminal conditions $\alpha(T) = 0, \beta(T) = 0$ where $\exp(\Gamma'\beta(t) + U)$ refers to an element-wise exponential function of $\Gamma'\beta(t) + U$, $\mathbf{1}$ is a vector of ones, and $U = [u_1, \dots, u_I]$.

Proof: This proposition is a special case of Duffie et al. (2000).

Proposition 3. Suppose the limiting distribution of $N_{i,T}$ exists for positive Z_T, then:

$$\mathbb{E}[N_{i,T}] = \alpha_{u_i}(t,0) + \beta_{u_i}(t,0)'\mathbb{E}[Z_0], \quad i = 1, \dots, I,$$

where $\alpha_{u_i}(t,0)$ and $\beta_{u_i}(t,0)$ are the partial derivatives of α and β with respect to u_i and α and β are the solution to (17)-(18).

Proof. See Appendix A.

Assuming that Z_t is stationary with finite expectation, I get $\mathbb{E}[Z_t] \equiv \mathbb{E}[Z_0]$ for all t. The term $\mathbb{E}[Z_0]$ is obtained from the calculation in section 2.2.1. The second and third moment of $N_{i,T}$ can be obtained similarly in sequence.

Corollary 4. There exists $\nu = (\mu^X, \mu^Y, \xi, \zeta, \phi, \sigma, \eta)$ such that $\mathbb{E}[Z_t]$ is stationary and its unconditional moments exist and are finite.

Proof. See Appendix B.

2.3. The inference problem

Let $V_{i,0}$ denote the number of credits in the portfolio i and is constant for an entire interval of T-year. Define the central moments of $\frac{N_{i,T}}{V_{i,0}}$ $i = 1, \dots, I$ for a T-year sampling interval as follows:

$$m_{i,T} = \mathbb{E}[\frac{N_{i,T}}{V_{i,0}}], \tag{19}$$

$$m_{ij,T} = \mathbb{E}[(\frac{N_{i,T}}{V_{i,0}} - \mathbb{E}[\frac{N_{i,T}}{V_{i,0}}])(\frac{N_{j,T}}{V_{j,0}} - \mathbb{E}[\frac{N_{j,T}}{V_{j,0}}])], \tag{20}$$

$$\begin{aligned} m_{ijk,T} = &\frac{1}{(m_{ii,T}m_{jj,T}m_{kk,T})^{\frac{1}{2}}}\mathbb{E}[(\frac{N_{i,T}}{V_{i,0}} - \mathbb{E}[\frac{N_{i,T}}{V_{i,0}}]) \\ &* (\frac{N_{j,T}}{V_{j,0}} - \mathbb{E}[\frac{N_{j,T}}{V_{j,0}}])(\frac{N_{k,T}}{V_{k,0}} - \mathbb{E}[\frac{N_{k,T}}{V_{k,0}}])], \end{aligned} \tag{21}$$

Let H denote the number of observation for a T sampling interval. Define the empirical central moments of $\frac{N_{i,T}}{V_{i,0}}$ $i = 1, \dots, I$ as follows:

$$M_{i,T} = \frac{1}{H}\sum_{h=1}^{H}[\frac{\hat{N}_{i,h}}{V_{i,0}}], \quad (22)$$

$$M_{ij,T} = \frac{1}{H}\sum_{h=1}^{H}[(\frac{\hat{N}_{i,h}}{V_{i,0(h)}} - M_{i,T})(\frac{j}{V_{j,0(h)}} - M_{j,T})], \quad (23)$$

$$M_{ijk,T} = \frac{1}{H(M_{ii,T}M_{jj,T}M_{kk,T})^{\frac{1}{2}}}\sum_{h=1}^{H}[\left(\frac{\hat{N}_{i,h}}{V_{i,0}} - M_{i,T}\right) * (\frac{\hat{N}_{j,h}}{V_{j,0}} - M_{j,T})(\frac{\hat{N}_{k,h}}{V_{k,0}} - M_{k,T})], \quad (24)$$

where $\hat{N}_{i,h}$ is the number of default during an interval of T-year observed at period h = $1, \dots, H$.

Let κ, θ, δ be the maximum likelihood estimators of the process parameters of X estimated separately, $\nu = (\mu^X, \mu^Y, \xi, \zeta, \phi, \sigma, \eta)$ and the parameter Θ be the set of admissible parameters for ν. The problem of moment conditions is

$$\inf_{\nu\in\Theta} \sum_{i=1}^{I}\sum_{j=i}^{I}\sum_{k=j}^{I}(m_{ijk,T} - M_{ijk,T})^2,$$

subject to

$$m_{i,T} = M_{i,T},$$

$$m_{ij,T} = M_{ij,T},$$

which must be hold for any $i, j, k \in \{1,2, \dots, I\}$.

To handle local solutions of multiple solutions, I perform the following procedure:

1) Find a set of parameter ν that matches the first and the second empirical moments of $N_{i,T}/V_{i,0}$, using a search algorithm.

2) Let the solution from 1) be the initial point and search for the parameter values that minimize the mean square error of the moment conditions problem. This may yield multiple solutions.

3) Simulate data according to parameter sets in 2) and apply Komolgorov-Smirnov (KS) test of the generated distribution compared to the empirical distribution. I choose the parameter set with the smallest KS statistic. Then I discard the sets of parameters for which the null hypothesis is rejected. That is, the distribution does not match the empirical one, at a defined confidence interval.

3. EMPIRICAL ANALYSIS, RESULTS, AND DISCUSSION

Without loss of generality, I select two portfolios with different characteristics of time series of default data to estimate the parameters in order to examine the effect of contagion risk between the portfolio pairs.

3.1. The data

The Global Financial Stress Index or GFSI from the Bank of America Merrill Lynch is a composite index that aggregates financial risk measures across five asset classes including credits, equities, interest rates, financial exchange rates, and commodities. It consists of three sub-indexes: Flow, Risk, and Skew. The Flow sub-index is a measure of asset price momentum for equities, bonds and money markets, calculated using investor flows (data from EPFR) and volumes. The Risk sub-index is a global measure of market, solvency and liquidity risk in the financial system. It is composed of the implied volatilities across asset classes, corporate and sovereign credit quality metrics and funding-related stress indicators. The Skew sub-index is a measure of relative demand for protection against large swings in major global equities and currencies.

The Global Financial Stress sub index - Flow, from January 4, 2000 to July 31, 2014[3], is used to represent the common observable risk factor in this study because the literatures show that the level of fund flow indicates the level of stress of the economy. The result in Table 2 confirms that the Flow sub-index describes the default rates differently between the two different economic regimes. Due to the limited availability of loan portfolio data, this paper takes the delinquency rate of corporate and retail loans from the Federal Reserve for the period from Q1, 2000 to Q2, 2014 as a proxy for the default rate in the estimation and simulation exercises. The Federal Reserve website provides the quarterly charge off and delinquency rate of loan portfolios of all U.S. commercial banks. To calculate the proxy of the quarterly default rates, I add back the quarterly charge off amount to the quarterly change in the delinquency amount. I normalize the sum by dividing it by the total loan outstanding in the corresponding category at the end of the period. For consistency, I name the delinquency rate as the loss amount (in the percentage of total outstanding) in the subsequent analysis.

There is a large body of literature supporting the relationship between the status of fund flow and financial crisis, for example, Gelos (2011), Kaminsky, Reinhart,

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[3] See more of the definition of GFSI and its sub-indexes in Bloomberg.

and Vegh (2003) and Kodres and Pritsker (2002). The fund flow characteristic generally discussed is that the amount of fund flow is increasingly high before the crisis and suddenly decreases and is becomes highly volatile during the crisis. Since the Flow sub index is constructed from a fund flow data of marketable securities including equities, bond and money markets, it is justifiable to apply this index as a common risk factor representing the investment momentum in the numerical example. The statistical test result in the subsequent section shows that the fitted model with the Flow sub index representing the common factor can reproduce the empirical default rate distribution.

Table 1. Summary statistics of quarterly loss amount (%) and the Flow sub-index.

	Average	Std Dev	Skewness	Excess Kurt	Median	Min	Max
Corporate	0.9056	0.6273	1.1156	0.5550	0.6000	0.0300	3.4400
Retail	3.0882	2.5191	0.2359	0.5472	3.0156	0.0000	6.5687
Flow	0.0415	0.2792	0.8631	1.0173	-0.0200	-1.0400	2.2900

This table provides summary statistics of quarterly loss amount (%) of corporate and retail loans of U.S. commercial banks. The data cover the quarterly data period from Q1, 2000 to Q2, 2014 (58 observations). The data presents in a fraction of 100 notional outstanding. The Flow data covers the daily data period from January 4, 2000 to July 31, 2014 (3803 observations).

Table 1 shows the summary statistics of the quarterly loss amount (%) of the corporate and retail portfolios of U.S. commercial banks and the GFSI Flow sub-index. The data cover the period from Q1, 2000 to Q2, 2014 (58 observations) and presents in a fraction of 100 notional outstanding. The daily Flow data cover the period from January 2000 to December 2013 (3803 observations). The historical data exhibits skewness and fat-tail, which coincide with general loss characteristic of credit portfolios. Specifically, it is a stylized fact that the corporate loan portfolio exhibits a lower loss amount with a lager probability of large loss compared to the retail loan portfolio.

As documented by Giesecke and Kim (2011) , the first crisis is less affected by the systemic risk from the financial market but is subject to the shock in the real economy. For the 2008 global financial crisis, Longstaff (2010) and Giesecke and Kim (2011) provide an evidence supporting the existence of systemic risk and show that the shock transmitted through the market and funding liquidity channel induces a large number of credit defaults during this crisis. This systemic risk transmission is regarded as the contagion risk phenomenon.

To test whether the two crises affect the loss of loan portfolios differently, I divide the data period into two sub-periods and test whether there is a relationship breakdown between the Flow sub-index and the default rate in the two sub-periods. The first sub-period covers the period from Q1, 2000 to Q4, 2007. This period includes the Internet bubble crisis in 2001. The second sub-period covers the period from Q1, 2008 to Q2, 2014, which includes the global financial crisis in 2008. I estimate the relationship between default rates and the Flow sub-index as follows:

$$d = c_1 + c_2 * Flow + c_3 * Flow * \mathbf{1}_{t\in(2000,2007)}, \tag{25}$$

where d denotes the loss amount (%) of corporate or retail portfolios.

The regression analysis provided in Table 2 shows that the coefficients c_3, which is a factor determining the structural breakdown, of both corporate and retail portfolios are significant at 5%. The peak of the default rate during 2008, which is in the second sub-period, is seemingly induced but be explained only by the Flow sub-index, which is in the first sub-period. Figure 1 illustrates the relationship breakdown between the two sub-periods, which is consistent with the result of Table 2.

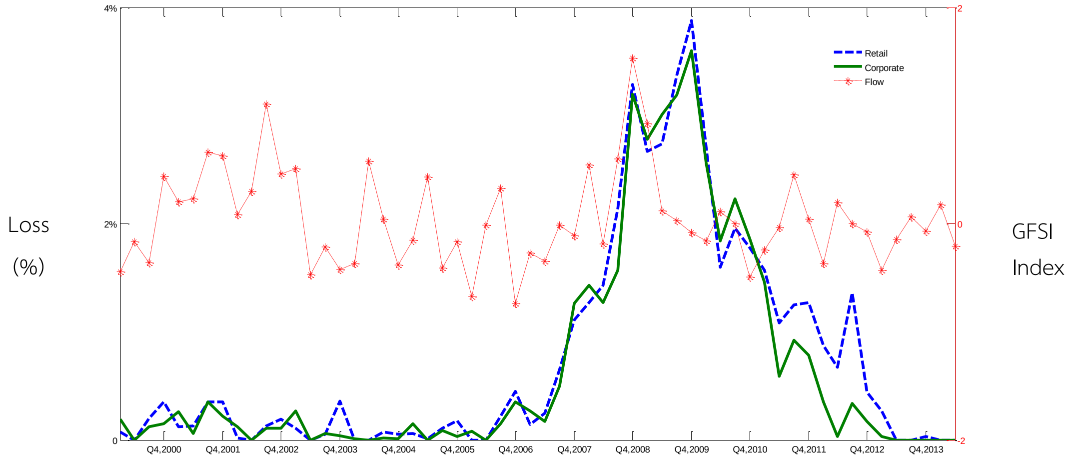
Table 2: The relationship between loss amount (%) and the Flow sub-index.

	Corporate			Retail		
	c_1	c_2	c_3	c_1	c_2	c_3
Coefficient	0.0070***	0.0131***	-0.0132**	0.0060***	0.0132***	0.0134**
t-statistic	5.5622	3.0417	-2.2977	4.9427	3.1174	-2.3539

This table provides the structural breakdown in the relationship between loss amount (%) of corporate or retail loans of U.S. commercial banks and Flow sub index as follows: $d = c_1 + c_2 * Flow + c_3 * Flow * \mathbf{1}_{t\in(2000,2007)}$ *where* d *denotes the default rate of corporate or retail portfolios. *** Significant at 1%, ** Significant at 5%, * Significant at 10%.*

Figure 1 shows the relationship between the quarterly aggregated data of the chosen common risk factor or the GFSI-Flow (star) and the loss amount of corporate (solid) and retail (dashed) loan portfolios during the period from Q1, 2000 to Q2, 2014. Observe that the co-movement in the distress index and the loss amount during the first sub-period is different from that during the second sub-period. The former accounts for Internet bubble followed by September 11 and in this period the stress index does not show its strong impact on the default rates while it does in the latter event of the 2008 global financial crisis.

Figure 1: Loss amount (%) of loans of U.S. commercial banks and Flow sub index.



The following analysis will take the second sub-period to study the impact of contagion risk in portfolio risk quantification.

3.2. The estimated parameters

Panel A) of Table 3 provides the estimated parameters of six default rate models for corporate and retail loan portfolios of U.S. banks using the method of moments using the data in the second sub-period. The full contagion model (FCM) includes self-exciting and cross-exciting effects of the specific risk factor. The no contagion model (NCM) excludes these effects ($\eta = 0$). The self-exciting model takes only contagion effects within the portfolio while the cross-exciting model takes only contagion across portfolios. The one-way contagion models include self-exciting and incorporate only the cross-exciting from corporate to retail (C-R) or from retail to corporate (R-C). Observing that the speed of adjustment of an unobservable risk factor Y (or ξ) from any models are higher than that of X (or θ), I regard Y as a factor representing the shorter cycle of economic variation. When contagion is ignored, the exposure to common risk factors, μ^X, increases to almost 7.5 times for the corporate portfolio and increases almost 20 percent for the retail loan (comparing between FCM and NCM). This increment is to compensate the dependency between portfolio default rates due to the absence of self-exciting and cross-exciting factors; however, this could make the portfolio risk become oversensitive to the common risk factors.

Table 3: Estimated parameters of loans portfolios.

	Full Contagion (FCM)		No Contagion(NCM)		Self-exciting		Cross-exciting		C-R Contagion		R-C Contagion	
	Q1,2000-Q2,20014: θ = 0.0373, κ = 0.6039 and δ= 0.3384											
ξ	1.905		1.262		1.591		1.690		3.573		2.474	
	Corporate	Retail	Corporate	Retail	Corporate	Retail	Corporate	Retail	Corporate	Retail	Corporate	Retail
ϕ	1.534	21.880	8.390	27.630	5.360	22.639	0.216	23.856	3.946	23.386		22.919
ζ	15.824	16.631	15.856	16.618	15.233	17.173	16.154	16.540	14.826	17.039	15.184	17.221
σ	0.582	0.477	0.594	0.482	0.582	0.4649	0.585	0.480	0.561	0.141	0.860	0.568
$\eta_{i,j}$	7.796	2.580			5.607	4.045			8.105	1.091	7.013	3.994
$\eta_{j,i}$	1.485	6.368					4.716	3.844		8.868	2.370	
μ^X	2.182	21.035	15.657	26.874	8.504	22.702	6.443	24.869	4.683	22.827	1.603	22.900
μ^Y	1.000	0.984	1.000	1.411	1.000	1.000	1.000	1.000	1.000	1.354	1.000	0.917
MSE	0.011		0.147		0.059		0.129		0.011		0.023	
p_{value}	0.190	0.101	0.025	0.002***	0.200	0.092**	0.194	0.002**	0.159	0.085*	0.149	0.089*

Panel A): Full period (Q1,2000-Q2,2014)

	Full Contagion (FCM-S)		No Contagion (NCM-S)		C-R Contagion	
	Q1,2008-Q2,2014: θ = 0.0565, κ = 0.4507and δ= 0.2128					
ξ	2.210		0.717		2.676	
	Corporate	Retail	Corporate	Retail	Corporate	Retail
ϕ	4.256	19.696	9.161	30.359	4.848	21.637
ζ	15.516	16.477	15.823	16.631	16.017	15.412
σ	0.700	0.374	0.604	0.483	0.723	0.376
$\eta_{i,j}$	7.407	3.118			7.851	2.032
$\eta_{j,i}$	0.283	9.722				9.789
μ^X	7.435	29.494	21.117	29.712	21.117	29.712
μ^Y	1.000	1.252	1.000	1.658	1.000	1.000
MSE	0.023	1.233	0.141			
p_{value}	0.203	0.224	0.186	0.073*	0.225	0.250

Panel B): The sub-periods (Q1,2008-Q2,2014)

*Panel A) provides the estimation results of the data period from Q1, 2000 to Q2, 2014. The full contagion model includes self-exciting and cross-exciting while the self-exciting and cross-exciting models include only contagion within and across portfolios respectively. The corporate to retail contagion model (C-R Contagion) incorporates contagion within portfolio and cross contagion from corporate to retail portfolio whereas the retail to corporate contagion (R-C Contagion) takes the opposite way of the effect. Panel B) provides the estimation results of period during Q1, 2008-Q2, 2004. The symbols * and ** show that the null is rejected at 90% and 95% confident interval respectively. The mean square error (MSE) presents the sum square of the difference between model and empirical moment of the first, the second and the third moments.*

I reconstruct the loss distribution of the loss amount of each portfolio and test whether the reconstructed distribution fits well with the data. The p-value represents the Komolgorov-Smirnov (KS) test of null hypothesis that the simulated data from the calibrated model and the observation are from the same distribution. Panel A) of Table 3 show that only the full contagion model fails to reject null hypothesis that the simulated and the actual portfolios are from the same distribution at 10% significance level. Therefore, I choose the full contagion model to describe the joint default rate dynamic of the entire sample portfolios.

Table 4: Moment reconstruction and empirical moments.

Full Contagion (FCM)					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	0.9097	0.6366	0.9267	1.3137	0.6068
	(0.0001)	(0.0007)	(0.0005)	(0.2201)	(0.1146)
Retail	3.0955	2.5591		0.3248	0.3059
	(0.0006)	(0.0010)		(0.0994)	(0.1528)
No Contagion (NCM)					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	0.9406	0.5748	0.9828	0.3020	0.0964
	(0.0309)	(0.0624)	(0.0567)	(0.7916)	(0.6250)
Retail	3.3765	2.3473		0.1554	0.0481
	(0.2809)	(0.2129)		(0.0656)	(0.4125)
Self-Contagion					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	0.8987	0.6324	0.9246	0.7517	0.2140
	(0.0110)	(0.0048)	(0.0015)	(0.3419)	(0.5074)
Retail	3.0728	2.5478		0.2776	0.0756
	(0.0228)	(0.0123)		(0.0567)	(0.3850)
Cross Contagion					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	0.8252	0.6187	1.0654	0.4286	0.2240
	(0.0845)	(0.0185)	(0.1392)	(0.6650)	(0.4974)
Retail	2.7604	2.1570		0.1675	0.1436
	(0.4252)	(0.4031)		(0.0535)	(0.3170)
C-R Contagion					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	0.9097	0.6366	0.9268	1.2950	0.5920
	(0.0001)	(0.0006)	(0.0002)	(0.2014)	(0.1393)
Retail	3.0958	2.5585		0.2735	0.2764
	(0.0002)	(0.0016)		(0.0525)	(0.1842)
R-C Contagion					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	0.9092	0.6378	0.9259	1.0529	0.3849
	(0.0005)	(0.0006)	(0.0002)	(0.0407)	(0.3367)
Retail	3.0958	2.5518		0.2769	0.1654
	(0.0002)	(0.0017)		(0.0558)	(0.2952)

Panel A): Full period (Q1,2000-Q2,2014)

Full Contagion (FCM-S)					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	1.0130	0.8699	1.8158	0.9621	0.4873
	(0.0004)	(0.0004)	(0.0004)	(0.0668)	(0.0477)
Retail	3.3735	5.3779		0.2627	0.2591
	(0.0016)	(0.0054)		(0.4313)	(0.1184)
No Contagion(NCM-S)					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	0.9987	1.0549	1.3453	0.3657	0.0667
	(0.0140)	(0.1854)	(0.4709)	(0.6632)	(0.4683)
Retail	3.1523	2.2103		0.0480	0.0268
	(0.2228)	(3.1729)		(0.2167)	(0.1139)
C-R Contagion					
Model Moments	Mean	Variance	Covariance	Skewness	Co-skewness
Corporate	0.1132	1.0467	2.0839	1.2250	0.5989
	(0.8994)	(0.1774)	(0.2677)	(0.1961)	(0.0639)
Retail	3.4625	5.6883		0.2660	0.3019
	(0.0874)	(0.3050)		(0.4346)	(0.1612)
Panel B): The sub-periods (Q1,2008-Q2,2014)					

The table shows the moment reconstruction from the estimated parameters obtained from Table 3. The absolute error between empirical and the model moments are shown in parentheses.

Table 4 shows the reconstructed moments and the absolute error between empirical and the model moments as shown in parentheses. The absolute error of the first moment ranges between 0.0001 and 0.8994, and that of the variance and covariance are between 0.0004 and 3.1729, and between 0.002 and 0.4709, respectively. The absolute error of the skewness and of the co-skewness are between 0.0407 and 0.7916, and between 0.0477 and 0.6250, respectively.

Table 2 shows that the co-movement between the Flow sub-index and the default rates demonstrates the structural break in Q1, 2008. The default rates overshooting during 2008 to 2010 could be subject to other explanatory variables. As demonstrated in the literature, Giesecke and Kim (2011) and Longstaff (2010), the distress in the portfolio in the first regime, covering Internet bubble period during 2001, is not affected by contagion while it is in the second regime during 2008. To examine the effect of contagion on the portfolio default behavior, I choose the second sub-period, and estimate the model with and without contagion coefficients. In addition, I choose the model specification with the one-way contagion from corporate to retail as the result from the full data period shows that the one-way contagion coefficient from corporate to retail is stronger than the opposite direction. The calibration results of the second sub-period models, from Q1, 2008 to Q2, 2014, are shown in panel B) of Table 3. From Table 3 panel B), the full contagion model (FCM-S) and the one-way contagion fail to reject the null hypothesis that the simulated data and the empirical data are from the same distribution. The no contagion model of the sub-period (NCM-S) rejects the null hypothesis at 10% significant. This confirms the existence of contagion in this sub-period data.

In order to evaluate the impact of the contagion risk in portfolio risk quantification during the crisis period, I select two models of the second sub-period data, which are FCM-S and NCM-S in the subsequent analysis. Although the NCM-S model rejects the null hypothesis at 10%, it fails to reject null at 5%.

In order to evaluate the impact of the common economic shock (X) and the specific risk (λ) on the loss amount of the corporate and retail portfolios when there is contagion risk, I select the FCM of the full data period and conduct the scenario analysis when the shocks to the portfolios are triggered from the different sources of contagion.

To understand portfolio behavior under contagion risk and answer the first two research questions, this study conducts simulation exercises for each stress scenarios on each comparative model.

3.3. The simulation exercises

3.3.1. Effects of Contagion on Capital Quantification

I investigate three scenarios of risk channels. The first scenario is the system-wide economic shock (System) defined as the event when the common risk factor moves toward its value at 99.9 percentile of its distribution. This represents a regime shift in the long-run target of common factor X. By construction, the model with the contagion risk allows the spillover effect from systematic risk factor to the specific risk factor, λ, through the contagion coefficients ($\eta_{i,j}$) which means the system-wide economic shock can induce extra default correlation between portfolio pair beyond those explained by the common risk factor. In addition, the default spillover can occur through the specific risk shock of one portfolio to another without the impact from the system-wide economy. Therefore, the second and the third scenarios refer to the test of contagion risk through this channel. Specifically, the second scenario is the default spillover from corporate to retail portfolio (SpeCo) and the third scenario is for the opposite direction (SpeRet). In the second and the third scenarios, the specific risk shock event is defined as the event when the risk factor λ moves toward the target at the 99.9 percentile of the distribution of the corresponding specific risk factor.

I apply the three scenarios to two contagion models obtained from the second sub-period data, which are FCM-S and NCM-S. I denote ELR_{severe} as the expected loss amount generated by the combination of all three stress events for the FCM-S, and NCM-S. The yearly expected loss amount as a percentage of total loan outstanding of the portfolio *(% ELR)* of each model is obtained from the mean of the simulated one-year loss distribution generated from that model. The incremental expected loss amount as a percentage of total loan outstanding of the portfolio *(% ΔELR)* is obtained from the increase in the mean of the simulated one-year loss distribution generated from the same model with the change in the long-run target parameter (θ or/and ϕ) to the 99.9 percentile of its original loss distribution before changing the long-run target parameter(s).

Table 5: Yearly expected loss amount (% ELR) and incremental expected loss amount (% Δ_{ELR}) by scenarios.

	%ELR	System. %ΔELR	SpeCo %ΔELR	SpeRet %ΔELR	CapResv $\%ELR_{Severe}$ - %ELR	CapResv Implied IRB
Full Contagion(FCM-S)						
Corporate	4.16	4.84	14.85	0.37	21.58	21.89
Retail	13.16	10.32	8.24	11.20	13.23	14.89
No Contagion(NCM-S)						
Corporate	4.35	6.60	0.05	0.00	6.60	22.25
Retail	12.60	8.89	0.00	0.04	8.94	14.63

This table shows the unconditional expected loss amount and expected loss amount increment by stress scenarios.

Table 5 provides the result of the change in the expected loss amount due to the distress from each scenario, ΔELR. If the true data generating process is FCM-S, the NCM-S model overestimates the common risk factor sensitivity of the corporate portfolio (from 4.84% increment to 6.6% increment) and underestimates the sensitivity factor of the retail portfolio (from 10.32% increment to 8.89% increment). The overestimation of the factor sensitivity in the corporate portfolio is because the higher factor sensitivity to the common risk factor (μ_X) is more than offset by an elimination of the contagion spillover (η) within and across portfolios. On the contrary, removing the contagion spillover factors ($\eta = 0$), with the higher effect of η_{ij} from corporate to retail, causes the underestimation of the factor sensitivity in the retail portfolio in NCM-S.

The FCM-S demonstrates significant default spillover from corporate to retail or the SpeCo stress event, which results in 8.24% of the default rate increment in the retail portfolio, but the reverse is not true; when the shock occurs from the retail portfolio or the SpeRet stress event, it results in only 0.37% of the default rate increment in the corporate portfolio.

As expected, the NCM-S cannot capture default spillover through the specific risk linkage. The maximum of the default rate increment from the specific risk factor triggered by a distress event is only 0.05%.

The required capital reserve of each model determined by the most severe default rate (ELR_{severe}) in excess of unconditional expected loss (ELR) of the same model is compared against the capital reserve implied by the IRB capital requirement. Appendix C shows the IRB capital formula used in this paper.

The two last columns of Table 5 show that the capital requirement from the contagion model of FCM-S closely matches the implied IRB capital. In fact, the IRB-implied capital reserve from all models are close because the ELR of each model, which is the only input of IRB, is estimated from the same historical data. However, the required capital reserve of models with contagion matches with the IRB-implied capital reserve much better. Therefore, the change in the ELR from stress testing a component of the models provides consistent capital reserve as suggested by the IRB formula.

In summary, the contagion model has the richer feature comparing to the non-contagion model, which is the ability to perform scenario analysis of distress from default spillover. This spillover effect significantly alters capital concentration and affects portfolio allocation differently with and without contagion shock expectation.

The following subsection demonstrates the implication of default spillover during the stress period through risk decomposition when the portfolio manager employs stress-testing as a risk management tool.

3.3.2. Impact from Sources of Contagion

Since the analysis in this section is not specific to the data period, I take the model estimated from the entire data period, which is FCM. Therefore, I assume that the data generating process is FCM. This section demonstrates that the default risk can be decomposed into its exposure to long-run component and the transient components due to systematic and specific risk factors. This decomposition helps demonstrate how the infectious default mechanism increases the total portfolio expected default rate induced by the different sources of risk that trigger the contagion.

Following Proposition 3, the expected loss amount conditional on the state of the economy is a linear combination of risk factors:

$$ELR(\pi) = b_0(\pi) + b_x(\pi)\mathbb{E}[X|\pi] + b_{\lambda_1}(\pi)\mathbb{E}[\lambda_1|\pi] + b_{\lambda_2}(\pi)\mathbb{E}[\lambda_2|\pi], \tag{26}$$

where π is the value(s) of the long-run target parameter(s), θ and/or ϕ, which determines the state of the economy, and $b_x(\pi), b_{\lambda_1}(\pi)$ and $b_{\lambda_2}(\pi)$ are factor sensitivities of the expected loss amount for a given interval. The parameter $b_0(\pi)$ is the factor determining the long-run component. The interval of this analysis is one year. The first term on the right-hand-side of (26) is a permanent component determined by the joint dynamic of risk factors whereas the last three terms are the transient components, in which their effects are decreasing in the time interval. All of the $b_0(\pi)$, $b_x(\pi), b_{\lambda_1}(\pi)$ and $b_{\lambda_2}(\pi)$ are state dependent. The notation λ_1 and λ_2 are associated with specific risk of corporate and retail portfolios subsequently.

Equation (26) is equivalent to Proposition 3 whereas $\frac{\alpha_{u_i}(t,0)}{N_{i,0}}$ is $b_0(\pi)$ and the factor loading $(b_x(\pi), b_{\lambda_1}(\pi), b_{\lambda_2}(\pi))'$ is $(\frac{\beta_{u_i,X}(t,0)}{N_{i,0}}, \frac{\beta_{u_i,\lambda_1}(t,0)}{N_{i,0}}, \frac{\beta_{u_i,\lambda_2}(t,0)}{N_{i,0}})$ where π is the value(s) of the long-run target parameter(s), θ and/or ϕ, under each scenario.

I denote π_{normal} as the set of parameters obtained from calibration exercise and $\pi_{distress}$ is the set of parameters under each distress regime. Specifically, the parameter $(\theta, \phi')'$ alters by each distress scenario. Under systematic distress event, I replace the calibrated parameter θ by the value at 99.9 percentile of the distribution of X under the normal economy. Under each specific distress event, I replace the element of ϕ correspondent with the portfolio that triggers the event with the value at 99.9 percentile of the marginal distribution of λ associated with that portfolio under the normal regime.

Table 6 decomposes the increase in the expected default rate ΔELR of distress scenario into the contribution from the long-run component (L-R), $b_0(\pi_{distress}) - b_0(\pi_{distress})$, from the common risk factors (CFactor), which is $b_x(\pi_{distress})(\mathbb{E}[X|\pi_{distress}] - \mathbb{E}[X|\pi_{normal}])$, and from the specific risk factors (Corp. and Retail), which are $b_x(\pi_{distress})\mathbb{E}[\lambda_1|\pi_{distress}] - \mathbb{E}[\lambda_1|\pi_{normal}])$ and $b_x(\pi_{distress})(\mathbb{E}[\lambda_2|\pi_{distress}] - \mathbb{E}[a_2|\pi_{normal}])$. The percent contribution in ΔELR of each element is calculated as the value of ΔELR of each element over the total ΔELR.

Considering the systematic distress event, the long-run component contributes 57.89% of the increase in the default rate of the corporate portfolio whereas the common risk factor contributes 39.43% and the specific risk factor of corporate and retail portfolios contribute 2.45% and 0.24% respectively. Observe that the long-run component has the highest contribution for the total default rate increment as the distress is created by the change in regime due to the systematic risk factor. The same is true when considering the distress event triggered by the specific risk of corporate or retail portfolio, the long-run component contributes the highest to the total increase in default rate. The long-run component contributes 93.89% of the total default under the normal regime and it contributes 96.59% of the total default rate increment under the corporate distress regime, and 95.40% of the total default rate increment under the retail distress regime. Table 6 shows that the highest default rate increment occurs when the shock is triggered from the specific risk event of the corporate portfolio. This is because the highest increment in the long-run component happens when the distress is triggered from the specific risk of the corporate portfolio.

At the same time, the long-run component contributes the highest at 58.43% of the increase in the expected default rate of the retail portfolio, a common risk factor contributes 40.80% and the specific risk components contribute marginally. Similar to the default rate of the corporate portfolio, the long-run risk component contributes the highest when the distress is created from the specific risk of corporate or retail portfolios. The table shows that the highest default rate increment in the retail portfolio occurs when the shock is created from the specific risk event of the retail portfolio. This is because the highest increment in the long-run component happens when the distress is triggered from the specific risk of the retail portfolio.

Table 6: Incremental loss amount of three origins of contagion.

	Corporate					Retail				
	Total	L-R.	C Factor.	Corp.	Retail	Total	L-R	C Factor.	Corp.	Retail
Normal										
%ELR	3.69	3.46	0.05	0.13	0.05	12.45	12.00	0.16	0.06	0.24
%Contribution		93.89	1.44	3.45	1.22		96.38	1.26	0.46	1.96
Systematic stress										
%ΔELR	3.20	1.85	1.26	0.08	0.01	8.95	5.23	3.65	0.03	0.04
%Contribution		57.87	39.43	2.45	0.24		58.43	40.80	0.38	0.39
SpeCo										
%ΔELR	19.64	18.97		0.68	0.03	8.06	7.73		0.03	0.15
%Contribution		96.59		3.45	0.16		95.88		3.67	1.91
SpeRet										
%ΔELR	1.80	1.71		0.05	0.40	9.67	9.49		0.02	0.18
%Contribution		95.40		2.97	1.98		98.10		0.24	1.89

This table shows risk factor contribution to yearly expected loss amount (%) in normal period and to change in yearly rate in distress scenario under the true contagion model (FCM). The scenarios are consistent with those applied in Table 5. The % contribution in normal ELR of each element is calculated as the value of ELR of in each element over the total EDR. The % contribution in ΔELR of each element is calculated as the value of ΔELR of in each element over the total ΔELR.

The long-run risk component plays an important role in the default rate increment under each stress regime because the long-run risk component creates a persistency of the default risk level. The high default risk that is persistent in the longer period is a key factor driving the cumulative loss amount of the portfolio.

The result of this decomposition shows that the persistency of the default risk is a key driver to the high loss amount as the contagion effect intensifies the effect of the distress in the risk factor. The persistency in the long-run component of corporate portfolio loss amount when the distress is created from the corporate portfolio is a result of the high self-exciting component. The persistency in the long-run component of the retail portfolio default when the distress is created from the retail portfolio is a result of the high cross-exciting component.

The following two subsections examines the level of importance of contagion risk to the risk capital quantification of loan portfolios.

3.3.3. Value at Risk Capital Estimation error

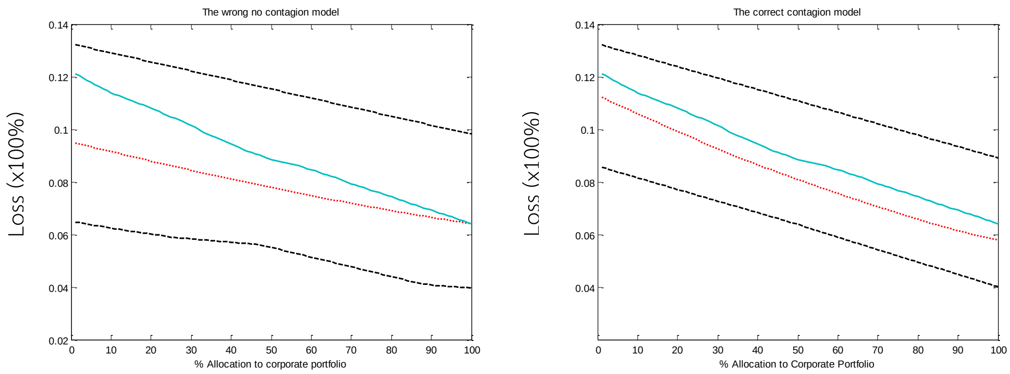
The T-year Value at Risk capital at q confidence level is defined as the loss amount at the value of $VaR_q(L)$, where $VaR_q(L) = \inf(l \in R: \mathrm{P}[L > l] \leq 1 - q)$ and L is the random variable representing the loss amount at the end of T-year horizon estimated from the distribution under the hypothetical risk model, beyond the expected value of loss amount estimated from the distribution under the true data generating process at the end of T-year horizon. The Value at Risk capital, or the VaRCapital, is the 1-year Value at Risk capital at 99.9% confidence level where the risk model is the re-estimated contagion (R-FCM) or no contagion (R-NCM) model.

To understand the effect of the model risk to the bias of the Value at Risk capital estimation, I use the contagion model, FCM, as the true data generating process to generate the data and use the generated data to estimate the contagion (R-FCM) and no contagion (R-NCM) models. I adopt the approach defined in Lim, Shanthikumarb, and Vahn (2011) to evaluate the model risk effect.

Consider the contagion model, FCM, as the true model:

1) Generate 80 observations of quarterly loss amount from the true model. This number of observations equals 20 year of time-series data, which is close to the data availability in the real case. Estimate the VaRCapital under the true FCM model.
2) Estimate the parameters under the true model (R-FCM) and the wrong no contagion model (R-NCM) using the generated data in 1).
3) For each portfolio allocation, estimate the VaRCapital by Monte-Carlo simulation under both R-FCM and R-NCM risk models.
4) Repeat step 1-3 for 25 times, and calculate the average, the maximum and the minimum of VaRCapital at each level of portfolio allocation between the corporate and retail portfolios.

Figure 2: Estimated VaRCapital of Loan Portfolio.



The solid line is the true VaRCapital; the dot middle line is the average VaRCapital and the upper and the lower dash lines are the maximum and the minimum estimated VaRCapitals.

Table 7: The estimated Value at Risk capital band from the default rate models.

%Allocation to corporate portfolio	1%	20%	40%	60%	80%	100%
Bandwidth under the wrong model	0.0674	0.0655	0.0617	0.0605	0.0610	0.0585
Bandwidth under the correct model	0.0465	0.0467	0.0469	0.0475	0.0483	0.0490

The band is the distance between the maximum and the minimum Value at Risk capital estimated under the wrong no contagion model and the correct contagion model.

The result of the estimation presents in Figure 2. The true VaRCapital of the true data generating process is in the solid line and the average VaRCapital estimated from the model is shown in the dot line. The upper and the lower dashed lines present the maximum and the minimum VaRCapital estimated from the model. Table 7 shows the width of the estimation band which is the distance between the maximum and the minimum Value at Risk capital estimated under the wrong no contagion model and the correct contagion model at various values of the allocation between the corporate and retail portfolios.

Ignoring the contagion effect results in prominent bias in the retail portfolio. As can be seen in Figure 2, it shows that the true Value at Risk capital deviates from the average Value at Risk capital when the portfolio allocation moves into the high level of allocation in the retail portfolio. This high level of deviation is the impact from the default spillover caused by the contagion coefficient (η), especially the cross-contagion. The compensation of the cross-contagion coefficient through the increase in the long-run specific risk (ϕ) of the retail portfolio does not substantially contribute to the risk capital under the stress event when the no contagion risk is ignored, as there is no mechanisms of loss spiral triggered by the specific risk component.

The bias is close to zero in the portfolio that only consists of the corporate portfolio because the compensation from missing the high self-contagion coefficient of the corporate portfolio is through the increase in the exposure to the systematic risk factor (μ^X) and the long-run specific risk of the corporate portfolio (ϕ). The oversensitivity to the systematic risk factor (μ^X) and the increase in the long-run specific risk (ϕ) under the wrong contagion model offsets the downward bias of high quantile estimation under the correct contagion model (right plot of Figure 2). This makes the bias under the wrong contagion model close to zero when the total allocation is the corporate portfolio.

In general, the average VaRCapital from the wrong no contagion model produces negative bias at the maximum of 2.11% when the portfolio is entirely allocated to the retail portfolio.

In addition, the wrong no contagion model produces the Value at Risk capital with a higher error band comparing to the Value at Risk capital estimated from the correct contagion model. The error band is higher under the portfolio with the higher allocation to the retail portfolio as shown in Table 7.

Generally, the wrong no contagion model is less parameterized comparing to the true model; therefore, the model is less flexible to match the tail of the generated distribution. That means, mistakenly ignoring the contagion effect increases the uncertainty in the Value at Risk capital.

4. CONCLUSION

In this chapter, the credit contagion model of loan portfolios and the estimation approach are developed. The model is a multi-factor model, which includes observable and unobservable risk factors. It allows for quantifying the portfolio Value at Risk conditional on different economic scenarios. The chapter contributes to the literature by providing the model that incorporates the contagion risk within and across portfolios in the loan context. The calibration methodology requires only the default data at the portfolio level. The model at the portfolio level is suitable for risk quantification and allocation problem concerning a set of portfolios with a large number of credits of similar risk profile within each portfolio.

By replacing the first moment with the manager's view about the probability of default, the calibrated model can generate loss distribution consistent with the default rate expectation while maintaining the volatility and the higher moment from that of the actual distribution.

The study examines the impact of ignoring contagion in the capital quantification and capital estimation error. The result shows that the contagion component substantially alters the capital quantification during the crisis period. The different level of capital determination to buffer the stress from different sources of distress events can be large if the contagion exists. Ignoring the contagion increases the estimation error in capital quantification to the different directions, which depends on the portfolio risk profile.

Because contagion risk is regime dependent, the horizontal effect could be substantial in determining appropriate risk capital that partly affects an optimal allocation of the loan portfolio under investment opportunity subject to the contagion risk. Since the loan portfolio cannot be sold down immediately in anticipating of the upcoming crisis, it is important to consider contagion risk in the loan portfolio planning with the long-term perspective.

APPENDIX

Appendix A.

Proof of Proposition 3:

By analogy to (11), I get:

$$\begin{aligned}\mathbb{E}[N_{i,T} \mid Z_t] &= \widetilde{\Psi}_{u_i}(u,t,T,Z_t)|_{u=0} \\ &= \widetilde{\Psi}(u,t,T,Z_t)\big(\alpha_{u_i}(t,u) + \beta_{u_i}(t,u)' Z_t\big)|_{u=0} \\ &= \alpha_{u_i}(t,0) + \beta_{u_i}(t,0)' Z_t, \end{aligned} \tag{A.1}$$

where $\widetilde{\Psi}(0,t,T,Z_t) = 1$, and $\widetilde{\Psi}_{u_i}$, α_{u_i} and β_{u_i} are the partial derivatives of $\widetilde{\Psi}$, α and β with respect to u_i. To obtain the unconditional first moment of $N_{i,T}$, I take the unconditional expectation on both sides of (A3), which yields:

$$\mathbb{E}[N_{i,T}] = \alpha_{u_i}(t,0) + \beta_{u_i}(t,0)'\mathbb{E}[Z], \quad i = 1, \dots, I.$$

Q.E.D.

Appendix B.

Proof of Corollary 4:

If $\eta = [0]$, the data generating process (4) reduces to the vector of the mean-reverting processes. This vector processes generate the multi-variated normal random variables; therefore, their unconditional moments are finite. Since (2) and (3) are also the mean-reverting processes, the sets of parameters $\nu = (\mu^X, \mu^Y, \xi, \zeta, \phi, \sigma, \eta)$ with $\eta = [0]$ are the special cases that make the conditional moments of $Z_{i,T}$ finite. For general cases, see Giesecke and Zhu (2013) and Cvitanic, Ma, and Zhang (2012).

Q.E.D.

Appendix C.

Formula of IRB Capital:

According to BCBS (2006), the IRB capital is given by:

$$\Phi\left(\frac{\Phi^{-1}\left(\overline{PD}\right) + \sqrt{\rho}\,\Phi^{-1}(0.999)}{\sqrt{1-\rho}}\right) - \Phi\left(\Phi^{-1}\left(\overline{PD}\right)\right), \qquad \text{(C.1)}$$

where Φ is the cumulative normal distribution, and ρ follows the corporate and retail functions indicated in BCBS (2006) which are given by:

Corporate:

$$\rho = \frac{0.12\left(1 - e^{-50\overline{PD}}\right)}{1 - e^{-50}} + \frac{0.24\left(1 - \left(1 - e^{-50\overline{PD}}\right)\right)}{\left(1 - e^{-50}\right)}; \qquad \text{(C.2)}$$

Retail:

$$\rho = \frac{0.03(1 - e^{-35\overline{PD}})}{1 - e^{-35}} + \frac{0.16(1 - (1 - e^{-35\overline{PD}}))}{1 - e^{-35}}; \qquad \text{(C.3)}$$

and $\overline{PD}$ is the expected default rate taken from the default rate distribution. In this chapter, the $\overline{PD}$ is the expected loss amount taken from the historical loss data.
Q.E.D.

CHAPTER 3: LENDING AND LEVERAGE DECISION OF BANK LOAN PORTFOLIO WITH CONTAGION RISK AND LOAN SALE MARKET

1. INTRODUCTION

There are a large body of literature in the portfolio theory providing the studies about the impact of the contagion effect on the optimal portfolio allocation. Thanks to the financial market interconnectedness, financial assets across markets and asset classes encounter systemic risk by exhibiting regime dependent return correlations, which strengthen during the bear market. This situation may create ripple and feedback effects of large multiple losses in asset values, one after of another, which illustrate the contagion phenomena. A large number of literatures support the premise that either financial contagion or systemic risk impair the diversification benefit. The long horizontal portfolios with buy and hold strategy (Ait-Sahalia, Cacho-Diaz, & Hurd, 2009; Ang & Bekaert, 2002; Branger, Kraft, & Meinerding, 2014; Buraschi, Porchia, & Trojani, 2010; Honda, 2003; Kraft & Steffensen, 2009) could forgo the optimality due to a loss in diversification. Because of the reduction in a diversification benefit, the optimal allocations under contagion risk may suggest less investment in risky assets as found by Kraft and Steffensen (2009).

While these literatures provide the studies about the effect of contagion risk on the tradable financial assets, bank loan portfolios, which are different as they are not tradable, and subject to credit review, may experience different outcome from the contagion effect on the optimal allocation policy. Since the interest rates charged on loans can be revaluated as usually done after certain periodic credit reviews (Fama, 1985), banks can adjust a spread premium corresponding with updated information, privately acquired by the banks, about the credit risk of that loan (Allen, 1990). In addition, banks cannot offload the pre-existing loan if the loan sale market is not available. They can only make decision on the new lending amount in which they can charge the interest rate based on the new perceived default risk. The focus in this chapter is on the lending and leverage decision when banks encounter contagion risk and the absence of the loan sale markets.

When the credit quality deteriorates, the new lending amount with the higher interest rate is more profitable and benefits banks by compensating the deteriorated value of the pre-existing loan. In contrast to its negative effect, the higher default risk due to the contagion, if observable and predictable, could create the value to the banks by providing the higher interest rate associated with the higher perceived level of risk. This is because banks can charge the high level of lending interest rate that compensates for both the high-expected default rate and the high chance of the downside of large loss in anticipation of the distress event due to the contagion risk. This may suggest an opposite effect of contagion on the optimal portfolio allocation of bank loan portfolio by increasing rather than decreasing the allocation to the risky asset. The question on which type of portfolio risk profiles that the contagion increases the value to the level that suggests the higher allocation of risky asset under the contagion risk serves as a basis for the first research question of this study.

Among the limited number of literatures about the effect of the secondary loan sale market on the lending activities, both Cebenoyan and Strahan (2004) and Purnanandam (2011) conclude in the same direction that a loan sale market increases lending activities. Drucker and Puri (2009) shows that the covenants increase the level of risk management, which helps enable the ease of loan sale activity. The loan sale activity helps endure the relationship lending because the original lenders can free up their lending capacity and facilitate the borrowers' future financial needs. Cebenoyan and Strahan (2004) finds that banks participate in the loan sale market because they want to maximize the risk management efficiency and reduce the overall portfolio risk level, while Purnanandam (2011), using the data during the U.S. subprime crisis period from Q2, 2006 to Q4, 2007, finds that banks face a capital constraint so they employ the originate-to-distribute model. It is the model of loan acquisition with an ultimate goal to sell down in the secondary market. However, none of the study proves if the secondary loan sale market helps bank maximize the shareholders' value. The study of the role of the secondary market to improve the shareholders' value serves as a basis for the second research question of this study.

This study proposes an infinite-horizon portfolio problem in which the bank maximize the expected discounted value of future dividend payments distributed from the profits in the lending of bank loan portfolio. This study provides a dynamic balance sheet decision model that integrates both portfolio decision and leverage decision with credit contagion for banks' loan portfolios. This study is the first study that suggests how banks should alter the portfolio mix between corporate and retail portfolios when the loan sale is not available, and there is contagion risk. Four important assumptions are imposed on this study. First, banks can observe defaults and contagion risk. The observability of the contagion risk implies no information asymmetries between banks and borrowers. This assumption is consistent with the role of bank capital that creates the monitoring incentive to mitigate the information asymmetries (Allen et al., 2011;

Mehran & Thakor, 2011; Purnanandam, 2011). Second, banks have the ability to adjust the interest rate charge on the new loans based on the updated information of the default risk. Therefore, banks' capital are important to banks' sustainability by securing the ongoing operations and by allowing banks to grasp the investment opportunities that might be undergone should banks lack of capital. Third, banks are required to comply with the risk management policy and maintain sufficient capital to absorb the loss from defaults. Fourth, there is no secondary loan sale market to offload the loans. Under this assumption, banks that anticipate their risk binding constraint in the future would limit themselves to grant more loan if they cannot sell down the loan in the future. The different type of risk and return profiles of loan portfolios may lead to the different conclusion of the optimal holding in that loan portfolios.

This study concerns two types of loan portfolios, the corporate and retail loan portfolios. The corporate loan portfolio potentially exhibits the strong infectious default within the corporate portfolio as there is a high plausibility that each firms will increase their degree of interconnectedness through the more closely interrelationship of their business operations. In addition, the increase in the interconnectedness may create the stronger infectious default from corporate loans to retail loan portfolios as it is highly likely that the defaults of corporations impact the unemployment rate which is an important macroeconomic factor determining the default rate in the loan portfolio (Jakubik & Schmieder, 2008). On the other hand, the retail loan portfolio should not exhibit the strong infectious default within its own portfolio nor to the corporate portfolio due to the lesser degree of importance of each retail obligor to the overall economy. This contagion behavior is asymmetry and it is the contagion feature adopted by this study.

The approach used in this study has key distinctive features from the literature. First, it is an infinite horizon portfolio problem. The objective function that maximizes the expected discounted value of the future dividend payments can directly represent a market value of equity without an assumption of the utility function of the representative agent. It provides the dynamic balance sheet model including dynamic loan issuance, loan repayment, loan default mechanism, and contagion feature. The optimization solves simultaneously the allocation and leverage decisions.

Some key features in the literature are similar to my approach. First, the value at risk constraint of Estrella (2004) is imposed; however, the problem of Estrella (2004) is multi-stage finite horizon with the optimal capital as the decision variable and there is no contagion in this model. Second, it is similar to Chambers and Charnes (1961) in terms of the optimal lending policy; however, their problem is a multi-stage finite horizon with linear risk constraint without contagion. Third, it is similar to Mehran and Thakor (2011) in terms of the shareholders' value maximization; however, their problem maximizes the expected value of utility function with the optimal capital and monitoring effort as the decision variables. The credit contagion model used in my study follows the affine-jump-diffusion model of Errais et al. (2010) that is used in credit derivative pricing; however, my study assumes the observability assumption of the default risk factors.

The researches examining how the bank capitals influence the lending decisions include Diamond and Rajan (2000) , Diamond and Rajan (2001), Calomiris and Wilson (2004), Purnanandam (2011), Allen et al. (2011), Mehran and Thakor (2011) and Košak et al. (2015). Mehran and Thakor (2011) and Košak et al. (2015) find that the bank's value increases with the bank's capital. This is because bank's capital alleviates moral hazard by forming the monitoring incentive, which mitigates the risk of information asymmetry. On the other hand, Diamond and Rajan (2000), and Diamond and Rajan (2001) argue that the capital decreases value as capital is not subject to run and then it does not discipline the banks to extract the rent. Calomiris and Wilson (2004) finds that the banks aim at maintaining the low level of risk to the depositors by controlling the level of risk associated with the credit supply at an appropriate level, which depends on the current level of equity, the ability to accumulate the capital, and the cost of raising new capital. In my study, a capital constraint is imposed as a requirement from the regulator.

In what follows, I describe the model and perform the analysis that answer the research questions.

2. METHODOLOGY

2.1. The dynamic model of bank's balance sheet

2.1.1. The default risk processes of loan portfolios

Consider a bank with I homogeneous-loan portfolios, each with different risk profiles. Each loan in the same portfolio is assume to have the same characteristic in the repayment rate and the default risk.

The default event of portfolio i is driven by the default intensity λ_i , which follows the affine-jump diffusion process with interacting default intensities under the physical measure:

$$d\Lambda_t = \zeta(\phi - \Lambda_t)dt + \sqrt{\Lambda_t}\Omega dB_t + \eta dN_t, \quad (1)$$

Where $\Lambda_t = (\lambda_{1t}, \dots, \lambda_{It})'$ is the vector of default intensities of all I portfolios, and $N_t = (N_{1,t}, N_{2,t}, \dots N_{I,t})'$ denotes a vector of I observable counting processes, each of which represents the number of default events of loans in portfolio i that occur by time t. The speed of mean reversion parameter ζ is an I by I diagonal matrix determining the rate of adjustment of the default intensity to the long-run risk level ϕ, which is an I dimensional vector. $\sqrt{\Lambda_t}$ is the diagonal matrix with an element-wise square root of Λ_t at the diagonal. The notation B is M dimensional standard Brownian motions representing the variations of M common economic risk factors, and Ω is an I by M matrix representing the exposures of each portfolio i to each economic risk factor.

The default risk between each portfolio is infectious through the characterization of the contagion coefficient matrix η, which is an I by I matrix with all elements being positive representing the self- and cross- exciting coefficients where the diagonal elements are the self-exciting coefficients and the off-diagonal elements are the cross-exciting coefficients. A self-exciting coefficient determines the level of the increase in the default intensity because of the default of the obligor in the same portfolio. A cross-exciting coefficient determines the level of the increase in the default intensity of a portfolio due to a default of an obligor in another portfolio.

The default intensity process (1) is similar to that of Errais et al. (2010) with additional assumptions that the risk factors driven by the vector of Brownian motion B, the default process vector N_t and the vector of default intensities Λ_t are observable.

I denote the loan outstanding of portfolio i at time t by L_{it}. The portfolio i is repaid continuously with repayment rate δ_i, as a proportion of the loan outstanding L_{it}- before any default occurs at time t. The default event, which is described by N_{it}, induces the loss of $\delta_i L_{ti}^-$ where l_i is the loss ratio of portfolio i.The rate of repayment determines the average time interval that the loan reaches the threshold of specific residual principal value. For example, with the rates of repayment 0.75 and 0.5, the loan principal values reach the 5% of the original principal values in 4 and 6 years subsequently. Therefore, the rate of repayment specifies the average effective maturity of the portfolio.

The dynamic process of the loan outstanding specification is as follows:

$$dL_{it} = -\delta_i L_{it^-} dt - l_i L_{it^-} dN_{it}, \quad (2)$$

2.1.2. The valuations of balance sheet elements

Let γ_i denote the risk premium for portfolio i that represents the compensation of the default events from N_{it}. It is generally used in the credit risk literatures such as Jarrow, Lando, and Yu (2005) and Duffie (2005) and Kraft and Steffensen (2009). These literatures including my study ignore the default risk compensation for taking the risk that arises from the uncertainty of the change in default intensity due to Brownian motion. The specification of the default risk premium on the Brownian motion is specified in Jarrow, Lando, and Yu (2005). Under the risk-neutral measure, N_{it} has intensity $\tilde{\lambda}_i = \gamma_i \lambda_i$.

I denote the discount factor of portfolio i by $F_i(\tilde{\Lambda}, \delta_i, l_i)$ where $\tilde{\Lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_I)'$. Therefore, the value of portfolio i denoted by $V_i(\tilde{\Lambda}, \delta_i, l_i)$ is given by:

$$V_i(\tilde{\Lambda}, \delta_i, l_i) \;=\; F_i(\tilde{\Lambda}, \delta_i, l_i)\, L_i. \qquad (3)$$

Following the transform analysis of Duffie et al. (2000), the discounted factor $F_i(\tilde{\Lambda}, \delta_i, l_i)$ is given by:

$$F_i(\tilde{\Lambda}, \delta_i, l_i) \;=\; \delta_i \int_0^\infty e^{-(r_d - r_f)t + A_i(0,t;l_i) + B_i'(0,t;l_i)\tilde{\Lambda}} dt. \qquad (4)$$

See the proof of (3) and proof of (4), and the function $A_i(0,t;l_i)$ and $B_i'(0,t;l_i)$ in Appendix A.

2.1.3. The dynamic balance sheet process

Assume that there are two types of asset in the asset side of the bank's balance sheet, which are cash with the value C_t and loans with the value $\sum_{i=1}^{I} V_{it}(\tilde{\Lambda}, \delta_i, l_i)$. On the liability side of the balance sheet, there are the deposit with value D_t and the equity with value E_t. The bank's balance sheet equation is given by:

$$C_t + \sum_{i=1}^{I} V_{it} = D_t + E_t. \qquad (5)$$

The bank makes a decision to lend in each portfolio i at each time t. The notional amount dW_{it} is provided to a loan in portfolio i at each time t by disbursing the cash amount at the discounted value of $F_i(\tilde{\Lambda}, \delta_i, \gamma_i) dW_{it}$ to the borrower. The bank may also make a decision to enlarge the balance sheet to lend more by increasing the value of the deposit or shrink the balance sheet to lend less by decreasing the value of the deposit. If the bank has the target leverage policy, the change in the value of the deposit or dD_t will be an endogenous variable rather than the decision variable.

The level of the cash on hand increases from (i) the repaid amount from loans, (ii) the interest rate bank earns from cash deposit is paid, and (iii) the new deposit amount. It decreases from (i) giving new loans, (ii) paying deposit interest, (iii) paying dividend, and (iv) the deposit withdrawal. Assume that the bank pays the cost of deposit at rate r_d and earns the interest from the cash on hand at rate r_f. The bank distributes the earning from lending business with the dividend payout ratio y. By assuming that there is no tax and operation cost, the change in the equity value at time t is equal to the net operating income. Assuming that the bank will only pay the dividend if the net operating income is positive. Therefore, the endogenous cash process of the bank is given by:

$$dC_t = -\sum_{i=1}^{I} F_{it} dW_{it} + \delta_i L_{it^-} dt + r_f C_t dt - r_d D_t dt + dD_t - y(max(dE_t, 0)), \quad (6)$$

2.2. The balance sheet decision model

The bank makes a lending decision for each loan in portfolio i through W_{it} and the deposit decision through D_t which is equivalent to making the leverage decision D_t/E_t as the value of E_t is known at the point of the decision. This assumes that the bank can manage the funding and reinvesting activities to get the value of D_t at the effective deposit rate r_d. The bank faces the risk constraint related to its Value at Risk on the loss of its capital:

$$VaR_q(E_{t_0} - E_t) \leq (1 - \alpha)E_{t_0}, \quad (7)$$

where $VaR_q(X) = \inf(x \in R: \mathrm{P}[X > x] \leq 1 - q)$ is the Value at Risk at q confidence level and the time interval between t and t_0 where $t_0 < t$ is the risk horizon period and in this study it is time between decision period. E_{t_0} is the value of equity at the beginning of the period and E_t is the value of equity at the end of the period. α is the risk constraint parameter determining the safety cushion of the portfolio value.

I assume that the bank maximizes the expected discounted value of the future dividend with the discount rate K, which is the required rate of return on equity of the bank's shareholders. Therefore, the bank solves the following infinite-horizon problem:

$$\mathrm{J}(\Gamma) = \max_{U \in A(\Gamma)} y\, \mathbb{E}[\int_0^{\infty} e^{-Kt} max(dE_t(\Gamma),0))], \quad (8)$$

subject to (1),(3) – (7), and

$$dL_{it} = -\delta_i L_{it^-} dt - l_i L_{it^-} dN_{it} + dW_{it}, i = 1, \dots I, \quad (9)$$

where $U = (W_1, W_2, W_3, \dots., W_I, D)'$ is the set of decision variables, and $A(\Gamma) = \{U: \Theta \times [0, \infty) \to \mathbb{R}^{I+1}: W_{it}$ is $F_t -$ measurable and nondecreasing in $t \,\forall\, i = 1, \dots I,$ D_t is $F_t -$ measurable$\}$ is the admissible set, where Θ is the sample space, and $\{F_t\}$ is the filtration generated by (B_t, N_t, W_t, D_t) where $W_t = (W_1, W_2, W_3, \dots., W_I)'$. $\Gamma = (\Lambda, E, D, V)$ is a vector of state variables. The optimal value function $\mathrm{J}(\Gamma)$ represents the market value of equity.

2.3. The implementation

I represent the state variables as $Z = (\Lambda, E, D_{multiple}, V_{multiple})$ where $D_{multiple} = D/E$, and $V_{multiple} = (\frac{V_1}{D+E}, \frac{V_2}{D+E}, \dots. \frac{V_I}{D+E})$. The value of E_0 or the initial equity value is normalized to 1. The values of D and V can be recovered from the values of Z.

The continuous time problem is solved by an approximated dynamic programming approach in which the discretization period is monthly. The implementation assumes the decision is made every quarter (decision period) while the loan default, loan repayment, and dividend payment occur monthly (event period). I denote k as the number of event periods between each decision period, which is three for this study. I denote the time interval between of each event period by $\Delta\tau$, which is one month for this study.

I denote $g_t = max(dE_t, 0)$ and the discretization counterpart of function $\int_t^{t+k\Delta\tau} g_s ds$ is given by:

$$G_t = \sum_{j=1}^{k} max\left(\Delta E_{t+j\Delta\tau}, 0\right),$$

where $\Delta E_{t+j\Delta\tau} = E_{t+j\Delta\tau} - E_{t+(j-1)\Delta\tau}$ and the value $y\sum_{j=1}^{k} max(\Delta E_{t+j\Delta\tau}, 0)$ is the total dividend paid out during each decision period, from t to $t + k\Delta\tau$. I make the assumption that the dividend is taken from the balance sheet at the end of each month and paid to the shareholders at the end of each quarter.

Therefore, the discrete time version of the optimization problem (8) is given by:

$$\mathrm{J}(Z) = \lim_{n\to\infty} \max_{\tilde{U}\in\tilde{A}(Z)} (y\, \mathbb{E}[\textstyle\sum_{j=1}^{n} e^{-jk\Delta\tau K}\, G_{jk\Delta\tau} | Z_0 = Z]. \quad (10)$$

Subject to (7) and the following constraints:

$$\Lambda_{t+\Delta\tau} = \Lambda_t + \zeta(\phi - \Lambda_t)\Delta\tau + \sqrt{\Lambda_t}\Omega(\mathrm{B}_{t+\Delta\tau} - \mathrm{B}_t) + \eta(\mathrm{N}_{t+\Delta\tau} - \mathrm{N}_t), \quad (11)$$

$$L_{it+\Delta\tau} = L_{it}(1 - \delta_i)\Delta\tau - l_i L_{it}(N_{it+\Delta\tau} - N_{it}) + \mathrm{w}_i, \quad (12)$$

$$C_t + \textstyle\sum_{i=1}^{I} V_{it} = D_t + E_t, \quad (13)$$

$$\begin{aligned} C_{t+\Delta\tau} = &-\sum_{i=1}^{I} F_{it} w_i + \delta_i L_{it}\Delta\tau + C_t(1 + r_f)\Delta\tau \\ &-r_d D_t \Delta\tau + d_t - y(max(E_{t+\Delta\tau} - E_t, 0)), \end{aligned} \quad (14)$$

where $\tilde{U} = (w_1, \ldots., w_I, d)', w_{it}$ is the notional amount of the new loan for portfolio i at time t, and d_t is the increase in the deposit amount determined by the bank at time t. The admissible set is $\tilde{A}(\mathrm{Z}) = \{\tilde{U}: \Theta \times \{0, k\Delta\tau, 2k\Delta\tau, \ldots\} \to \mathbb{R}^{I+1}: w_{it}$ is $F_t -$ measurable and $w_{it} \geq 0$ for $t \in \{0, k\Delta\tau, 2k\Delta\tau \ldots\}$ and $w_{it} = 0$ otherwise, and $d_t = 0$ for $t \neq jk\Delta\tau, j \in \{0,1,2 \ldots\}\}$.

According to Puterman (2005) , problem (9) has an equivalent infinite stage, finite states Markov discount problem as follows:

$$\mathrm{J}(Z) = \max_{\tilde{U}\in\tilde{A}(Z)} (\mathbb{E}\left[G | f\left(Z, \tilde{U}(Z)\right)\right] + e^{-kt\Delta\tau K} Q^*(f\left(Z, \tilde{U}(Z)\right))), \quad (15)$$

where $k\Delta\tau$ is the time between each decision period, and $f\left(Z, \tilde{U}(Z)\right)$ is the deterministic function determining the post-decision state, m, which depends on the pre-decision vector of state variables, Z and the policy $\tilde{U}(Z)$. Denote by $Q^*(f\left(Z, \tilde{U}(Z)\right))$ the post-decision value function, which is given by:

$$Q^*(m_t) = \sum_{s=1}^{S} p(Z_s|m_t)\mathrm{J}(Z_s), \tag{16}$$

where $m_t = f\left(Z_t, \widetilde{U}(Z)\right)$, S is the number of states, and $p(Z_s|m_t)$ is the state transition probability from the post-decision state m_t at time t to the next period state variable Z_s at time $t + k\Delta\tau$. The dynamic programming algorithm that solves (15) subject to (7) and (11) – (14) is described in Appendix B.

3. ANALYSIS, RESULTS, AND DISCUSSION

3.1. The problem definitions

There are two major types of loan portfolio characteristics in this study. First is the corporate loan portfolio that endows with a lower expected default rate but a higher loss per default, and a higher the default rate volatility. Second is the retail loan portfolio that endows with a higher expected default rate but a lower loss per default, and a lower default rate volatility.

Since a corporate loan typically has a larger size, the loss per default is usually higher comparing to a retail loan. So the loss ratio (l) is set to a higher value for the corporate loan portfolio to represent the larger size as compared to that of the retail portfolio. In addition, the nature of the loan products to a corporate loan has a shorter term to maturity comparing to a retail loan portfolio. The funding needs of corporations usually are for the short-term working capital and medium-term corporate investment. The retail loans, on the other hand, may comprise a mix of funding needs for the medium-term car loan and the long-term mortgage loans. As a consequence, the rate of repayment (δ) is set to a higher value for the corporate loan portfolio. The risk premium (γ) of the corporate loan portfolio is set at a lower value comparing to the retail loan portfolio to represent the higher premium bank can earn in the less competitive retail segment.

To generate the default rate distribution, the process parameters describing the default behavior with the contagion effect in the following problems are taken as a guideline from the calibration result in Chapter 2. Since the dynamic process of the default intensity Λ used in this chapter is a square root process with jumps, there are some adjustments in the process parameters to get the desired characteristics, which characterize the higher first moment of the default rate of the retail portfolio but the lower third moment comparing to that of the corporate portfolio.

I assume that the process parameters described yearly default rate. The default rate process parameters $\zeta, \phi, \eta,$ which represent the speed of adjustment, the target default intensity, and the contagion coefficient, are taken directly from Chapter 2. The parameter Ω is chosen to have the default correlation between portfolio pairs less influenced by the diffusion risk but highly influenced by the contagion effect. The overall volatility of the default intensity due to the diffusion risk are maintained to be closely to that of the original parameter in Chapter 2.

There are eight optimization problems in this study. Table 1 describes the detail of the problems. The optimization results of problem 1-4 help answer the research question on how banks should alter the portfolio mix of the corporate and retail portfolios, when there is the contagion risk. Problems 1 and 2 allow both self- and cross- contagion while problems 3 and 4 do not have neither of them. In problems 1-4, the bank solves for the optimal allocation between the corporate and retail loan portfolios. Problems 1 and 3 assume loan sales are not possible, while problems 2 and 4 allow loan sales. To answer this research question, the portfolio decision policies are evaluated against the different levels of default risk, equity value and the initial loan position with the contagion risk, which is problem 1, and without the contagion risk, which is problem 3. In order to understand the optimal lending decision when there is no non-tradable constraint of the pre-existing loan position, the results from problems 2 and 4 are given as benchmarks.

In problem 5-8, the bank solves for the optimal leverage and the amount of investment in a single loan portfolio, with self-exciting contagion. Problems 5 and 6 consider the corporate loan portfolio, while problems 7 and 8 consider the retail loan portfolio. Loan sales are allowed in problems 6 and 8, while they are not in problems 5 and 7. The comparative results between problems 5 and 6, and between problems 7 and 8, provide the analysis that give the answer to the second research problem. To answer this research question, the difference in the optimal lending quantities between the problems with loan sale and without loan sale are used to evaluate the impact of loan sale market to the lending decision of the bank holding for each specific risk profile (corporate and retail).

Table 1: Problem definition.

Problem	No. Port	Loan Sale	Portfolio	Default Risk Process	Self-Exciting	Cross-Exciting	Decision Variables
1	2	No	Corporate Retail	$\tilde{\lambda}_{corp}$ $\tilde{\lambda}_{retail}$	Yes	Yes	Lending allocation w_{corp} Lending allocation w_{retail}
2	2	Yes	Corporate Retail	$\tilde{\lambda}_{corp}$ $\tilde{\lambda}_{retail}$	Yes	Yes	Lending allocation w_{corp} Lending allocation w_{retail}
3	2	No	Corporate Retail	$\ddot{\lambda}_{corpo}$ $\ddot{\lambda}_{retail}$	No	No	Lending allocation w_{corp} Lending allocation w_{retail}
4	2	Yes	Corporate Retail	$\ddot{\lambda}_{corpo}$ $\ddot{\lambda}_{retail}$	No	No	Lending allocation w_{corp} Lending allocation w_{retail}
5	1	No	Corporate	λ_{corp}	Yes	No	Lending allocation w_{corp} Leverage decision d
6	1	Yes	Corporate	λ_{corp}	Yes	No	Lending allocation w_{corp} Leverage decision d
7	1	No	Retail	λ_{retail}	Yes	No	Lending allocation w_{retail} Leverage decision d
8	1	Yes	Retail	λ_{retail}	Yes	No	Lending allocation w_{retail} Leverage decision d

This table defines the set up to the problem 1-4 and the problem 5-8.

I refer to the default intensity when none of self- and cross-contagion exists as $\ddot{\lambda}$, when only self-exciting contagion exists as λ, and when both self- and cross-exciting contagion exists as $\tilde{\lambda}$.

3.2. The set up parameters

Table 2 describes the process parameters of the default risk used for the analysis of problems 1-8. The risk premium (γ) of the corporate and retail loan portfolios are set to 1.2 and 2.0 respectively to represent the higher premium the bank can earn in the less competitive retail segment. The rate of repayment (δ) of the corporate loan portfolio is set at the higher rate comparing to that of the retail loan in order to represent the nature in their maturities of loan products, which has been discussed.

Table 2: Process parameters.

Process	ζ		ϕ	Ω		η		δ_i	γ_i	l_i
$\tilde{\lambda}_{corp}$	15.82	-	1.53	2.99	-	7.80	1.48	0.75	1.20	0.005
$\tilde{\lambda}_{retail}$	-	16.63	21.88	4.06	20.61	6.37	2.58	0.5	2.00	0.001
$\ddot{\lambda}_{corp}$	15.82	-	1.53	2.99	-	-	-	0.75	1.20	0.005
$\ddot{\lambda}_{retail}$	-	16.63	21.88	4.06	20.61	-	-	0.5	2.00	0.001
λ_{corp}	15.82	-	1.53	2.99	-	7.80	-	0.75	1.20	0.005
λ_{retail}	-	16.63	21.88		20.61	-	2.58	0.5	2.00	0.001

Table 3 describes the parameter sets used for the analysis of problems 1-8. The risk constraint parameter α for problems 1-4 is set at the lower value comparing to that for problems 5-8 because the control of the risk limit for the problem with multiple portfolios at the target leverage is more difficult than that of the single portfolio with the flexible leverage.

Table 3: The set up parameters.

r_d	r_f	K	y	$\Delta\tau$	q	d	α	
						Problem 1-4	Problem1-4	Problem5-8
0.01	0.0025	0.2	0.4	0.25	0.99	50	0.2	0.4

3.3. The discretization

The implementation follows the infinite stage, finite states Markov discount problem (14) which requires the discretization of the state space and construct the transition probability matrix $p(Z_s|m_t)$ defined in (15).

The discretized grids of the default risk, Λ, is taken from the stationary distribution of the simulated distribution Λ. From the stationary distribution, the difference between the maximum and the minimum value of the risk factor is divided into 5 equal bands and the lower bound of each band is chosen for each grid point. In total, there are five grids for each default intensity. Table 4 shows the range of the default intensities.

The lowest discretized grid point of the equity is chosen as 0.1 to represent the near bankruptcy status and the others grid points are 0.25, 0.5, 1, 2, 3, and 5. The maximum grid value at five represents the long-run value creation of typical banks, which could grow up to 50 times from the state where banks are close to the insolvency status.

The discretized grid points of the leverage multiple $D_{multiple}$ ranges from five to ninety with the grid points as follows : 5, 10, 20, 30, 50, 60, 70, 80, and 90. According to González (2005), banks during 1995-1999 around the world take the maximum leverage at 25. This is the period before the boom of the originate-to-distribute model, which is the period of much higher leverage. The data in Wharton Research Data Service during 2000 to 2013 shows that the U.S. banks take the leverage up to 70. By approximating the actual banks leverage with some buffer, the study chooses the maximum leverage value at 90.

The discretized grid points of the value of loan multiple $V_{multiple}$ range from 0.07 to 0.63. The grid spacing is 0.07. Therefore, there are nine grid points for each loan type.

Table 4: Range of default intensity.

Process/ Grid	Lowest	2	3	4	Highest
$\tilde{\lambda}_{corp}$	3.8913	5.4370	6.9827	8.5285	10.0742
$\tilde{\lambda}_{retail}$	51.4726	59.5604	67.6482	75.7360	83.8238
$\ddot{\lambda}_{corp}$	1.5193	1.8541	2.1889	2.5236	2.8584
$\ddot{\lambda}_{retail}$	72.928	79.5552	86.1819	92.8087	99.4355
λ_{corp}	3.6384	5.0458	6.4533	7.8607	9.2681
λ_{retail}	46.5293	55.8997	65.2702	74.6406	84.0110

The values at each grid points of risk factors $\lambda_{corp}, \lambda_{retail}, (\tilde{\lambda}_{corp}, \tilde{\lambda}_{retail})'$, and $(\ddot{\lambda}_{corp}, \ddot{\lambda}_{retail})'$ are determined by dividing the range of the simulated values into 5 equal bands. The grid points are chosen from the lower bounds of each band. The steady state distribution is obtained by letting the simulation run for 30 years of data and the distribution is collected from the cross-sectional distribution at the end of year 30. At the end of year 30, the residual value of the loan outstanding is less than 1.0e-6 for any loan characteristics involved in this study.

3.4. The generated loss distributions

Table 5 presents the default rate and loss distribution over the 6-year horizon. The 6-year horizon is the time when the loan portfolio with the slowest rate of repayment ($\delta_i = 0.5$) are paid out with 99% of the original principal.

Table 5: The default rate and cumulative loss distributions at the end of horizon.

Process		6-year			
		Average (%) x10^2	Standard Deviation	Skewness	Kurtosis
$\tilde{\lambda}_{corp}$	Default	0.2914	6.6252	0.1080	-0.0200
	Loss	0.1354	0.0287	0.0102	-0.0479
$\tilde{\lambda}_{retail}$	Default	0.5471	2.6097	0.0329	-0.0990
	Loss	0.0533	0.0025	0.0252	-0.0996
$\ddot{\lambda}_{corp}$	Default	0.0953	2.8598	0.2897	0.0318
	Loss	0.0465	0.0136	0.2482	-0.0116
$\ddot{\lambda}_{retail}$	Default	0.5335	2.5819	0.0280	-0.0364
	Loss	0.0520	0.0024	0.0203	-0.0360
λ_{corp}	Default	0.1918	6.1687	0.3097	0.0995
	Loss	0.0919	0.0279	0.1917	0.0089
λ_{retail}	Default	0.5407	2.5993	0.0359	-0.0560
	Loss	0.0527	0.0024	0.0358	-0.0319

The simulated default rate and loss distribution, with 10000 sample paths, from process parameters described in Table 2.

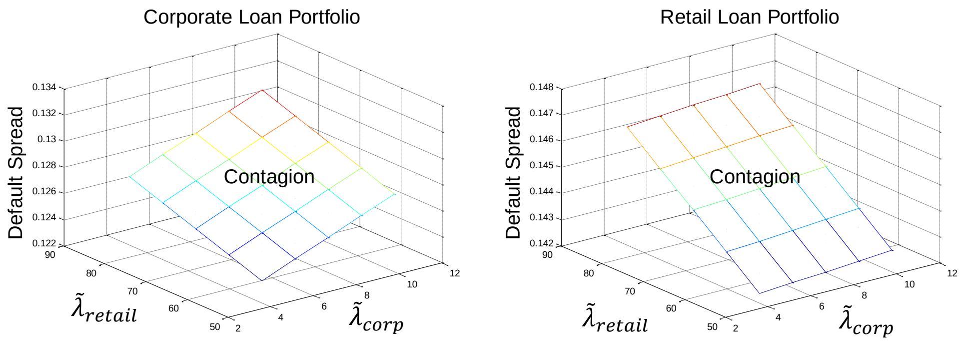
Table 5 shows that the corporate portfolios demonstrate the lower expected default rate and the higher volatility and skewness of default rate as compared to the retail portfolios. However, the corporate portfolios have the higher loss ratio. The higher loss ratio in corporate portfolios makes the expected loss rate of the corporate portfolios higher than that of the retail portfolios. In addition, the contagion effect makes the expected loss rate of the corporate portfolios even higher as compared to those of the retail loan portfolios. Consequently, the contagion factor creates the stronger cascading default effect in the corporate portfolios that make the loss concentrated at the body of the loss distribution. As a result, the loss distributions of the corporate portfolios become less skew. Table 5 shows that the loss distribution of the corporate portfolio with self- and cross-exciting contagion becomes less skew than that of the retail loan portfolio. The loss distribution of the corporate portfolio with only self-exciting contagion becomes less skew but the level of the skewness is still higher than that of the retail loan portfolio, which is subject to the same contagion effect.

3.5. The default spreads of loan portfolios

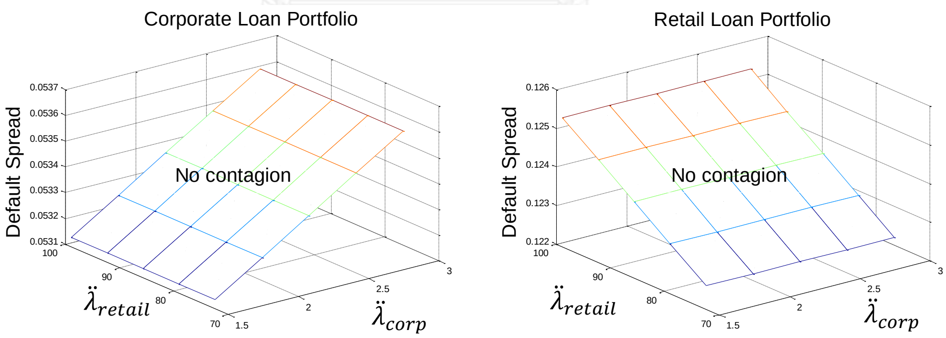
Figure 1 presents the curves of default spread, which represents the interest rate on loan portfolio, generated from the process parameters$(\tilde{\lambda}_{corp}, \tilde{\lambda}_{retail})'$,

$(\ddot{\lambda}_{corp}, \ddot{\lambda}_{retail})'$ and $\lambda_{corp}, \lambda_{retail}$ at each level of the default intensities. The default spread is the value of one minus discount factor, which is the cumulative compensation bank earns from issuing a loan over the entire life of the loan. The computation for the discount factor (F) is calculated over a 10-year horizon as an approximation to the infinite horizon. The 10-year horizon gives the residual of portfolios with repayment rate 0.75 and 0.5 at 0.0005 and 0.0067 respectively.

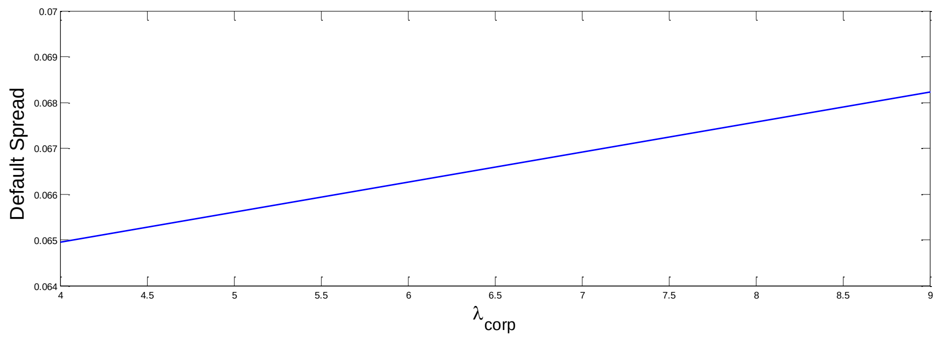
Figure 1: Default spread of loan portfolios.



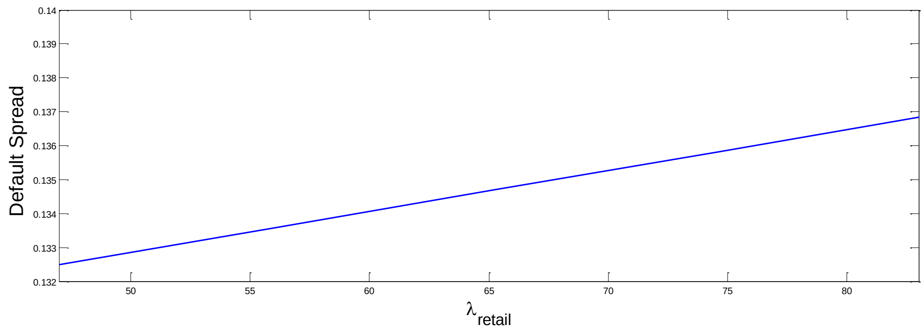
Panel a Default spread of corporate and retail loan portfolio with intensity $(\tilde{\lambda}_{corp}, \tilde{\lambda}_{retail})'$.



Panel b) Default spread of corporate and retail loan portfolio with intensity $(\ddot{\lambda}_{corp}, \ddot{\lambda}_{retail})'$.



Panel c) Default spread of corporate loan portfolio with intensity λ_{corp}.



Panel d) Default spread of retail loan portfolio with intensity λ_{retail}.

The comparison between the default spread and the loss distribution generated by the dynamic default risk processes provides an evaluation of the level of the value generated from each loan portfolio over an entire life of loan. The low default spread translates into the low required rate of return on the issued loan portfolio as the bank expects a low level of loss from that loan portfolio. The expected loss rate and the value of the default risk premium determine the value of the default spread. Because the valuation of loan value is done under the risk neutral measure, the level of skewness of the loss distribution does not directly determine the default spread.

The high level of the skewness of the loss distribution of the corporate portfolios under the no contagion and under the self-contagion only is more than offset by the low value of the risk premium. In contrast, the higher default risk premium in the retail portfolios make the values of the default spreads much higher than the level of the expected loss. With the lower level of the skewness in the retail portfolios comparing to that in the corporate portfolios, the corporate portfolios face the chance of experiencing the larger loss from default, which cannot be compensated by a lower spread. This makes the long-run value of the retail loan portfolios superior to that of the corporate portfolios.

Not only have the lower level of the skewness, the retail portfolios exhibit the lower level of the second moments of the loss distributions. These help illustrate how lending in the retail portfolios should be more profitable as compared to the lending in the corporate portfolios considering the long-term perspective. For the same token, it helps explain how the contagion risk improves the lending profitability over that of the portfolios without contagion when the portfolios are priced according to their riskiness.

The maximum default spread of corporate loan portfolios with the intensity $\ddot{\lambda}_{corp}, \lambda_{corp},$ and $\tilde{\lambda}_{corp}$ are at 6.8%, 5.37%, and 13.00% respectively, while their upper three standard deviations of the loss rate are at 17.56%, 8.73%, and 22.15% respectively. This means there is a chance that the cumulative losses highly exceed the cumulative profits for these corporate loan portfolios. These outcomes give the chance of a deep negative profitability from issuing the loans to these portfolios. However, the loss distribution of the corporate portfolios under without the contagion and under with only the self-contagion have the higher skewness comparing to the loss distribution associated with the corporate loan under with both self- and cross-contagions. This means banks that lend to corporate the portfolios under without the contagion and under with only the self-contagion face a chance of experiencing a very large loss that exceeds the cumulative profit over the entire life of loan to the higher level than the corporate portfolio under the full contagion, which includes both cross- and self-exciting.

On the other hand, the maximum default spread of retail loan portfolios with intensity $\ddot{\lambda}_{corp}, \lambda_{corp},$ and $\tilde{\lambda}_{corp}$ are at 12.20%, 14.62%, and 13.68% respectively, while their upper three standard deviations of the loss rate are at 5.92%, 6.08%, and 5.99% respectively. This means it is highly likely that banks mostly earn a positive profit from lending to the retail customer as the cumulative profit exceed the cumulative loss over the entire life of loan.

Panel a) of figure1 shows that the default spread under the contagion model of the corporate loan portfolio depends on the risk level of both corporate and retail portfolio and the same is true for the retail loan portfolio but to the lesser degree. Although the cross-exciting coefficient from the corporate to retail is higher than that from the retail to corporate, the default spread of corporate loan demonstrates the stronger dependency to the intensity of retail loan than that of the corporate loan does to the intensity of the retail loan. This is because the higher loss per default of the corporate loan portfolio makes its default spread more sensitive to the change in default intensities of both corporate and retail loan portfolios, comparing to what the lower loss per default of the retail loan portfolio does to the default spread of its own. The relationship in (A.8) in the appendix A explains how the loss ratio (l) explains the default spread mathematically. The likelihood to default of the corporate portfolio under the full contagion effect depends on both the likelihood to default of both corporate and retail portfolios but the factor determining the size of the loss is only specific to the loss ratio of the corporate portfolio. The size of the loss directly relates to the expected loss rate that determines the default spread. As a result, the loss ratio is a factor determining the sensitivity of the default spread with the risk level of the related portfolios.

Although the risk premium of the retail portfolio is higher and the default intensity of the retail portfolio is subject to a higher cross-contagion effect from the corporate default, the loss ratio is lower comparing to that of the corporate loan. This is more than offset by the higher cross-contagion effect and the higher risk premium, which compensate the losses, makes the retail loan portfolio less sensitive to the default risk of the corporate loan portfolio.

Under the contagion risk, the higher the default risk either resides in the corporate or retail portfolio would create the increment in the default spread in the corporate portfolio to the higher level than that in the retail portfolio.

The following optimization result demonstrates the portfolio allocation that answers the research questions.

3.6. The optimization result

3.6.1. Portfolio problem under contagion risk

3.6.1.1. Optimal allocation of multiple loan portfolios under contagion and no contagion risk

Problems 1 and 3 help answer the following research question: How banks should alter the portfolio mix of the two major bank loan portfolios, corporate and retail portfolios, when there exists the contagion phenomenon if the contagion risk is observable? The problem 2 and 4 are similar to the problems 1 and 3 respectively with only the difference is that the bank can reduce the holding in loan portfolios by selling down the loans in the secondary market. The purpose of introducing the problem 2 and 4 is to get the benchmark of the optimal level that does not depend on the pre-existing allocation.

According to the problem set up, the maximum allocation limit of each loan portfolio is 0.63 of the total asset value and the minimum allocation limit of each loan portfolio is 0.07 of the total asset value.

Under both with and without contagion, the result shows that the bank chooses maximum position limit of loan allocation at 0.63 in all state variables for the retail loan portfolio when there is no loan sale restriction or the constraint of the existing position is not binding under the loan sale restriction. When there is loan sale restriction the position may be binding at the existing position constraint because the bank cannot rebalance the portfolio by reducing the position of the other loan and increasing the position of the retail loan to reach the non-constraint optimal position. Table 6 shows the results of the optimal holdings of the retail loan portfolios, which do not depend on the level of equity. It shows that bank holds the retail loan at the maximum possible position because the retail loan portfolio gives the higher risk-adjusted return comparing to the corporate loan portfolio. In addition, the retail loan portfolio creates a long-term value because of the higher risk premium and the lower positive skewness of the loss distribution. The result shows that the optimal holding does not depend on the value of equity.

Table 6: The optimal allocation to retail portfolios of the minimum pre-existing position.

	Position 0.07,0.07	Optimal allocation No loan sale Equity Value ≥ 0.1					Optimal allocation Available loan sale Equity Value ≥ 0.1				
Contagion	$\tilde{\lambda}_{corp}$ / $\tilde{\lambda}_{retail}$	3.89	5.43	6.98	8.52	10.07	3.89	5.43	6.98	8.52	10.07
	51.4726	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	59.5604	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	67.6482	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	75.736	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	83.8238	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	Average	0.63					0.63				

	Position 0.07,0.07	Optimal allocation No loan sale Equity Value ≥ 0.1					Optimal allocation Available loan sale Equity Value ≥ 0.1				
	$\ddot{\lambda}_{corp}$ $\ddot{\lambda}_{retail}$	1.54	2.04	2.54	3.03	3.53	1.54	2.04	2.54	3.03	3.53
Without contagion	53.70	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	55.64	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	57.59	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	59.53	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	61.47	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
	Average	0.63					0.63				

Table 6 shows that the bank chooses the maximum position limit with and without contagion. This is because the retail loan portfolio creates the higher value to the bank than the corporate loan does under both with and without contagion.

Table 7 shows the optimal allocation to the corporate portfolios. The table describes the optimal holding in the corporate loan portfolios under both with and without the contagion and with and without the loan sale market available for any value of equity and for the level of current position at 0.07 and 0.07 in the corporate and retail loan portfolios respectively. The optimal holding of the corporate portfolios under the availability of a loan sale market does not depend on the pre-existing position. The optimal holding of the corporate portfolios under a loan sale restriction depends on the pre-existing position of both corporate and retail portfolios, as the bank cannot sell down the current positions to reach the optimal holding when the current position of the corporate loan is higher than the optimal allocation.

Both under with and without the loan sale restriction, the lending level to the corporate loan portfolio under the contagion risk depends on the current level of default intensity and the level of equity when the equity is low. This is because the bank may face the future risk binding constraint from experiencing a large loss from defaults. As discussed in previous section, under the contagion risk the higher the default risk resides in the corporate or retail portfolio would create the higher default spread in the corporate portfolio, which in turn increases the profitability on the risk-adjusted basis. The study shows that the value of equity is limited at the value less than 0.5. The result demonstrates that with the equity value less than 0.5, the bank anticipates a risk binding constraint in the future; therefore, the bank lends less as compared to the optimal holding without the loan sale restriction. The optimal holding under without the loan sale restriction is the optimal holding that the bank faces the current risk binding constraint and this holding level is higher than the level under the loan sale restriction. With the equity value greater than or equal to 0.5, the bank does not face the current risk binding constraint nor anticipate a risk binding constraint in the near future; therefore, the bank lends at the maximum possible position under the contagion risk taking into consideration that the bank puts the lending priority at the retail portfolio. This means after bank allocates the highest possible allocation to the retail portfolios, which is at 0.63, bank lends as much as possible within the available capacity left, which is 0.35, as the lending creates value.

Table 7: The optimal allocation to corporate portfolios of the minimum pre-existing position.

	Position 0.07,0.07	Optimal allocation No loan sale Equity Value = 0.1					Optimal allocation Available loan sale Equity Value = 0.1				
Contagion	$\tilde{\lambda}_{corp}$ $\tilde{\lambda}_{retail}$	3.89	5.43	6.98	8.52	10.07	3.89	5.43	6.98	8.52	10.07
	51.4726	0.21	0.21	0.07	0.21	0.28	0.21	0.21	0.35	0.21	0.35
	59.5604	0.07	0.14	0.28	0.35	0.35	0.07	0.14	0.28	0.35	0.35
	67.6482	0.21	0.14	0.28	0.14	0.28	0.35	0.35	0.35	0.14	0.35
	75.736	0.28	0.07	0.07	0.28	0.28	0.35	0.07	0.07	0.35	0.35
	83.8238	0.28	0.28	0.21	0.28	0.21	0.35	0.35	0.21	0.35	0.35
	Average	0.2184					0.2744				
	Position 0.07,0.07	Optimal allocation No loan sale Equity Value = 0.1					Optimal allocation Available loan sale Equity Value = 0.1				
Without Contagion	$\ddot{\lambda}_{corp}$ $\ddot{\lambda}_{retail}$	1.54	2.04	2.54	3.03	3.53	1.54	2.04	2.54	3.03	3.53
	53.70	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	55.64	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	57.59	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	59.53	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	61.47	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	Average	0.07					0.07				
	Position 0.07,0.07	Optimal allocation No loan sale Equity Value = 0.25					Optimal allocation Available loan sale Equity Value = 0.25				
Contagion	$\tilde{\lambda}_{corp}$ $\tilde{\lambda}_{retail}$	3.89	5.43	6.98	8.52	10.07	3.89	5.43	6.98	8.52	10.07
	51.4726	0.21	0.21	0.07	0.21	0.35	0.21	0.21	0.35	0.21	0.35

	Position 0.07,0.07	Optimal allocation No loan sale Equity Value = 0.25					Optimal allocation Available loan sale Equity Value = 0.25				
	$\tilde{\lambda}_{corp}$ $\tilde{\lambda}_{retail}$	3.89	5.43	6.98	8.52	10.07	3.89	5.43	6.98	8.52	10.07
	59.5604	0.07	0.14	0.28	0.35	0.35	0.35	0.14	0.28	0.35	0.35
	67.6482	0.21	0.14	0.35	0.14	0.35	0.35	0.35	0.35	0.14	0.35
	75.736	0.35	0.35	0.07	0.35	0.35	0.35	0.35	0.07	0.35	0.35
	83.8238	0.35	0.35	0.21	0.35	0.21	0.35	0.35	0.21	0.35	0.35
	Average	0.2548					0.2968				
	Position 0.07,0.07	Optimal allocation No loan sale Equity Value = 0.25					Optimal allocation Available loan sale Equity Value = 0.25				
Without Contagion	$\ddot{\lambda}_{corp}$ $\ddot{\lambda}_{retail}$	1.54	2.04	2.54	3.03	3.53	1.54	2.04	2.54	3.03	3.53
	53.70	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	55.64	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	57.59	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	59.53	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	61.47	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	Average	0.07					0.07				
	Position 0.07,0.07	Optimal allocation No loan sale Equity Value $\geq$ 0.5					Optimal allocation Available loan sale Equity Value $\geq$ 0.5				
Contagion	$\tilde{\lambda}_{corp}$ $\tilde{\lambda}_{retail}$	3.89	5.43	6.98	8.52	10.07	3.89	5.43	6.98	8.52	10.07
	51.4726	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	59.5604	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	67.6482	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	75.736	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	83.8238	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	Average	0.35									
	Position 0.07,0.07	Optimal allocation No loan sale Equity Value $\geq$ 0.5					Optimal allocation Available loan sale Equity Value $\geq$ 0.5				
Without Contagion	$\ddot{\lambda}_{corp}$ $\ddot{\lambda}_{retail}$	1.54	2.04	2.54	3.03	3.53	1.54	2.04	2.54	3.03	3.53
	53.70	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	55.64	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	57.59	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	59.53	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	61.47	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
	Average	0.07					0.07				

Total allocation to corporate loan portfolio as a fraction of total asset.

Table 7 shows that bank increases the holding in the corporate portfolio as the equity becomes higher if there exists the contagion. The higher value of equity increases the risk taking capacity for the bank to hold a higher level of the risky asset that yields the positive value creation. The contagion component creates the value as the higher default risk turns into the higher default spread while the contagion component makes the loss concentrate at the body of the loss distribution, which leads to the decrease in the skewness of the loss distribution. This may allow the bank to earn a cumulative profit over a cumulative loss over the entire life of the loan on average at the higher level than that from lending to the corporate portfolio under without the contagion.

Table 8: The optimal allocation to retail portfolios of the maximum pre-existing position of corporate portfolio.

	Position 0.63,0.07	Optimal allocation No loan sale Equity Value ≥ 0.1					Optimal allocation Available loan sale Equity Value ≥ 0.1				
	$\ddot{\lambda}_{corp}$ $\ddot{\lambda}_{retail}$	1.54	2.04	2.54	3.03	3.53	1.54	2.04	2.54	3.03	3.53
Without Contagion	53.70	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	55.64	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	57.59	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	59.53	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	61.47	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
	Average	0.35					0.35				

Total allocation to retail loan portfolio as a fraction of total asset.

On the contrary, the optimal holding in the corporate loan portfolio without contagion is set at the minimum position limit for any value of equity and any value of the default risk under with and without the loan sale market. Two possible reasons explain the holdings at the minimum. First, the bank may face the current capital binding constraint even though the bank holds the corporate loan at the minimum lending limit. Second, the bank does not face the current capital constraint but lending to the corporate loan under no contagion decreases the value creation and the level of value destruction from this lending is higher that the value destruction from holding the cash, which requires the effective cost of holding, cash at the rate $r_d - r_f$. The result in Table 8 disproves the first conjecture as the bank with the pre-existing position of 0.07 in the retail portfolio and 0.63 in the corporate portfolio still choose to lend as much as possible in the retail portfolio for the available lending capacity. If the bank faces the capital constraint at the level of the minimum pre-existing position, which is 0.07 in the retail portfolio and 0.07 in the corporate portfolio, the bank cannot increase the holding to retail if the pre-existing position in the corporate portfolio is as high as 0.63 as the risk biding constraint will be more certain. Hence, the results support the reason that lending to the corporations under the no contagion decreases the bank's value creation. Without the contagion, the low expected loss rate results in the low default spread while the high level of skewness in the loss distribution is more than compensated by the default risk premium. This makes the cumulative loss higher than the cumulative profit the bank earns on average over the entire life of loan.

3.6.1.2. Factors determining optimal allocation for multiple loans portfolio

The following discussion focuses on the optimal allocation in the corporate loan portfolios for various values of the level of equity and default risk. Since the allocation to the retail loan portfolios does not change with the level of equity or default risk if the current position is not binding, I do not include the data for the retail loan in this discussion.

For the corporate loan portfolios, the results are similar for each set of state variables. I present the average of the optimal allocation across some state variables in Table 9. The average optimal allocation to the corporate loan portfolio increases with the equity value as shown in Panel a) of Table 9. In addition, the bank holds more corporate loan under the contagion risk. Comparing to the situation under without the loan sale restriction, it shows that the average optimal holding under the loan sale restriction is higher because the bank cannot sell down the loan when the current holding exceeds the optimal level.

Panel b) and Panel c) of Table 9 show that the bank slightly increases the holding in the corporate loan at the increasing level of $\tilde{\lambda}_{corp}$ and $\tilde{\lambda}_{retail}$, which confirms that the contagion risk improves the value creation and the higher level of the default risk means the higher value creation the bank gets from a higher level of lending. The optimal lending amounts in both portfolios do not exhibit a sharp increase in the level of default risk. This is because the bank is restricted with the current leverage so the bank cannot fully utilize the available capital for profitable lending during the situation of high risk and high return period, which is the period of high default loss that the bank can charge for the high default spread.

Table 9: Average optimal allocation to corporate loan and portfolio characteristics.

Equity Value	Contagion		No Contagion	
	No loan sale	Available loan sale	No loan sale	Available loan sale
0.1	0.3602	0.2744	0.3204	0.07
0.25	0.3801	0.2968	0.3204	0.07
0.5	0.4077	0.3472	0.3204	0.07
1	0.4092	0.3500	0.3204	0.07
2	0.4092	0.3500	0.3204	0.07
3	0.4092	0.3500	0.3204	0.07
5	0.4092	0.3500	0.3204	0.07

Panel a) the relationship between equity value and average allocation to corporate loan as a fraction of total asset, from optimal solution of problem5-6 for contagious loan portfolio and from optimal solution of problem7-8 for non-contagious loan portfolio.

$\tilde{\lambda}_{corp}$	Contagion		$\ddot{\lambda}_{corp}$	No Contagion	
	No loan sale	Available loan sale		No loan sale	Available loan sale
3.8913	35.2064	30.0543	1.5449	27.6639	6.0435
5.4370	35.2057	29.9982	2.0400	27.6639	6.0435
6.9827	35.2161	29.9676	2.5350	27.6639	6.0435
8.5285	35.2452	30.0390	3.0301	27.6639	6.0435
10.0742	35.2807	30.2175	3.5252	27.6639	6.0435

Panel b) the relationship between default intensity, $\tilde{\lambda}_{corp}$ and $\ddot{\lambda}_{corp}$, and average allocation in absolute value to corporate loan, from optimal solution of problem5-6 for contagious loan portfolio and from optimal solution of problem7-8 for non-contagious loan portfolio.

$\tilde{\lambda}_{retail}$	Contagion		$\ddot{\lambda}_{retail}$	No Contagion	
	No loan sale	Available loan sale		No loan sale	Available loan sale
51.4726	35.1669	29.9523	53.6980	27.6639	6.0435
59.5604	35.2439	30.0339	55.6419	27.6639	6.0435
67.6482	35.2187	30.1104	57.5858	27.6639	6.0435
75.7360	35.2756	30.0339	59.5297	27.6639	6.0435
83.8238	35.2490	30.1461	61.4736	27.6639	6.0435

Panel c) the relationship between default intensity, $\tilde{\lambda}_{retail}$ and $\ddot{\lambda}_{retail}$, and average allocation in absolute value to corporate loan, from optimal solution of problem5-6 for contagious loan portfolio and from optimal solution of problem7-8 for non-contagious loan portfolio.

3.6.1.3. Factors determining optimal value of multiple loan portfolios

The bank with the higher equity level can invest more on the risky asset. As long as the lending creates value, the higher equity value allows the bank to lend more, which helps increase the shareholders' value.

For the total bank portfolio, the results are similar across each set of the state variables. I present the average of the optimal value across state variables in Table 10. Panel a) of Table 10 shows that the optimal allocated portfolio creates shareholders' value and it increases with the level of equity in every case. Panels b) and c) of Table 10 show that the value increases with the increase in the level of default risk. This is true under both the contagion and without the contagion and both with and without the loan sale restriction. This is because the contagion changes the portfolio risk profile by making the loss concentrate at the body, which increases the expected loss rate and decreases the tail risk. As a result, the bank can charge the interest rate in correspondent with the high-expected loss rate, which yields the positive cumulative net profit over the entire life of the loan on average at the level that is higher than under no contagion.

The findings give an opposite view of the portfolio decision of the tradable asset under contagion risk. Instead of reducing the position, the contagion influences the bank to increase the position when the bank is free from the capital and position limit. The bank increases allocation to the less profitable portfolio for the higher-level risk when a high opportunity to create the higher profit exists. The corporate portfolio represents the less profitable portfolio, which provides the bank with the lower risk premium. The contagion effect makes the bank increase the holding in the less profitable loan when the high risk is more likely to earn more profit through the higher compensation in the higher default spread and the lower tail risk.

Table 10: Average optimal shareholders' value and portfolio characteristics.

Equity Value	Contagion		No Contagion	
	No loan sale	Available loan sale	No loan sale	Available loan sale
0.1	38.9848	39.0723	18.0340	18.5473
0.25	43.0829	43.2310	19.7303	20.4770
0.5	47.6334	47.8470	21.8033	22.6326
1	52.3225	52.6751	23.9619	24.8044
2	57.0494	57.6690	25.9884	26.7442
3	59.6607	60.4802	27.5932	28.2366
5	61.8363	63.0172	28.5749	28.9055

Panel a) the relationship between equity value and average optimal value, from optimal solution of problem5-6 for contagious loan portfolio and from optimal solution of problem7-8 for non-contagious loan portfolio.

$\tilde{\lambda}_{corp}$	Contagion		$\ddot{\lambda}_{corp}$	No Contagion	
	No loan sale	Available loan sale		No loan sale	Available loan sale
3.8913	51.3990	51.6172	1.5449	23.6451	24.5338
5.4370	51.4390	51.6544	2.0400	23.6569	24.5481
6.9827	51.4994	51.7093	2.5350	23.6509	24.5292
8.5285	51.5645	51.7645	3.0301	23.6786	24.5494
10.0742	51.6479	51.8431	3.5252	23.7157	24.5742

Panel b) the relationship between default intensity,$\tilde{\lambda}_{corp}$and $\ddot{\lambda}_{corp}$, and average optimal value, from optimal solution of problem5-6 for contagious loan portfolio and from optimal solution of problem7-8 for non-contagious loan portfolio.

$\tilde{\lambda}_{retail}$	Contagion		$\ddot{\lambda}_{retail}$	No Contagion	
	No loan sale	Available loan sale		No loan sale	Available loan sale
51.4726	51.2365	51.4569	53.6980	23.6253	24.5191
59.5604	51.3667	51.5761	55.6419	23.6632	24.5465
67.6482	51.5292	51.7310	57.5858	23.6554	24.5278
75.7360	51.6347	51.8402	59.5297	23.6987	24.5671
83.8238	51.7828	51.9845	61.4736	23.7047	24.5743

Panel c) the relationship between default intensity, $\tilde{\lambda}_{retail}$and $\ddot{\lambda}_{retail}$, and average optimal value, from optimal solution of problem5-6 for contagious loan portfolio and from optimal solution of problem7-8 for non-contagious loan portfolio.

3.6.2. Lending decision and the loan sale market

3.6.2.1. Optimal allocation of single loan portfolio and factors determining optimal allocation

The problems 5-8 help answer the second research question: should banks participate in the loan sale market to maximize the shareholders' value? If participating in the loan sale market increases the shareholders' value, banks should lend more if banks do not currently face the risk binding constraint but anticipate the binding constraint in the future. In addition, they should offload the loan in the secondary loan sale market if they face the current risk binding constraint. Table 11 and Table 12 present the results of optimal holding of loan portfolio of particular loan types in absolute amount, which is equal to $(D_{leverage} + 1) * V_i$. The value of $D_{leverage}$ and V_i are the optimal value getting from the optimization result.

The objectives of the optimization problems 5-8 are to choose the optimal absolute lending decision by adjusting the leverage and the lending allocation as a fraction of the total asset. The optimization result shows that the bank chooses the maximum leverage at 90 in any optimization problem and chooses the optimal level of position as a fraction of the total asset. Choosing the maximum leverage and adjusting the lending allocation is optimal as the effective cost of holding cash is positive, which is $r_d - r_f$, so the bank utilizes cash from the deposit for lending as much as possible. Therefore, the bank chooses the maximum deposit utilization by choosing the maximum leverage.

Panel a) of Table 11 shows the optimal holding of the bank that only lend to the corporate portfolio with and without loan sale restriction. The table on the left-hand side reports the optimal holding under the loan sale restriction when the bank does not face the current position constraint, which means the pre-existing position is no greater than the optimal holding. The table on the right-hand side reports the optimal holding without the loan sale restriction. The result shows that the banks that cannot access to the loan sale market lend less comparing to when they can. Banks that cannot access to the loan sale market anticipate the risk constraint binding in the future; therefore, they lend less than their current risk taking capacity. On the other hand, the banks that can access to the loan sale market lend as much as possible up to their risk limit if the lending creates value. It is confirmed by the result in Table 12 that the banks that participate in the loan sale market invest up to their maximum risk taking capacity, as their holdings are less that the maximum position limit. Table 12 shows the maximum position limit at each level of equity. The maximum position limit is the value of $(D_{leverage} + 1) * V_i$ when $D_{leverage}$ is maximum at 90 and V_i is maximum at 0.63.

The result in Panel a) of Table 11 shows that banks tend to increase the lending with the higher level of equity as they have more risk taking capacity. In addition, they increase the lending with the higher level of default risk, which demonstrates that the higher default risk creates the higher value to banks than holding the cash. The higher value of the default risk also requires more capital resource to cushion the loss; therefore, the loan with the higher default risk level utilizes more risk capacity. The trade-off between the value creation and the risk utilization at each value of the default risk and each value of equity determines the optimal holding of the corporate loan.

Holding the corporate loan portfolio improves the shareholders' value better than holding the cash because there is a cost of holding cash at rate $r_d - r_f$. This demonstrates that investing in the corporate loan portfolio generates the cumulative profit over the cumulative loss at the rate greater than $r_d - r_f$.

The result in Panel a) of Table 11 shows that banks holding the corporate portfolio that can access to the loan sale market lend at least as much as banks that cannot access to the loan sale market for every value of equity and every value of default risk. The difference in the optimal lending value of those between banks with and without the loan sale restriction tends to increase with equity. This shows that the weaker banks take more risk than the stronger capital based banks.

Panel b) of Table 11 shows the optimal holding of the banks that only lend to the retail portfolio with and without the loan sale restriction. The table on the left-hand side reports the optimal holding under the loan sale restriction when banks do not face the current position constraint, which means the pre-existing position is no greater than the optimal holding. The table on the right-hand side reports the optimal holding without the loan sale restriction. The result shows that banks that cannot access to the loan sale market lend at the same level with banks that can access to the loan sale market. Banks that cannot access to the loan sale market do not anticipate the risk constraint binding in the future; therefore, they lend as much as possible up to their maximum position limit or risk taking capacity limit similar to banks that can access to the loan sale market. This is because the retail portfolio is less subject to tail risk, in which its loss distribution is described by the negative kurtosis, and low skewness. It is confirmed by the result in Table 12 that banks that participate in the loan sale market invest up to their maximum risk position limit when their equity values are less than 3 and they invest up to their maximum risk capital limit when their equity is greater than or equal to 3.

Table 11: The optimization result of problem 5-8 without the current position constraint under loan sale restriction.

Corporate Loan Portfolio	Absolute target lending No loan sale					Absolute target lending Available loan sale				
λ_{corp} Equity	3.64	5.05	6.45	7.86	9.27	3.64	5.05	6.45	7.86	9.27
0.1	3.84	4.47	4.47	4.47	5.10	3.84	4.47	4.47	4.47	5.10
0.25	8.03	9.61	9.61	9.61	11.18	9.61	9.61	9.61	11.18	11.18
0.5	16.07	16.07	16.07	19.22	19.22	19.22	19.22	19.22	22.37	25.52
1	32.13	32.13	32.13	38.43	38.43	38.43	38.43	38.43	38.43	51.03
2	64.26	64.26	64.26	76.86	76.86	76.86	76.86	76.86	76.86	76.86
3	77.49	77.49	96.39	96.39	96.39	96.39	96.39	96.39	96.39	115.3
5	129.15	129.15	160.65	160.65	160.65	160.7	160.7	192.2	192.2	223.7
Average	53.18					62.80				

Panel a) On the left-hand side, the table shows total absolute lending to corporate loan portfolio when the current position and leverage of corporate loan under loan sale restriction is less than one of the followings: 0.28 and 90, 0.35 and 70, 0.42 and 60, 0.49 and 50, 0.56 and 40, 0.63 and 40. On the right-hand side, the table shows total absolute lending to corporate loan portfolio when the current position and leverage of corporate loan under loan sale available is less than 0.63 and 90.

Retail Loan Portfolio	Absolute target lending No loan sale					Absolute target lending Available loan sale				
λ_{retail} Equity	3.64	5.05	6.45	7.86	9.27	3.64	5.05	6.45	7.86	9.27
0.1	5.73	5.73	5.73	5.73	5.73	5.73	5.73	5.73	5.73	5.73
0.25	14.33	14.33	14.33	14.33	14.33	14.3325	14.3325	14.3325	14.3325	14.3325
0.5	28.67	28.67	28.67	28.67	28.67	28.665	28.665	28.665	28.665	28.665
1	57.33	57.33	57.33	57.33	57.33	57.33	57.33	57.33	57.33	57.33
2	114.66	114.66	114.66	114.66	114.66	114.66	114.66	114.66	114.66	114.66
3	152.88	152.88	152.88	152.88	152.88	152.88	152.88	152.88	152.88	152.88
5	254.80	254.80	254.80	254.80	254.80	254.80	254.80	254.80	254.80	254.80
Average	89.77					89.77				

Panel b) On the left-hand side, the table shows total absolute lending to retail loan portfolio when the current position and leverage of corporate loan under loan sale restriction is less than one of the followings: 0.56 and 90, 0.63 and 70. On the right-hand side, the table shows total absolute lending to retail loan portfolio when the current position and leverage of corporate loan under loan sale available is less than 0.63 and 90.

The result in Panel a) of Table 11 shows that banks tend to increase the lending with the higher level of equity as they have more risk taking capacity. However, the optimal lending does not increase with the default risk because banks already invest up to the maximum position limit, which shows in Table 12. The fact that banks invest more in the retail loan portfolio when they have the higher risk capacity help confirm that the retail loan portfolio creates the value higher than holding the cash.

Holding the retail loan portfolio improves shareholders' value better than holding the cash because there is a cost of holding cash at rate $r_d - r_f$. This demonstrates that investing in the retail loan portfolio generates the cumulative profit over the cumulative loss at the rate greater than $r_d - r_f$.

Table 12: The absolute maximum lending position limit.

Equity	0.1	0.25	0.5	1	2	3	5
Absolute lending	5.73	14.33	28.67	57.33	114.66	171.99	286.65

The table shows absolute lending to corporate and retail loan portfolio at the maximum position limit

3.6.2.2. Factors determining optimal shareholders' value for single loan portfolio

Previous discussion describes that holding the corporate or retail loan portfolio improves the value better than holding the cash. Panel a) of Table 13 shows the total value of banks holding the corporate loan or retail loan portfolio compared with the book value of equity. The total value of the bank is the optimal value from investing optimally in each type of loan, which represents the market value of equity, and the book value of equity is the capital resource that the bank has for its risk taking capacity. The result in Panel a) of Table 13 shows that banks only lending to corporations create value over the book values of equity when their equity values are at 0.1, which is the near bankruptcy status; however, their market values of equity deteriorate when their equity values are greater than 0.1. The market values of equity of banks only lending to corporations are less than their book values of equity for their book values greater than 0.1. The result in Table 12 shows that although investing only in the corporate loan portfolio deteriorates the market value, holding the cash diminishes the value even deeper. In contrast, banks only lending to the retail customers create value over the book values of equity for every values of book equity.

Panel b) of Table 13 shows that the values of banks holding either the corporate or retail loan portfolio increase with the increase in the level of default risk. This is because the higher level of default risk creates the higher values by allowing banks to lend at the higher rate in excess of the expected loss from default.

Panel b) of Table 13 shows that the banks holding the corporate loan can improve their shareholders' values by participating in the loan sale market. This is because the loan sale market helps increase their risk taking capacities as they exploit the loan sale market as a risk management tool that mitigates their risk binding constraint in the future.

Table 13: The determinants to optimal shareholders' value.

Equity Value	Corporate Loan Portfolio			Retail Loan Portfolio		
	No loan sale	Available loan sale	%Value gain	No loan sale	Available loan sale	%Value gain
0.1	0.1504	0.1663	10.57%	25.5731	25.5731	0.00%
0.25	0.1667	0.1834	10.02%	28.3795	28.3795	0.00%
0.5	0.2071	0.2265	9.37%	31.5258	31.5258	0.00%
1	0.2976	0.3249	9.17%	34.7672	34.7672	0.00%
2	0.4770	0.5212	9.27%	38.0632	38.0632	0.00%
3	0.6774	0.7429	9.67%	39.8305	39.8306	0.00%
5	1.0375	1.1595	11.76%	41.1428	41.1439	0.00%

Panel a) the relationship between equity value and the optimal shareholders' value, average across the current positions, the leverage and the default risk level, from optimal solution of problems 5-6 for corporate loan portfolio and from optimal solution of problems7-8 for retail loan portfolio.

λ_{corp}	Corporate Loan Portfolio			λ_{retail}	Retail Loan Portfolio		
	No loan sale	Available loan sale	%Value gain		No loan sale	Available loan sale	%Value gain
3.6384	0.3756	0.4189	11.53%	43.5318	33.9727	33.9728	0.00%
5.0458	0.3957	0.4384	10.79%	54.2051	34.1411	34.1412	0.00%
6.4533	0.4250	0.4674	9.98%	64.8784	34.1603	34.1605	0.00%
7.8607	0.4525	0.4982	10.10%	75.5517	34.2695	34.2697	0.00%
9.2681	0.5038	0.5520	9.57%	86.2250	34.3722	34.3725	0.00%

Panel b) the relationship between default intensity and the optimal shareholders' value, average across the current positions, the leverage and the default risk level, from optimal solution of problems5-6 for corporate loan portfolio and from optimal solution of problems7-8 for retail loan portfolio.

The analysis shows that loan sale market increases an opportunity to lend more for banks with the capital limit and banks are required to maintain the sufficient capital to comply with the risk management policy.

4. CONCLUSION

This study examines the effect of contagion and loan sale market in banks' portfolio decision making. It answers to the research question that banks alter the portfolio mix when there exists the contagion phenomenon by increasing the allocation to the lower profitable lending segment, which is the corporate loan portfolio, with the higher allocation during the higher risk period as long as the risk capital is sufficient. If bank can alter the interest rate charged on the corporate loan portfolio based on the change in the perceived default risk of all the risk factors affecting the interest rate, the contagion increases the profitability and alters the less attractive loan portfolio to be more attractive. In general, bank lends more with the contagion effect and the level of lending increases with the default risk and the equity value.

It also answers the research question that the loan sale market increases the lending when the capital is limited taking into consideration that the bank conforms to risk management requirement and shareholders' value maximizing objective. The corporate loan represents the lower risk-adjusted return on investment, which potentially makes the risk constraint binding in a certain state of economy for the bank with limited capital. The retail loan represents the higher risk-adjusted return investment and less likely makes the risk constraint binding. Accessible to the loan sale market has an influential effect for banks holding corporate loan portfolio but does not affect banks holding only the retail loan portfolio. As banks with the corporate loan portfolio anticipate risk binding constraint in the future, banks participate in the loan sale market for a risk management purpose. This finding is consistent with the finding from Cebenoyan and Strahan (2004). Besides supporting the argument of Cebenoyan and Strahan (2004), I further argue that bank maximizes shareholders' value by participating in the loan sale market that enhances the risk management effectiveness. The study of the validity of the secondary loan sale market could lead to the future study that suggests the optimal lending activities in the system and the optimal participation level in the loan sale market, which lies between the over-participation in the loan sale market or the under-participation. In addition, it serves as a basis for further study on the equilibrium market value of loans, which influences an optimal level of the participation in the loan sale market. The over-participation would be a sign of the dominance of the originate-to-distribute model in which banks grant a high level of loan beyond the current risk taking capacity limit while the under-participation would be the evidence of an inefficient resource allocation due to the lower level of lending than the optimal value under loan sale availability. The high level of under-participation could lead to the event of credit crunch in the primary loan market.

The study relies on the assumption that the risk premium is constant. When the risk premium varies by the risk factor, the attractiveness of loan portfolios randomly changes by the state of economy. As a result, I expect to get a different optimal portfolio policy. Varying risk premium by a level of risk factor could represent the different level of the default risk premium to compensate the surprise loss during the different economic environment under the different level of risk. Therefore, this suggests a future research for the effect of contagion on the varying risk premium and portfolio decision.

APPENDIX

Appendix A.

Considering the loan of outstanding $L_{i\tau}$ at time τ with rate of repayment δ_i and with the default process $N_{i\tau}$ with intensity:

$$\tilde{\lambda}_{it} = \gamma_i \lambda_{it},$$

and λ_{it} follows (1). Assume the net funding cost for bank is $r_d - r_f$, which is the interest rate paid on the deposit over the rate of return on risk free rate. Define the numeraire under the risk-neutral measure as $e^{(r_d - r_f)\tau}$, the value of the repayment element $\delta_i L_i$ under risk-neutral measure is given by:

$$V_{\delta_i L_{i\tau}} = \delta_i \mathbb{E}^Q[e^{-(r_d - r_f)\tau} L_{i\tau}]. \tag{A.1}$$

Assuming the loan outstanding at the current period is L_{i0} , the remaining outstanding under risk neutral at time τ, taking into consideration of repayment and loss from default, which can be derived from (5), is given by:

$$L_{i\tau} = e_i^{-\tau\delta_i + \ln(1-l_i)^{\tilde{N}_{i\tau}}} L_{i0}. \tag{A.2}$$

Therefore, the valuation of loan portfolio i can be taken by integrating $V_{\delta_i L_{i\tau}}$ over τ from the current time to infinity to get the summation of all value of the piece of cash repayment at each point in time τ. That is,

$$V_i = \delta_i \int_0^\infty \mathbb{E}^Q[e^{-(r_d - r_f)\tau} L_{i\tau}] d\tau. \tag{A.3}$$

From (A.2) and (A.3) I get:

$$V_i = \delta_i \int_0^\infty \mathbb{E}^Q[e_i^{-(r_d - r_f)\tau - \tau\delta_i} (1 - l_i)^{\tilde{N}_{i\tau}} L_i] d\tau. \tag{A.4}$$

By integrability assumption, I get:

$$V_i = \delta_i \mathbb{E}^Q \left[\int_0^\infty e_i^{-(r_d - r_f)\tau - \tau\delta_i} (1 - l_i)^{\tilde{N}_{i\tau}} L_i \, d\tau \right]. \tag{A.5}$$

Apply iterative law of expectation:

$$V_i = \delta_i \mathbb{E}^Q \left[\mathbb{E}^Q \left[\int_0^\infty e_i^{-(r_d - r_f)\tau - \tau\delta_i} (1 - l_i)^{\tilde{N}_{i\tau}} L_i \, d\tau \middle| \lambda_{i\tau}; \tau \ \geq 0 \right] \right]. \tag{A.6}$$

Apply the probability generating function of Poisson random variable,
$\mathbb{E}^Q\left[(1 - l_i)^{\tilde{N}_{i\tau}} | \lambda_{i\tau}; \tau \ \geq 0\right] = e^{-\int_0^\tau l_i \tilde{\lambda}_{it} dt}$,

$$V_i = \delta_i L_i \mathbb{E}^Q [\int_0^\infty e_i^{-(r_d - r_f)\tau - \int_0^\tau l_i \tilde{\lambda}_{it} dt} \, d\tau]. \tag{A.7}$$
$$V_i = \delta_i L_i \int_0^\infty e_i^{-(r_d - r_f)\tau} \, \mathbb{E}^Q \left[e^{-\int_0^\tau l_i \tilde{\lambda}_{it} dt} \right] d\tau. \tag{A.8}$$

According to Duffie et al. (2000),

$$\mathbb{E}^Q \left[e^{-\int_0^\tau l_i \tilde{\lambda}_{it} dt} \right] = e^{A_i(0,\tau;l_i) + B_i'(0,\tau;l_i)\tilde{\Lambda}}, \tag{A.9}$$

where $A_i(0, \tau; l_i)$ and $B_i(0, \tau; l_i)$ solve the system of ordinary differential equations, with the terminal condition $B_i(\tau, \tau; l_i) = \vec{0}$ and $A_i(\tau, \tau; l_i) = 0$, as follows:

$$\frac{\partial B_i(0,\tau,t;l_i)}{\partial t} = l + \zeta B_i - \left(e^{\eta' B_i} - \vec{1}\right), \tag{A.10}$$
$$\frac{\partial A_i(0,\tau,t;l_i)}{\partial t} = \ -\zeta \phi B_i, \tag{A.11}$$

where $\vec{0}$ is the I – dimensional vector of zeros and $\vec{1}$ is the I – dimensional vector of ones. The I – dimensional vector l is the vector with all elements are zero excepts the i^{th} element, which is the the value l_i. The term $e^{\eta' B_i}$ is an element-wise exponential function of $\eta' B_i$.

From (A.8) and (A.9) I get (3) and (4) accordingly.

Q.E.D.

Appendix B.

The optimization problem (15) is the infinite stage, finite states discount problem (Bertsekas, 2012). To solve (15), I apply the modified policy iteration approach from Puterman (2005) by building the state transition probability matrix $p(Z_s|m_s)$ shown in (16) constructed from the simulation of 500 sample paths. The policy iteration and value iteration are executed one after the other. The steps are as follows:

1) Determine the initial policy $\widetilde{U}_0(Z)$ and the initial value function $\mathrm{J}_0(Z)$.
2) Calculate $\mathrm{J}_\mathrm{m}(Z)$ according to the following equation.

$$\mathrm{J}_\mathrm{m}(Z) = \mathbb{E}\left[G|f\left(Z, U_0(Z)\right)\right] + e^{-k\Delta\tau K}Q^*(f\left(Z, \widetilde{U}_0(Z)\right)), \quad \text{(B.1)}$$

   for 120 iterations with the initial policy $\widetilde{U}_0(Z)$.
3) Search for the optimal policy for the latest updated value function $J_m(Z)$.
4) Update the value function $J_{update}(Z)$ with the new updated policy.
5) Check if the value improvement satisfies $\| J_{update}(Z) - J_m(Z) \| < \epsilon(1 - e^{-k\Delta\tau K})e^{2k\Delta\tau K}$; if satisfies, stop and get the optimal value function and optimal policy.; if does not satisfy, go back to step 1) with the updated policy and the updated value function.

With the discount factor, $e^{-k\Delta\tau K}$ and parameters k, K and time step $\Delta\tau$, used in the study, the result converges, the change in value less than $1.2e^{-17}$, after 10 policy iterations.

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APPENDIX

VITA

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