CHAPTER II

SHAPIRO'S RESULTS

As stated in the introduction, we are interested in finding a suitable condition on a sequence of independent random variable $\{X_n\}$ so that there exist sequences of real numbers $\{A_n\}$ and $\{B_n\}$ such that the distribution functions of

$$\frac{1}{B_n}\sum_{k=1}^{n}\frac{1}{|X_k|^r}-A_n$$

converge to a limit, where r is a positive real number. It is obvious that the choice of $\{A_n\}$ and $\{B_n\}$ depend on the parameter r, hence, for clarity, we will denote them by $\{A_n(r)\}$ and $\{B_n(r)\}$ for each r, respectively.

In this chapter we state the results obtained by Shapiro.

1. Identically Distributed Summands

When the problem was first examined by Shapiro (c.f.[2] and [3]), the independent random variables were assumed to be identically distributed and the distribution function was assumed to be absolutely continuous, that is, they possess a common distribution function and a common density function. We shall now give the results in the following two theorems.

Theorem 4. Let $X_1, X_2, \ldots$ be a sequence of identically distributed independent random variables with probability density function f. Assume that f is continuous at 0 and f(0) is nonzero. Then for each positive real number r, there exist sequences of real constants $\{A_n(r)\}$ and $\{B_n(r)\}$ such that the distribution of the sum

$$\frac{1}{B_n(r)} \sum_{k=1}^{n} \frac{1}{|X_k|^r} - A_n(r)$$

converges to a limit, and the limit distribution is a normal law for $0 < r \le \frac{1}{2}$ and is a stable law with characteristic exponent $\frac{1}{r}$ for $r > \frac{1}{2}$.

Theorem 5 Under the assumption of Theorem 4, the limit distribution of

$$\frac{1}{n} \sum_{k=1}^{n} \frac{1}{X_k} - A_n$$

converges to the Cauchy law with parameter $\lambda = \pi f(0)$ and $\nu = 0$. That is,

$$\lim_{n\to\infty} P\left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{X_k} - A_n \le x\right) = \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan \frac{x}{\pi f(0)}\right).$$

Moreover, the constants A_n can be chosen to be

$$A_n = \int_{|u|>\frac{1}{n}} \frac{f(u)}{u}\, du .$$

2. A More General Case.

The problem becomes more interesting if the conditions on the random variables are less restrictive.

Shapiro has extended his study to the more general case where the random variables are not assumed to be identically distributed. He obtained the following partial results.

Theorem 6 Let $X_1, X_2, \ldots$ be a sequence of independent random variables and f_k be the probability density function of X_k. Assume that

1) the family $\{f_k\}$ is equicontinuous at 0, and
2) the Cesàro limit L of the sequence $\{f_k(0)\}$ exists and is positive.

Then for $r > \frac{1}{2}$ and for suitably chosen $A_n(r)$ the distribution function of

$$\frac{1}{n^r} \sum_{k=1}^{n} \frac{1}{|X_k|^r} - A_n(r)$$

converges to a stable law with characteristic exponent $\frac{1}{r}$.

Theorem 7 Under the same hypotheses as in Theorem 6, the distribution function of $\frac{1}{n} \sum_{k=1}^{n} \frac{1}{X_k} - A_n$ converges to the Cauchy distribution

$$F(x) = \frac{1}{\pi} \left(\frac{\pi}{2} + \arctan \frac{x}{\pi L}\right),$$

where the constants A_n are given by

$$A_n = \frac{1}{n} \sum_{k=1}^{n} \int_{|y| > \frac{1}{n}} \frac{f_k(y)}{y} \, dy.$$