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Generalization of Prime Ideals in Γ-semirings

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คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

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Generalization of Prime Ideals in Γ-semirings

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ในแกมมากึ่งริงสลับที่ เรานิยาม 2-แอบซอร์บบิงไอดีล, 2-แอบซอร์บบิงไพรแมรีไอดีล และ 2-แอบซอร์บบิงเดลตาไพรแมรีไอดีลซึ่งต่างเป็นการวางนัยทั่วไปของไอดีลเฉพาะ นอกจากนี้ เราศึกษาสมบัติต่างๆของ 2-แอบซอร์บบิงไพรแมรีไอดีล และ2-แอบซอร์บบิงเดลตาไพรแมรีไอดีล ยิ่งไปกว่านั้น เราให้เงื่อนไขที่จำเป็นและเพียงพอในการเป็น 2-แอบซอร์บบิงไพรแมรีไอดีล และ2-แอบซอร์บบิงเดลตาไพรแมรีไอดีล

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต . . . .

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NUTTAWUT SANGJAER: GENERALIZATION OF PRIME IDEALS IN Γ-SEMIRINGS. ADVISOR: ASSOC. PROF. SAJEE PIANSKOOL, PH. D. , 41 PP.

In commutative Γ-semirings, the definition of 2-absorbing ideals, 2-absorbing primary ideals and 2-absorbing δ-primary ideals which are generalizations of prime ideals are given. Then, various properties of 2-absorbing primary ideals and 2-absorbing δ-primary ideals are investigated. Moreover, characterizations of ideals to be 2-absorbing primary ideals and 2-absorbing δ-primary ideals are provided.

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Contents

Chapter 1

Introduction

As a generalization of rings, Γ-rings were introduced by N. Nobusawa [7] in 1964. Also, as a generalization of semirings and Γ-rings, the notion of Γ-semirings was introduced by M. K. Rao [8] in 1995. Some properties of ideals and k-ideals in a Γ-semiring were also discussed by M. K. Rao [8] in 1995 and T. K. Dutta and S. K. Sardar [3] in 2000. T. K. Dutta and S. K. Sardar [4] in 2001 gave the definition of prime ideals in Γ-semirings and studied some of their properties. In 2017, M. K. Rao and B. Venkateswarlu [11] initiated the definition of primary ideals in Γ-semirings which is a generalization of prime ideals in Γ-semirings.

The concept of 2-absorbing ideals in commutative rings was introduced by A. Badawi [1] in 2007 which is a generalization of prime ideals in commutative rings. Recently, A. Badawi [2] in 2014 introduced the concept of 2- absorbing primary ideals in commutative rings and gave some characterizations related to it. This was also extended to commutative semiring. The notion of 2-absorbing primary ideals in commutative semirings was introduced by P. Kumar [6] in 2016. Moreover, M. Y. Elkettani and A. Kasem [5] in 2016 extended the concept of prime ideals and primary ideals in Γ-rings to 2-absorbing δ-primary ideals in Γ-rings which unify 2-absorbing ideals and 2-absorbing primary ideals in Γ-rings.

These inspired us to generalize those concepts to commutative Γ-semiring. Our main goal is to provide the notion of 2-absorbing primary ideals and 2-absorbing δ-primary ideals in commutative Γ-semirings. Also, we study these properties and provide some of their characterizations.

This report is organized as follows:

In Chapter 2, we provide some basic results which will be applied later.

In Chapter 3, radical ideals are introduced and are studied.

In Chapter 4, one of our main results, 2-absorbing primary ideals are provided. Moreover, some properties are studied. At the end, some characterizations are given.

In Chapter 5, expansion of ideals are proposed. These will be a main tool for Chapter 6.

In Chapter 6, the other of our main results, 2-absorbing δ-primary ideals are given. We investigate their properties and characterizations.

Chapter 2

Preliminaries

In this chapter, we recall some of fundamental concepts and definitions which are necessary for this project.

Definition 2.1. [8] For any commutative semigroups $(R, +)$ and $(\Gamma, +)$, R is called a **Γ-semiring** if there exists a function $\cdot$ from $R \times \Gamma \times R$ into R, where $\cdot(x, \gamma, y)$ is denoted by $x\gamma y$ for all $x, y \in R$ and $\gamma \in \Gamma$, satisfying the following properties: for all $x, y, z \in R$ and $\gamma, \beta \in \Gamma$,

1. $x\gamma(y + z) = x\gamma y + x\gamma z$ and $(x + y)\gamma z = x\gamma z + y\gamma z$;
2. $x(\gamma + \beta)y = x\gamma y + x\beta y$; and
3. $(x\gamma y)\beta z = x\gamma(y\beta z)$.

Throughout this project, let $\mathbb{Z}_0^+$ be the set of non-negative integers. Then $\mathbb{Z}_0^+$ is a semigroup under the usual addition. For a Γ-semiring R, $A, B \subseteq R$ and $\beta \in \Gamma$, let $A\Gamma B = \{ a\gamma b \mid a \in A, \gamma \in \Gamma \text{ and } b \in B \}$ and $A\beta B = \{ a\beta b \mid a \in A \text{ and } b \in B \}$.

Example 2.2. (1) Let R be the commutative semigroup containing all $m \times n$ matrices over $\mathbb{Z}_0^+$ under the usual addition and Γ be the commutative semigroup containing all $n \times m$ matrices over $\mathbb{Z}_0^+$ under the usual addition. Then R is a Γ-semiring where $a\gamma b$ is the usual matrix product for any $a, b \in R$ and $\gamma \in \Gamma$.

(2) For each $n \in \mathbb{N}$, recall that $n\mathbb{Z}_0^+ = \{ na \mid a \in \mathbb{Z}_0^+\}$ is a commutative semigroup under the usual addition of integers. Then $n\mathbb{Z}_0^+$ is an $m\mathbb{Z}_0^+$-semiring for all $m, n \in \mathbb{N}$ where $x\gamma y$ is the usual multiplication of integers for all $x, y \in n\mathbb{Z}_0^+$ and $\gamma \in m\mathbb{Z}_0^+$.

Definition 2.3. [8] A Γ-semiring R is said to have a **zero** element if there exists an element $0 \in R$ such that $x + 0 = x$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in R$ and $\alpha \in \Gamma$.

Definition 2.4. [10] A Γ-semiring R is said to have a **unity** element if there exists an element $1 \in R$ such that for all $x \in R$, there exists $\alpha \in \Gamma$ such that $1\alpha x = x = x\alpha 1$.

Definition 2.5. [8] A Γ-semiring R is said to be **commutative** if $x\alpha y = y\alpha x$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Definition 2.6. [8] Let R be a Γ-semiring and A be a subset of R. Then A is called a Γ**-subsemiring** of R if A is a subsemigroup of $(R, +)$ and $A\Gamma A \subseteq A$.

Proposition 2.7. [8] *Let R_i be a Γ_i-semiring for all $i \in \{1, 2, \ldots, n\}$. Then $R_1 \times R_2 \times \cdots \times R_n$ is a $(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n)$-semiring where*

$$(x_1, x_1, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \text{ and}$$
$$(x_1, x_1, \ldots, x_n)(\gamma_1, \gamma_2, \ldots, \gamma_n)(y_1, y_2, \ldots, y_n) = (x_1\gamma_1 y_1, x_2\gamma_2 y_2, \ldots, x_n\gamma_n y_n)$$

for all $x_i, y_i \in R_i$, $\gamma_i \in \Gamma_i$ and $i \in \{1, 2, \ldots, n\}$.

Moreover, if R_i is commutative for all $i \in \{1, 2, \ldots, n\}$, then the $(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n)$-semiring is also commutative.

Definition 2.8. [8] A subset I of a Γ-semiring R is called an **ideal** in R if I is a subsemigroup of $(R, +)$, $I\Gamma R \subseteq I$ and $R\Gamma I \subseteq I$.

It is clear that a Γ-semiring R is an ideal in R. Moreover, if R is a Γ-semiring with zero 0, then $0 \in I$ for all ideal I in R.

Definition 2.9. [8] An ideal I in a Γ-semiring R is called a k**-ideal** in R if for all $x, y \in R, x + y \in I$ and $x \in I$ implies $y \in I$.

Example 2.10. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a $5\mathbb{Z}_0^+$-semiring. Then $3\mathbb{Z}_0^+$ is a k-ideal in $\mathbb{Z}_0^+$. However, $3\mathbb{Z}_0^+ - \{3\}$ is an ideal in $\mathbb{Z}_0^+$ but it is not a k-ideal in $\mathbb{Z}_0^+$ because $6 + 3 \in 3\mathbb{Z}_0^+ - \{3\}, 6 \in 3\mathbb{Z}_0^+ - \{3\}$ but $3 \notin 3\mathbb{Z}_0^+ - \{3\}$.

Proposition 2.11. *[4] Let R be a Γ-semiring with zero and $a \in R$. Define*

$$\langle a \rangle = \{\ na + \sum_{j=1}^{p} a\eta_j t_j + \sum_{k=1}^{q} u_k\delta_k a + \sum_{l=1}^{s} v_l\mu_l a\lambda_l w_l \mid$$
$$n \in \mathbb{Z}_0^+, p, q, s \in \mathbb{Z}^+, t_j, u_k, v_l, w_l \in R \text{ and } \eta_j, \delta_k, \mu_l, \lambda_l \in \Gamma\ \}.$$

Then $\langle a \rangle$ is an ideal in R containing a.

Proposition 2.12. *Let R_i be a commutative Γ_i-semiring for all $i \in \{1, 2\}$. Then I is an ideal in the $(\Gamma_1 \times \Gamma_2)$-semiring $R_1 \times R_2$ if and only if $I = I_1 \times I_2$ for some ideals I_1 in R_1 and I_2 in R_2.*

Proof. First, suppose that I is an ideal in $R_1 \times R_2$. Let $I_1 = \{x \mid (x, y) \in I$ for some $y \in R_2\}$ and $I_2 = \{y \mid (x, y) \in I$ for some $x \in R_1\}$. Since $I \neq \emptyset$, it follows that $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. Let $a, b \in I_1, \gamma \in \Gamma_1$ and $r \in R_1$. Since $I_2 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$, let $y \in I_2$ and $\beta \in \Gamma_2$. Since I is an ideal in $R_1 \times R_2$, we have $(a + b, y + y) = (a, y) + (b, y) \in I$, $(a\gamma r, y\beta y) = (a, y)(\gamma, \beta)(r, y) \in I$ and $(r\gamma a, y\beta y) = (r, y)(\gamma, \beta)(a, y) \in I$. So, $a + b \in I_1$, $a\gamma r \in I_1$ and $r\gamma a \in I_1$. Hence, I_1 is a subsemigroup of $(R_1, +)$, $I_1\Gamma_1 R_1 \subseteq I_1$ and $R_1\Gamma_1 I_1 \subseteq I_1$. Therefore, I_1 is an ideal in R_1. Similarly, I_2 is an ideal in R_2.

Next, suppose that $I = I_1 \times I_2$ for some ideals I_1 in R_1 and I_2 in R_2. Since $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$, we have $I = I_1 \times I_2 \neq \emptyset$. Let $x_1, x_2 \in I_1, y_1, y_2 \in I_2, \gamma \in \Gamma_1, \beta \in \Gamma_2, r_1 \in R_1$ and $r_2 \in R_2$. Since $x_1 + x_2 \in I_1$ and $y_1 + y_2 \in I_2$, it follows that $(x_1 + x_2, y_1 + y_2) \in I_1 \times I_2 = I$. Since $x_1\gamma r_1, r_1\gamma x_1 \in I_1$ and $y_1\beta r_2, r_2\beta y_1 \in I_2$, $(x_1, y_1)(\gamma, \beta)(r_1, r_2) \in I_1 \times I_2 = I$ and $(r_1, r_2)(\gamma, \beta)(x_1, y_1) \in I_1 \times I_2 = I$. Hence, I is a subsemigroup of $(R_1 \times R_2, +)$, $(I)(\Gamma_1 \times \Gamma_2)(R_1 \times R_2) \subseteq I$ and $(R_1 \times R_2)(\Gamma_1 \times \Gamma_2)(I) \subseteq I$. Therefore, I is an ideal in $R_1 \times R_2$. □

Chapter 3

Radical Ideals

Throughout this chapter, properties of radical ideals in commutative Γ-semirings are investigated. However, we focus on those which are involving with 2-absorbing primary ideals and 2-absorbing δ-primary ideals which will be applied later. In this chapter, let R be a commutative Γ-semiring.

We begin this chapter by providing the definition of the radical ideal.

Proposition 3.1. [11] *Let I be an ideal in R. Then*

$$\sqrt{I} := \{x \in R \mid \text{ there exists } n \in \mathbb{N} \text{ such that } (x\gamma)^{n-1}x \in I \text{ for all } \gamma \in \Gamma\}$$

*is an ideal in R containing I where $(x\gamma)^0 x = x$ and $(x\gamma)^n x = (x\gamma)^{n-1}x\gamma x$ for all $x \in R, \gamma \in \Gamma$ and $n \in \mathbb{N}$. The ideal $\sqrt{I}$ is called the **radical ideal** of I.*

Proof. If $x \in I$, then $(x\gamma)^{1-1}x = x \in I$ so that $x \in \sqrt{I}$. Hence, $I \subseteq \sqrt{I}$.

To show that $\sqrt{I}$ is a subsemigroup of $(R, +)$, let $x, y \in \sqrt{I}$. Then there exist $n, m \in \mathbb{N}$ such that $(x\gamma)^{n-1}x \in I$ and $(y\beta)^{m-1}y \in I$ for all $\gamma, \beta \in \Gamma$. So,

$$\begin{aligned}
[(x+y)\gamma]^{(m+n)-1}(x+y) &= \sum_{k=0}^{(m+n)-1} \binom{(m+n)-1}{k} (x\gamma)^k (y\gamma)^{(m+n)-1-k}(x+y) \\
&= \sum_{k=0}^{(m+n)-1} \binom{(m+n)-1}{k} (x\gamma)^k (y\gamma)^{(m+n)-1-k}x + \\
&\quad \sum_{k=0}^{(m+n)-1} \binom{(m+n)-1}{k} (x\gamma)^k (y\gamma)^{(m+n)-1-k}y \\
&\in I
\end{aligned}$$

for all $\gamma \in \Gamma$. Thus, $x + y \in \sqrt{I}$. Hence, $\sqrt{I}$ is a subsemigroup of $(R, +)$.

Next, let $x \in \sqrt{I}, \gamma \in \Gamma$ and $r \in R$. Then there exists $n \in \mathbb{N}$ such that $(x\beta)^{n-1}x \in I$ for all $\beta \in \Gamma$. Since R is commutative, we have $[((x\gamma r)\beta)^{n-1}x\gamma r] \in I$ for all $\beta \in \Gamma$. Hence, $\sqrt{I}\Gamma R \subseteq \sqrt{I}$. Since R is commutative, we have $R\Gamma\sqrt{I} \subseteq \sqrt{I}$.

Therefore, $\sqrt{I}$ is an ideal in R containing I. □

Note that $\sqrt{R} = R$.

Proposition 3.2. *Let I and J be ideals in R. If $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$.*

Proof. Suppose $I \subseteq J$. Let $x \in \sqrt{I}$. Then there exists $n \in \mathbb{N}$ such that $(x\gamma)^{n-1}x \in I$ for all $\gamma \in \Gamma$. Since $I \subseteq J$, it follows that $(x\gamma)^{n-1}x \in J$ for all $\gamma \in \Gamma$. So, $x \in \sqrt{J}$. Hence, $\sqrt{I} \subseteq \sqrt{J}$. □

Proposition 3.3. *Let I and J be ideals in R such that $I \cap J \neq \emptyset$. Then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.*

Proof. Clearly, $I \cap J$ is an ideal in R. Since $\sqrt{I \cap J} \subseteq \sqrt{I}$ and $\sqrt{I \cap J} \subseteq \sqrt{J}$, we have $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$.

Next, let $x \in \sqrt{I} \cap \sqrt{J}$. Then there exist $n, m \in \mathbb{N}$ such that $(x\gamma)^{n-1}x \in I$ for all $\gamma \in \Gamma$ and $(x\beta)^{m-1}x \in J$ for all $\beta \in \Gamma$. So, $(x\gamma)^{m+n-1}x \in I \cap J$ for all $\gamma \in \Gamma$. Hence, $x \in \sqrt{I \cap J}$. Thus, $\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I \cap J}$.

Therefore, $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. □

Proposition 3.4. *Let R_i be a commutative Γ_i-semiring for all $i \in \{1, 2\}$. If I_1 and I_2 are ideals in R_1 and R_2, respectively, then $\sqrt{I_1} \times \sqrt{I_2} = \sqrt{I_1 \times I_2}$.*

Proof. Let I_1 and I_2 be ideals in R_1 and R_2, respectively.

First, let $a \in \sqrt{I_1}$ and $b \in \sqrt{I_2}$. Then there exist $n, m \in \mathbb{N}$ such that $(a\alpha_1)^{n-1}a \in I_1$ and $(b\alpha_2)^{m-1}b \in I_2$ for all $\alpha_1 \in \Gamma_1$ and $\alpha_2 \in \Gamma_2$. So,

$$((a, b)(\alpha_1, \alpha_2))^{n+m-1}(a, b) \in I_1 \times I_2 \qquad \text{for all } \alpha_1 \in \Gamma_1 \text{ and } \alpha_2 \in \Gamma_2.$$

Thus, $(a, b) \in \sqrt{I_1 \times I_2}$. Hence, $\sqrt{I_1} \times \sqrt{I_2} \subseteq \sqrt{I_1 \times I_2}$.

Next, let $(p, q) \in \sqrt{I_1 \times I_2}$. Then, there exists $m \in \mathbb{N}$ such that

$$((p, q)(\alpha_1, \alpha_2))^{m-1}(p, q) \in I_1 \times I_2 \qquad \text{for all } \alpha_1 \in \Gamma_1 \text{ and } \alpha_2 \in \Gamma_2.$$

Hence, $(p\alpha_1)^{m-1}p \in I_1$ and $(q\alpha_2)^{m-1}q \in I_2$ for all $\alpha_1 \in \Gamma_1$ and $\alpha_2 \in \Gamma_2$. So, $(p,q) \in \sqrt{I_1} \times \sqrt{I_2}$. Thus, $\sqrt{I_1 \times I_2} \subseteq \sqrt{I_1} \times \sqrt{I_2}$.

Therefore, $\sqrt{I_1} \times \sqrt{I_2} = \sqrt{I_1 \times I_2}$. □

For each proper ideal I in a commutative Γ-semiring R, the commutative Γ-semiring R/I is proposed by T.K. Dutta and S.K. Sardar in [4].

Definition 3.5. [4] Let ρ be an equivalence relation on a commutative Γ-semiring R. Then ρ is called a **Γ-congruence** on R if $x\rho x'$ and $y\rho y'$ implies $(x+y)\rho(x'+y')$ and $(x\gamma y)\rho(x'\gamma y')$ for all $\gamma \in \Gamma$ and $x, y, x', y' \in R$.

Definition 3.6. [4] Let I be a proper ideal in a commutative Γ-semiring R and ρ_I be a Γ-congruence on R. Then ρ_I is called **the Bourne Γ-congruence** on R if for all $x, y \in R$, $x\rho_I y$ if and only if $x + i_1 = y + i_2$ for some $i_1, i_2 \in I$.

The Bourne Γ-congruence class of an element r of R is denoted by r/ρ_I or simply r/I and the set of all such Γ-congruence classes of the elements of R is denoted by R/ρ_I or simply by R/I.

For any proper ideal I in R, R/I is a commutative Γ-semiring where

$$r/I + r'/I = (r + r')/I \quad \text{and} \quad (r/I)\alpha(r'/I) = (r\alpha r')/I$$

for all $\alpha \in \Gamma$ and $r, r' \in R$.

Proposition 3.7. [9] *If I and J are ideals in R and $I \subsetneq J$, then*

(i) I is also an ideal in the Γ-subsemiring J; and

(ii) J/I is an ideal in the Γ-semiring R/I.

Lemma 3.8. *Let I be a proper ideal in R and P be a k-ideal in R such that $I \subsetneq P$. Then, for all $a \in R$, $a/I \in P/I$ if and only if $a \in P$.*

Proof. Note that P/I is a commutative Γ-semiring because $I \subsetneq P$. Let $a \in R$. If $a \in P$, then it is obvious that $a/I \in P/I$.

Next, let $a/I \in P/I$. Then, $a/I = p/I$ for some $p \in P$. Thus, there exist $i_1, i_2 \in I$ such that $a + i_1 = p + i_2$. Since $i_1, i_2 \in I \subseteq P$ and P is a k-ideal, $a \in P$. □

Proposition 3.9. *Let I be a proper ideal in R and P be a k-ideal in R such that $I \subsetneq P$. Then $\sqrt{P}/I \subseteq \sqrt{P/I}$.*

Proof. Note that $I \subsetneq \sqrt{P}$ so that both $\sqrt{P}/I$ and P/I are commutative Γ-semirings. Let $r \in \sqrt{P}$. Then there exists $n \in \mathbb{N}$ such that $(r\alpha)^{n-1}r \in P$ for any $\alpha \in \Gamma$. So, $((r/I)\alpha)^{n-1}(r/I) = ((r\alpha)^{n-1}r)/I \in P/I$ for any $\alpha \in \Gamma$. Thus, $r/I \in \sqrt{P/I}$.

Therefore, $\sqrt{P}/I \subseteq \sqrt{P/I}$. □

We end this chapter by introducing homomorphism between Γ-semirings.

Definition 3.10. [8] Let R_1 and R_2 be Γ-semirings (not necessary commutative). Then $g : R_1 \to R_2$ is called a **homomorphism** if $g(x + y) = g(x) + g(y)$ and $g(x\gamma y) = g(x)\gamma g(y)$ for all $x, y \in R_1$ and $\gamma \in \Gamma$.

Definition 3.11. Let R_1 and R_2 be Γ-semirings (not necessary commutative) and $g : R_1 \to R_2$ be a homomorphism. Then g is called an **epimorphism** if g is surjective.

Example 3.12. Note that $3\mathbb{Z}_0^+$ and $\mathbb{Z}_0^+/7\mathbb{Z}_0^+$ are $5\mathbb{Z}_0^+$-semirings. Define $f : 3\mathbb{Z}_0^+ \to \mathbb{Z}_0^+/7\mathbb{Z}_0^+$ by $f(x) = x/7\mathbb{Z}_0^+$ for all $x \in 3\mathbb{Z}_0^+$. Then f is an epimorphism.

Proposition 3.13. *Let R_1 and R_2 be commutative Γ-semirings, $g : R_1 \to R_2$ be a homomorphism and I be an ideal in R_2. Then $g^{-1}(\sqrt{I}) = \sqrt{g^{-1}(I)}$.*

Proof. Clearly, $g^{-1}(I)$ is an ideal in R_1.

First, let $a \in g^{-1}(\sqrt{I})$.Then, $g(a) \in \sqrt{I}$. So, there exists $n \in \mathbb{N}$ such that $(g(a)\alpha)^{n-1}g(a) \in I$ for all $\alpha \in \Gamma$. Thus, $g((a\alpha)^{n-1}a) = (g(a)\alpha)^{n-1}g(a) \in I$ for all $\alpha \in \Gamma$. Then, $(a\alpha)^{n-1}a \in g^{-1}(I)$ for all $\alpha \in \Gamma$. So, $a \in \sqrt{g^{-1}(I)}$. Hence, $g^{-1}(\sqrt{I}) \subseteq \sqrt{g^{-1}(I)}$.

Next, let $a \in \sqrt{g^{-1}(I)}$. Then, there exists $n \in \mathbb{N}$ such that $(a\alpha)^{n-1}a \in g^{-1}(I)$ for all $\alpha \in \Gamma$. Thus, $(g(a)\alpha)^{n-1}g(a) = g((a\alpha)^{n-1}a) \in I$ for all $\alpha \in \Gamma$. So, $g(a) \in \sqrt{I}$. Then, $\sqrt{g^{-1}(I)} \subseteq g^{-1}(\sqrt{I})$.

Therefore, $g^{-1}(\sqrt{I}) = \sqrt{g^{-1}(I)}$. □

Chapter 4

2-absorbing Primary Ideals

In this chapter, we introduce the concept of 2-absorbing primary ideals in a commutative Γ-semiring and investigate some results related to it. Throughout this chapter, let R be a commutative Γ-semiring.

Definition 4.1. [4] A proper ideal I in a commutative Γ-semiring R is called a **prime ideal** in R if whenever $a, b \in R$, $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$.

Example 4.2. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then $2\mathbb{Z}_0^+$ is a prime ideal in $\mathbb{Z}_0^+$.

Proof. Let $x, y \in \mathbb{Z}_0^+$ be such that $x\Gamma y \subseteq 2\mathbb{Z}_0^+$. So, $2 \mid (x)(5)(y)$. Hence, $2 \mid x$ or $2 \mid y$. Thus, $x \in 2\mathbb{Z}_0^+$ or $y \in 2\mathbb{Z}_0^+$. Therefore, $2\mathbb{Z}_0^+$ is a prime ideal in $\mathbb{Z}_0^+$. $\square$

Definition 4.3. [11] A proper ideal I in a commutative Γ-semiring R is called a **primary ideal** in R if whenever $a, b \in R$, $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in \sqrt{I}$.

The following is the immediate result obtained from the definitions.

Remark 4.4. *Every prime ideal in R is a primary ideal in R.*

The following definitions that are given in the context of Γ-semirings were inspired by [5].

Definition 4.5. A proper ideal I in a commutative Γ-semiring R is called a **2-absorbing ideal** in R if whenever $x, y, z \in R$, $\gamma, \beta \in \Gamma$ and $x\gamma y\beta z \in I$ implies $x\gamma y \in I$ or $x\beta z \in I$ or $y\beta z \in I$.

Definition 4.6. A proper ideal I in a commutative Γ-semiring R is called a **2-absorbing primary ideal** in R if whenever $x, y, z \in R,\ \gamma, \beta \in \Gamma$ and $x\gamma y\beta z \in I$, then $x\gamma y \in I$ or $x\beta z \in \sqrt{I}$ or $y\beta z \in \sqrt{I}$.

Definition 4.5 and Definition 4.6 lead to the following remark.

Remark 4.7. *Every 2-absorbing ideal in R is a 2-absorbing primary ideal in R.*

However, the converse of the above remark does not hold.

Example 4.8. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then $8\mathbb{Z}_0^+$ is a 2-absorbing primary ideal in $\mathbb{Z}_0^+$ but it is not a 2-absorbing ideal in $\mathbb{Z}_0^+$.

Proof. Let $x, y, z \in \mathbb{Z}_0^+$ and $\gamma, \beta \in 5\mathbb{Z}_0^+$ be such that $x\gamma y\beta z \in 8\mathbb{Z}_0^+$. If $8 \mid x\gamma y$, we are done. Suppose $8 \nmid x\gamma y$. Then, $2 \mid \beta z$. So, $8 \mid (x\beta z\alpha)^2 x\beta z$ for all $\alpha \in \Gamma$, that is $x\beta z \in \sqrt{8\mathbb{Z}_0^+}$. Thus, $8\mathbb{Z}_0^+$ is a 2-absorbing primary ideal in $\mathbb{Z}_0^+$.

Since $(2)(5)(2)(5)(2) \in 8\mathbb{Z}_0^+$ and $(2)(5)(2) \notin 8\mathbb{Z}_0^+$, it follows that $8\mathbb{Z}_0^+$ is not 2-absorbing ideal in $\mathbb{Z}_0^+$. $\square$

We can see from the next example that primary ideals need not be 2-absorbing ideals or prime ideals.

Example 4.9. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then $27\mathbb{Z}_0^+$ is a primary ideal in $\mathbb{Z}_0^+$ but it is not a 2-absorbing ideal so that it is not a prime ideal in $\mathbb{Z}_0^+$.

Proof. Let $x, y \in \mathbb{Z}_0^+$ be such that $x\Gamma y \subseteq 27\mathbb{Z}_0^+$. If $27 \mid x$, then $x \in 27\mathbb{Z}_0^+$. Suppose $27 \nmid x$. Since $x\Gamma y \subseteq 27\mathbb{Z}_0^+$, $3 \mid \alpha y$ for all $\alpha \in 5\mathbb{Z}_0^+$. Hence, $27 \mid (y\alpha)^3 y$ for all $\alpha \in 5\mathbb{Z}_0^+$. So, $y \in \sqrt{27\mathbb{Z}_0^+}$. Thus, $27\mathbb{Z}_0^+$ is a primary ideal in $\mathbb{Z}_0^+$.

Since $(3)(5)(3)(5)(3) \in 27\mathbb{Z}_0^+$ and $(3)(5)(3) \notin 27\mathbb{Z}_0^+$, it follows that $27\mathbb{Z}_0^+$ is not a 2-absorbing ideal in $\mathbb{Z}_0^+$. $\square$

Next, we present a relationship between prime ideals and 2-absorbing ideals as well as a relationship between primary ideals and 2-absorbing primary ideals.

Proposition 4.10. *Every prime ideal in R is a 2-absorbing ideal in R and then it is a 2-absorbing primary ideal in R.*

Proof. Suppose that I is a prime ideal in R. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in I$. Then, $x\gamma y\Gamma y\beta z \subseteq I$. Since I is a prime ideal, we have $x\gamma y \in I$ or $y\beta z \in I$.

Therefore, I is a 2-absorbing ideal in R. □

Proposition 4.11. *Every primary ideal in R is a 2-absorbing primary ideal in R.*

Proof. Suppose that I is a primary ideal in R. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in I$. Then, $x\gamma y\Gamma y\beta z \subseteq I$. Since I is a primary ideal, we have $x\gamma y \in I$ or $y\beta z \in \sqrt{I}$. Thus, I is a 2-absorbing primary ideal in R. □

We can see from the next example that 2-absorbing ideals and 2-absorbing primary ideals need not be primary ideals.

Example 4.12. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then $10\mathbb{Z}_0^+$ is a 2-absorbing ideal in $\mathbb{Z}_0^+$ so that it is a 2-absorbing primary ideal. However, it is not a primary ideal in $\mathbb{Z}_0^+$.

Proof. Let $x, y, z \in \mathbb{Z}_0^+$ and $\gamma, \beta \in 5\mathbb{Z}_0^+$ be such that $x\gamma y\beta z \in 10\mathbb{Z}_0^+$. Then, $10 \mid x\gamma y\beta z$. So, $2 \mid x$ or $2 \mid \gamma$ or $2 \mid y$ or $2 \mid \beta$ or $2 \mid z$. If $2 \mid x$ or $2 \mid y$, then $10 \mid x\gamma y$. If $2 \mid z$, then $10 \mid y\beta z$. If $2 \mid \gamma$ or $2 \mid \beta$, then $10 \mid x\gamma y$ or $10 \mid x\beta z$. Hence, $x\gamma y \in 10\mathbb{Z}_0^+$ or $x\beta z \in 15\mathbb{Z}_0^+$ or $y\beta z \in 10\mathbb{Z}_0^+$. Thus, $10\mathbb{Z}_0^+$ is a 2-absorbing ideal in $\mathbb{Z}_0^+$ and then it is a 2-absorbing primary ideal.

Since $2(5\mathbb{Z}_0^+)1 \subseteq 10\mathbb{Z}_0^+$, $2 \notin 10\mathbb{Z}_0^+$ and $1 \notin \sqrt{10\mathbb{Z}_0^+}$, it follows that $10\mathbb{Z}_0^+$ is not a primary ideal in $\mathbb{Z}_0^+$. □

The following results are inspired by results in [2] and [6]. The next result shows a relationship between 2-absorbing primary ideals in R and 2-absorbing primary ideals in R/I.

Theorem 4.13. *Let I be a proper ideal in R and P be a k-ideal in R such that $I \subsetneq P$. Then P is a 2-absorbing primary ideal in R if and only if P/I is a 2-absorbing primary ideal in R/I.*

Proof. First, suppose that P is a 2-absorbing primary ideal in R. Then, P/I is a proper ideal in R/I. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $(x/I)\gamma(y/I)\beta(z/I) \in$

P/I. Hence, $(x\gamma y\beta z)/I \in P/I$ and then by Lemma 3.8, $x\gamma y\beta z \in P$. Since P is a 2-absorbing primary ideal, $x\gamma y \in P$ or $x\beta z \in \sqrt{P}$ or $y\beta z \in \sqrt{P}$. Hence, $(x/I)\gamma(y/I) = (x\gamma y)/I \in P/I$ or $(x/I)\beta(z/I) = (x\beta z)/I \in \sqrt{P}/I \subseteq \sqrt{P/I}$ or $(y/I)\beta(z/I) = (y\beta z)/I \in \sqrt{P}/I \subseteq \sqrt{P/I}$. Thus, P/I is a 2-absorbing primary ideal in R/I.

Next, suppose that P/I is a 2-absorbing primary ideal in R/I. By Lemma 3.8, P is a proper ideal in R. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in P$. Hence, $(x/I)\gamma(y/I)\beta(z/I) = (x\gamma y\beta z)/I \in P/I$. Since P/I is a 2-absorbing primary ideal, $(x\gamma y)/I = (x/I)\gamma(y/I) \in P/I$ or $(x\beta z)/I = (x/I)\beta(z/I) \in \sqrt{P/I}$ or $(y\beta z)/I = (y/I)\beta(z/I) \in \sqrt{P/I}$. If $(x\gamma y)/I \in P/I$, then $x\gamma y \in P$ by Lemma 3.8. Suppose $(x\beta z)/I \in \sqrt{P/I}$. Then there exists $n \in \mathbb{N}$ such that $((x\beta z\alpha)^{n-1}x\beta z)/I = (((x\beta z)/I)\alpha)^{n-1}((x\beta z)/I) \in P/I$ for all $\alpha \in \Gamma$. By Lemma 3.8, $(x\beta z\alpha)^{n-1}x\beta z \in P$ for all $\alpha \in \Gamma$. Hence, $x\beta z \in \sqrt{P}$. Similarly, if $(y\beta z)/I \in \sqrt{P/I}$, then $y\beta z \in \sqrt{P}$. Thus, P is a 2-absorbing primary ideal in R. □

Proposition 4.14. *If I is an ideal in R, then*

$$(I : x) := \{\ r \in R \mid r\gamma x \in I \text{ for all } \gamma \in \Gamma\ \}$$

is an ideal in R containing I for all $x \in R$.

Proof. Let I be an ideal in R and $x \in R$. If $r \in I$, then $r\gamma x \in I$ for all $\gamma \in \Gamma$. Hence, $I \subseteq (I : x)$.

Next, let $a, b \in (I : x)$ and $\gamma \in \Gamma$. Then, $a\gamma x, b\gamma x \in I$. So, $(a + b)\gamma x \in I$, i.e., $a + b \in (I : x)$. Hence, $(I : x)$ is a subsemigroup of $(R, +)$.

Moreover, let $a \in (I : x), \alpha \in \Gamma$ and $r \in R$. Then, $a\gamma x \in I$ for all $\gamma \in \Gamma$. So, $a\alpha r\gamma x = r\alpha a\gamma x \in I$ for all $\gamma \in \Gamma$. Thus, $a\alpha r \in (I : x)$. Hence, $(I : x)\Gamma R \subseteq (I : x)$. Since R is commutative, we have $R\Gamma(I : x) \subseteq (I : x)$.

Therefore, $(I : x)$ is an ideal in R containing I. □

We provide examples of 2-absorbing primary ideals in R.

Theorem 4.15. *Let I be a 2-absorbing primary ideal in R and $\sqrt{I}$ be a prime ideal in R. Then $(I : x)$ is a 2-absorbing primary ideal in R for all $x \in R \setminus \sqrt{I}$.*

Proof. Let $x \in R \setminus \sqrt{I}$ and $a \in (I : x)$. Then $a\Gamma x \subseteq I \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal and $x \notin \sqrt{I}$, it follows that $a \in \sqrt{I}$. So, $(I : x) \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a proper ideal in R, it follows that $(I : x)$ is a proper ideal in R.

Next, let $a, b, c \in R$ and $\gamma, \beta \in \Gamma$ be such that $a\gamma b\beta c \in (I : x)$. Hence, $a\gamma(b\beta c)\beta x \in I$. Since I is a 2-absorbing primary ideal, $a\gamma b\beta c \in I$ or $a\beta x \in \sqrt{I}$ or $b\beta c\beta x \in \sqrt{I}$.

Case 1. $a\gamma b\beta c \in I$. Since I is a 2-absorbing primary ideal, $a\gamma b \in I \subseteq (I : x)$ or $a\beta c \subseteq \sqrt{I} \subseteq \sqrt{(I : x)}$ or $b\beta c \subseteq \sqrt{I} \subseteq \sqrt{(I : x)}$.

Case 2. $a\beta x \in \sqrt{I}$. Hence, $a\beta c\Gamma x \subseteq \sqrt{I}$. Since $x \notin \sqrt{I}$ and $\sqrt{I}$ is a prime ideal, we have $a\beta c \in \sqrt{I} \subseteq \sqrt{(I : x)}$.

Case 3. $b\beta c\beta x \in \sqrt{I}$. Hence, $b\beta c\beta c\Gamma x \subseteq \sqrt{I}$. Since $x \notin \sqrt{I}$ and $\sqrt{I}$ is a prime ideal, we have $b\beta c\beta c \in \sqrt{I}$. Then there exists $n \in \mathbb{N}$ such that $(b\beta c\beta c\alpha)^{n-1}b\beta c\beta c \in I$ for all $\alpha \in \Gamma$. So, $(b\beta c\alpha)^{2n-1}b\beta c \in I$ for all $\alpha \in \Gamma$. It follows that $b\beta c \in \sqrt{I} \subseteq \sqrt{(I : x)}$.

Therefore, $(I : x)$ is a 2-absorbing primary ideal in R. □

Example 4.16. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then $8\mathbb{Z}_0^+$ and $10\mathbb{Z}_0^+$ are 2-absorbing primary ideals by Example 4.8 and Example 4.12, respectively. And so $\sqrt{8\mathbb{Z}_0^+} = \sqrt{10\mathbb{Z}_0^+} = 2\mathbb{Z}_0^+$. Moreover, $\sqrt{8\mathbb{Z}_0^+} = \sqrt{10\mathbb{Z}_0^+} = 2\mathbb{Z}_0^+$ is a prime ideal in $\mathbb{Z}_0^+$ by Example 4.2.

Proof. To show that $\sqrt{8\mathbb{Z}_0^+} = 2\mathbb{Z}_0^+$, let $x \in \sqrt{8\mathbb{Z}_0^+}$. So, there exists $n \in \mathbb{N}$ such that $(x\gamma)^{n-1}x \in 8\mathbb{Z}_0^+$ for all $\gamma \in 5\mathbb{Z}_0^+$. Hence, $8 \mid [(x)(5)]^{n-1}(x)$. Then, $2 \mid x$. Thus $\sqrt{8\mathbb{Z}_0^+} \subseteq 2\mathbb{Z}_0^+$. Next, let $x \in 2\mathbb{Z}_0^+$. Hence, $[(x)(\gamma)]^{3-1}x \in 8\mathbb{Z}_0^+$ for all $\gamma \in 5\mathbb{Z}_0^+$. Then, $x \in \sqrt{8\mathbb{Z}_0^+}$. Thus, $2\mathbb{Z}_0^+ \subseteq \sqrt{8\mathbb{Z}_0^+}$. Therefore, $\sqrt{8\mathbb{Z}_0^+} = 2\mathbb{Z}_0^+$.

To show that $\sqrt{10\mathbb{Z}_0^+} = 2\mathbb{Z}_0^+$, let $x \in \sqrt{10\mathbb{Z}_0^+}$. So, there exists $n \in \mathbb{N}$ such that $(x\gamma)^{n-1}x \in 10\mathbb{Z}_0^+$ for all $\gamma \in 5\mathbb{Z}_0^+$. Hence, $10 \mid [(x)(5)]^{n-1}(x)$. Then, $2 \mid x$. Thus $\sqrt{10\mathbb{Z}_0^+} \subseteq 2\mathbb{Z}_0^+$. Next, let $x \in 2\mathbb{Z}_0^+$. Hence, $[(x)(\gamma)]^{2-1}x \in 10\mathbb{Z}_0^+$ for all $\gamma \in 5\mathbb{Z}_0^+$. Then, $x \in \sqrt{10\mathbb{Z}_0^+}$. Thus, $2\mathbb{Z}_0^+ \subseteq \sqrt{10\mathbb{Z}_0^+}$. Therefore, $\sqrt{10\mathbb{Z}_0^+} = 2\mathbb{Z}_0^+$. □

This chapter is ended by providing 2-absorbing primary ideals in a commutative $(\Gamma_1 \times \Gamma_2)$-semiring $R_1 \times R_2$.

Theorem 4.17. *Let R_i be a commutative Γ_i-semiring for all $i \in \{1, 2\}$.*

(i) If I_1 is a 2-absorbing primary ideal in R_1, then $I_1 \times R_2$ is a 2-absorbing primary ideal in $R_1 \times R_2$.

(ii) If I_2 is a 2-absorbing primary ideal in R_2, then $R_1 \times I_2$ is a 2-absorbing primary ideal in $R_1 \times R_2$.

Proof. Note that $R_1 \times R_2$ is a commutative $(\Gamma_1 \times \Gamma_2)$-semiring.

(i) Suppose that I_1 is a 2-absorbing primary ideal in R_1. Then, $I_1 \times R_2$ is a proper ideal in $R_1 \times R_2$. Let $x_1, y_1, z_1 \in R_1$, $x_2, y_2, z_2 \in R_2$, $\gamma_1, \beta_1 \in \Gamma_1$ and $\gamma_2, \beta_2 \in \Gamma_2$ be such that $(x_1, x_2)(\gamma_1, \gamma_2)(y_1, y_2)(\beta_1, \beta_2)(z_1, z_2) \in I_1 \times R_2$. Hence, $x_1\gamma_1y_1\beta_1z_1 \in I_1$. Since I_1 is a 2-absorbing primary ideal, $x_1\gamma_1y_1 \in I_1$ or $x_1\beta_1z_1 \in \sqrt{I_1}$ or $y_1\beta_1z_1 \in \sqrt{I_1}$. If $x_1\gamma_1y_1 \in I_1$, then $(x_1, x_2)(\gamma_1, \gamma_2)(y_1, y_2) \in I_1 \times R_2$. If $x_1\beta_1z_1 \in \sqrt{I_1}$, then $(x_1, x_2)(\beta_1, \beta_2)(z_1, z_2) \in \sqrt{I_1} \times R_2 \subseteq \sqrt{I_1 \times R_2}$. Similarly, if $y_1\beta_1z_1 \in \sqrt{I_1}$, then $(y_1, y_2)(\beta_1, \beta_2)(z_1, z_2) \in \sqrt{I_1 \times R_2}$. Therefore, $I_1 \times R_2$ is 2-absorbing primary ideal in $R_1 \times R_2$.

The proof of (ii) is similar to the proof of (i). □

Theorem 4.18. *Let R_i be a commutative Γ_i-semiring with zero 0_{R_i} and unity 1_{R_i} such that $0_{R_i} \neq 1_{R_i}$ for all $i \in \{1, 2\}$. If I is a 2-absorbing primary ideal in $R_1 \times R_2$, exactly one of these holds:*

(i) $I = I_1 \times R_2$ for some 2-absorbing primary ideal I_1 in R_1;

(ii) $I = R_1 \times I_2$ for some 2-absorbing primary ideal I_2 in R_2;

(iii) $I = I_1 \times I_2$ for some primary ideal I_i in R_i for all $i \in \{1, 2\}$.

Proof. Suppose that I is a 2-absorbing primary ideal in $R_1 \times R_2$. Then, $I = I_1 \times I_2$ for some ideals I_1 in R_1 and I_2 in R_2. Assume $I_2 = R_2$. Then I_1 must be a proper ideal in R_1. Let $x, y, z \in R_1$ and $\gamma, \beta \in \Gamma_1$ be such that $x\gamma y\beta z \in I_1$. Let $a \in R_2$ and $\delta \in \Gamma_2$. So $(x, a)(\gamma, \delta)(y, a)(\beta, \delta)(z, a) \in I_1 \times R_2$. Since $I = I_1 \times R_2$ is a 2-absorbing primary ideal, $(x, a)(\gamma, \delta)(y, a) \in I_1 \times R_2$ or $(x, a)(\beta, \delta)(z, a) \in \sqrt{I_1 \times R_2}$ or $(y, a)(\beta, \delta)(z, a) \in \sqrt{I_1 \times R_2}$. If $(x, a)(\gamma, \delta)(y, a) \in I_1 \times R_2$, then $x\gamma y \in I_1$. If $(x, a)(\beta, \delta)(z, a) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times R_2$, then $x\beta z \in \sqrt{I_1}$. Similarly, if $(y, a)(\beta, \delta)(z, a) \in \sqrt{I_1 \times R_2}$, then, $y\beta z \in \sqrt{I_1}$. Thus, I_1 is a 2-absorbing primary ideal in R_1.

By similar argument, if $I_1 = R_1$, then I_2 is a 2-absorbing primary ideal in R_2.

Now, suppose that $I_1 \neq R_1$ and $I_2 \neq R_2$. Suppose that I_1 is not a primary ideal in R_1. If $1_{R_2} \in \sqrt{I_2}$, then $1_{R_2} \in I_2$, so $I_2 = R_2$ which is a contradiction. Hence, $1_{R_2} \notin \sqrt{I_2}$. Since I_1 is not a primary ideal in R_1, there exist $b, c \in R_1$ such that $b\Gamma c \subseteq I_1$ but neither $b \in I_1$ nor $c \in \sqrt{I_1}$. Since 1_{R_1} and 1_{R_2} are unities, there exists $\alpha, \alpha' \in \Gamma_1$ and $\alpha'' \in \Gamma_2$ such that $b\alpha 1_{R_1} = b_{R_1}$, $1_{R_1}\alpha' c = c$ and $1_{R_2}\alpha'' 1_{R_2} = 1_{R_2}$. Since $b\Gamma c \subseteq I_1$, we have $b\alpha c \in I_1$. Hence, $(b, 1_{R_2})(\alpha, \alpha')(1_{R_1}, 0_{R_2})(\alpha', \alpha'')(c, 1_{R_2}) = (b\alpha c, 0_{R_2}) \in I_1 \times I_2 = I$. Since I is a 2-absorbing primary ideal, we have

$$
\begin{aligned}
(b, 0_{R_2}) &= (b, 1_{R_2})(\alpha, \alpha')(1_{R_1}, 0_{R_2}) \in I \text{ or} \\
(b\alpha c, 1_{R_2}) &= (b, 1_{R_2})(\alpha, \alpha')(c, 1_{R_2}) \in \sqrt{I} = \sqrt{I_1} \times \sqrt{I_2} \text{ or} \\
(c, 0_{R_2}) &= (1_{R_1}, 0_{R_2})(\alpha', \alpha'')(c, 1_{R_2}) \in \sqrt{I} = \sqrt{I_1} \times \sqrt{I_2}.
\end{aligned}
$$

Hence $b \in I_1$ or $1_{R_2} \in \sqrt{I_2}$ or $c \in \sqrt{I_1}$, which is a contradiction. So I_1 is a primary ideal in R_1. Analogously, I_2 is a primary ideal in R_2. $\square$

Chapter 5

Expansion of Ideals

In a commutative ring, D. Zhao [12] defined a mapping δ to substitute $\sqrt{I}$ by $\delta(I)$ in the definition of primary ideals, called δ-primary ideals, which are more general than primary ideals. In this chapter, properties of expansion of ideals in commutative Γ-semirings are investigated. However, we focus on those which are involving with 2-absorbing δ-primary ideals which will be applied in the sixth chapter. Throughout this chapter, let R be a commutative Γ-semiring and $\mathsf{J}(R)$ be the set of all ideals in R.

First, we give the definition of ideal expansions in the context of Γ-semirings which were inspired by [12].

Definition 5.1. An **expansion of ideals** in a commutative Γ-simiring R (or an ideal expansion of R) is defined to be a function $\delta : \mathsf{J}(R) \to \mathsf{J}(R)$ such that:
1. $I \subseteq \delta(I)$ for all $I \in \mathsf{J}(R)$; and
2. $I \subseteq J$ implies $\delta(I) \subseteq \delta(J)$ for all $I, J \in \mathsf{J}(R)$.

Example 5.2. (1) The constant function $c : \mathsf{J}(R) \to \mathsf{J}(R)$ defined by $I \mapsto R$ is an ideal expansion of R.
(2) The identity function $I_d : \mathsf{J}(R) \to \mathsf{J}(R)$ defined by $I \mapsto I$ is an ideal expansion of R.
(3) The radical function $\sqrt{\ } : \mathsf{J}(R) \to \mathsf{J}(R)$ defined by $I \mapsto \sqrt{I}$ is an ideal expansion of R.

The following definitions that are given in the context of Γ-semirings were inspired by [12].

Definition 5.3. Let δ be an ideal expansion of a commutative Γ-semiring R. A proper ideal I in a commutative Γ-semiring R is called a **δ-primary ideal** in R if whenever $a, b \in R$, $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in \delta(I)$.

Definition 5.3 leads to the following remark.

Remark 5.4. *Let δ be an ideal expansion of R. Then every prime ideal in R is a δ-primary ideal in R.*

Definition 5.5. An ideal expansion δ of a commutative Γ-semiring R is said to be **intersection preserving** if for any ideals I and J in R with $I \cap J \neq \emptyset$,

$$\delta(I \cap J) = \delta(I) \cap \delta(J).$$

Definition 5.6. Let δ_{R_1} and δ_{R_2} be ideal expansions of commutative Γ-semirings R_1 and R_2, respectively. Then $(\delta_{R_1}, \delta_{R_2})$ is said to be **global** if for any homomorphism $g : R_1 \to R_2$ and ideal I in R_2,

$$\delta_{R_1}(g^{-1}(I)) = g^{-1}(\delta_{R_2}(I)).$$

Example 5.7. Let R_1 and R_2 be commutative Γ-semirings.
(1) According to Example 5.2 (2), let $I_{d_{R_1}}$ and $I_{d_{R_2}}$ be the identity functions which are ideal expansions of R_1 and R_2, respectively. Then $I_{d_{R_1}}$ is intersection preserving and $(I_{d_{R_1}}, I_{d_{R_2}})$ is global.
(2) According to Example 5.2 (3), let $\sqrt{}_{R_1}$ and $\sqrt{}_{R_2}$ be the radical functions which are ideal expansions of R_1 and R_2, respectively. Then $\sqrt{}_{R_1}$ is intersection preserving by Proposition 3.3 and $(\sqrt{}_{R_1}, \sqrt{}_{R_2})$ is global by Proposition 3.13.

Lemma 5.8. *Let R_1 and R_2 be commutative Γ-semirings, $g : R_1 \to R_2$ be an epimorphism and I be an ideal in R_2. Then $I = g(g^{-1}(I))$.*

Proof. Clearly, $g(g^{-1}(I)) \subseteq I$.

Next, let $y \in I$. Since g is surjective, there exists $x \in R_1$ such that $g(x) = y \in I$. So, $x \in g^{-1}(I)$. Then, $y = g(x) \in g(g^{-1}(I))$. Hence, $I \subseteq g(g^{-1}(I))$.

Therefore, $I = g(g^{-1}(I))$. □

Lemma 5.9. *Let R_1 and R_2 be commutative Γ-semirings, $g : R_1 \to R_2$ be a homomorphism and I be a k-ideal in R_1 such that $\{ x \in R_1 \mid \exists\, a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$. Then $I = g^{-1}(g(I))$.*

Proof. Clearly, $I \subseteq g^{-1}(g(I))$.

Next, let $y \in g^{-1}(g(I))$. Then, $g(y) \in g(I)$. So, there exists $p \in I$ such that $g(p) = g(y)$. Then, $p + y \in \{ x \in R_1 \mid \exists\, a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$. Since I is a k-ideal and $p \in I$, we have $y \in I$. Hence, $g^{-1}(g(I)) \subseteq I$.

Therefore, $I = g^{-1}(g(I))$. □

The next result presents the homomorphic image of an ideal is an ideal under some conditions.

Proposition 5.10. *Let R_1 and R_2 be commutative Γ-semirings, δ_i be an ideal expansion of R_i for all $i \in \{1, 2\}$ such that (δ_1, δ_2) is global, $g : R_1 \to R_2$ be an epimorphism and I be a k-ideal in R_1 such that $\{ x \in R_1 \mid \exists\, a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$. Then $g(I)$ is an ideal in R_2 and $g(\delta_1(I)) = \delta_2(g(I))$.*

Proof. Since $I \neq \emptyset$ and g is surjective, we have $g(I) \neq \emptyset$. Let $u, v \in g(I), \gamma \in \Gamma$ and $r \in R_2$. Then, there exist $p, q \in I$ such that $g(p) = u$ and $g(q) = v$. Hence, $u + v = g(p) + g(q) = g(p + q)$ and $p + q \in I$. So, $u + v \in g(I)$. Since g is surjective, there exists $t \in R_1$ such that $g(t) = r$. So, $u\gamma r = g(p)\gamma g(t) = g(p\gamma t)$, $r\gamma u = g(t)\gamma g(p) = g(t\gamma p)$, $p\gamma t \in I$ and $t\gamma p \in I$. So, $u\gamma r$, $r\gamma u \in g(I)$. Hence, $g(I)$ is an ideal in R_2.

Since (δ_1, δ_2) is global, by Lemma 5.9, we have $\delta_1(I) = \delta_1(g^{-1}(g(I))) = g^{-1}(\delta_2((g(I)))$. By Lemma 5.8, $g(\delta_1(I)) = g(g^{-1}(\delta_2(g(I))) = \delta_2(g(I))$. □

Chapter 6

2-absorbing δ-primary Ideals

This final chapter is another main result of our project. We introduce the concept of 2-absorbing δ-primary ideals in a commutative Γ-semiring and investigate some results related to it. These ideals are generalization of 2-absorbing primary ideals. Throughout this chapter, let R be a commutative Γ-semiring and $\mathsf{J}(R)$ be the set of all ideals in R.

The following definitions that are given in the context of Γ-semirings were inspired by [5].

Definition 6.1. Let δ be an ideal expansion of a commutative Γ-semiring R. A proper ideal I in R is called a **2-absorbing δ-primary** in R if whenever $x, y, z \in R,\ \gamma, \beta \in \Gamma$ and $x\gamma y\beta z \in I$, then $x\gamma y \in I$ or $x\beta z \in \delta(I)$ or $y\beta z \in \delta(I)$.

Example 6.2. (1) Let R be a commutative Γ-semiring and I_d be the identity function in Example 5.2 (2). Then I is a 2-absorbing I_d-primary ideal in R if and only if I is a 2-absorbing ideal in R.

(2) Let R be a commutative Γ-semiring and $\sqrt{}$ be the radical function in Example 5.2 (3). Then I is a 2-absorbing $\sqrt{}$-primary ideal in R if and only if I is a 2-absorbing primary ideal in R.

The following results are inspired by results in [5].

Proposition 6.3. *Let δ be an ideal expansion of R. Then every δ-primary ideal in R is a 2-absorbing δ-primary ideal in R.*

Proof. Suppose that I is a δ-primary ideal in R. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in I$. Then, $x\gamma y\Gamma y\beta z \subseteq I$. Since I is a δ-primary ideal, we have $x\gamma y \in I$

or $y\beta z \in \delta(I)$.

Therefore, I is a 2-absorbing δ-primary ideal in R. □

Proposition 6.4. *Let δ and η be ideal expansions of R. If $\delta(I) \subseteq \eta(I)$ for all $I \in \mathsf{J}(R)$, then every 2-absorbing δ-primary ideal in R is also a 2-absorbing η-primary ideal in R.*

Proof. Suppose that $\delta(I) \subseteq \eta(I)$ for all $I \in \mathsf{J}(R)$. Let I be a 2-absorbing δ-primary ideal in R. Let $a, b, c \in R$ and $\alpha, \beta \in \eta$ be such that $a\alpha b\beta c \in I$. Then, $a\alpha b \in I$ or $a\beta c \in \delta(I)$ or $b\beta c \in \delta(I)$. Since $\delta(I) \subseteq \eta(I)$, we have $a\alpha b \in I$ or $a\beta c \in \eta(I)$ or $b\beta c \in \eta(I)$. Therefore, I is a 2-absorbing η-primary ideal in R. □

Proposition 6.5. *Let δ be an ideal expansion of R. For any subset S of R, let $\mathsf{J}_\delta(S)$ be the intersection of all 2-absorbing δ-primary ideals in R containing S. Then the function $h : \mathsf{J}(R) \to \mathsf{J}(R)$, given by*

$$h(I) = \begin{cases} \mathsf{J}_\delta(I) & \text{if there exists a 2-absorbing } \delta\text{-primary ideal in } R \text{ containing } I, \\ R & \text{otherwise,} \end{cases}$$

for all $I \in \mathsf{J}(R)$, is an ideal expansion of R.

Proof. Let I and J be ideals in R. If there are no 2-absorbing δ-primary ideals in R containing I, then $h(I) = R$ is an ideal in R.

Suppose there exists a 2-absorbing δ-primary ideal in R containing I. Then, $h(I) = \mathsf{J}_\delta(I) \neq \emptyset$. To show that $J_\delta(I)$ is an ideal in R, let $x, y \in \mathsf{J}_\delta(I), \gamma \in \Gamma, r \in R$ and H be a 2-absorbing δ-primary ideal in R containing I. Then, $x, y \in H$. Since H is an ideal in R, we have $x + y \in H$, $x\gamma r \in H$ and $r\gamma x \in H$. Note that H is arbitrary. So, $x + y \in \mathsf{J}_\delta(I), R\Gamma\mathsf{J}_\delta(I) \subseteq \mathsf{J}_\delta(I)$ and $\mathsf{J}_\delta(I)\Gamma R \subseteq \mathsf{J}_\delta(I)$. Thus $h(I) = \mathsf{J}_\delta(I)$ is an ideal in R containing I.

This shows that $h(I)$ is an ideal in R for any cases.

Clearly, $I \subseteq h(I)$.

Next, suppose $I \subseteq J$. If there are no 2-absorbing δ-primary ideals in R containing J, then $h(I) \subseteq R = h(J)$. Now, we assume that there exists a 2-absorbing δ-primary ideal in R containing J. Since $I \subseteq J$, that 2-absorbing δ-primary ideal in R must contain I. So,

$$\begin{aligned} h(I) = \mathsf{J}_\delta(I) &= \bigcap\{H \in \mathsf{J}(R) \mid I \subseteq H \text{ and } H \text{ is a 2–absorbing } \delta\text{–primary ideal in } R\} \\ &\subseteq \bigcap\{H \in \mathsf{J}(R) \mid J \subseteq H \text{ and } H \text{ is a 2–absorbing } \delta\text{–primary ideal in } R\} \\ &= \mathsf{J}_\delta(J) = h(J). \end{aligned}$$

Therefore, h is an ideal expansion of R. □

We show that any finite intersections of 2-absorbing δ-primary ideals are 2-absorbing δ-primary ideals with some conditions.

Proposition 6.6. *Let δ be an ideal expansion of R which is intersection preserving. If $I_1, I_2, \ldots, I_n$ are 2-absorbing δ-primary ideals in R, $\delta(I_1) = \delta(I_2) = \ldots = \delta(I_k)$ for all $k \in \{ 1, 2, \ldots, n \}$ and $\bigcap_{k=1}^n I_k \neq \emptyset$, then $\bigcap_{k=1}^n I_k$ is a 2-absorbing δ-primary ideal in R.*

Proof. Suppose that $I_1, I_2, \ldots, I_n$ are 2-absorbing δ-primary ideals in R, $J = \delta(I_k)$ for all $k \in \{ 1, 2, \ldots, n \}$ and $\bigcap_{k=1}^n I_k \neq \emptyset$. Clearly, $\bigcap_{k=1}^n I_k$ is a proper ideal in R. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in \bigcap_{k=1}^n I_k$ and $x\gamma y \notin \bigcap_{k=1}^n I_k$. Then, $x\gamma y \notin I_m$ for some $m \in \{ 1, 2, \ldots, n \}$. Since $x\gamma y\beta z \in \bigcap_{k=1}^n I_k \subseteq I_m$ and I_m is a 2-absorbing δ-primary ideal, $x\beta z \in \delta(I_m)$ or $y\beta z \in \delta(I_m)$. Since δ is intersection preserving, we have

$$\delta(\bigcap_{k=1}^n I_k) = \bigcap_{k=1}^n \delta(I_k) = \delta(I_m).$$

So, $x\beta z \in \delta(\bigcap_{k=1}^n I_k)$ or $y\beta z \in \delta(\bigcap_{k=1}^n I_k)$.

Therefore, $\bigcap_{k=1}^n I_k$ is a 2-absorbing δ-primary ideal in R. □

Next, we present a characterization of 2-absorbing δ-primary ideals. However, the following lemma is needed.

Lemma 6.7. *Let δ be an ideal expansion of R, I be a 2-absorbing δ-primary ideal in R and $\delta(I)$ be a k-ideal in R. Suppose that there exist $a, b \in R$, an ideal J in R and $\gamma, \beta \in \Gamma$ such that $a\gamma b\beta J \subseteq I$. If $a\gamma b \notin I$, then $a\beta J \subseteq \delta(I)$ or $b\beta J \subseteq \delta(I)$.*

Proof. Suppose $a\gamma b \notin I$, $a\beta J \nsubseteq \delta(I)$ and $b\beta J \nsubseteq \delta(I)$. Then, there exist $j_1, j_2 \in J$ such that $a\beta j_1 \notin \delta(I)$ and $b\beta j_2 \notin \delta(I)$. Since $a\gamma b\beta j_1 \in I$, $a\gamma b \notin I$ and $a\beta j_1 \notin$

$\delta(I)$, we have $b\beta j_1 \in \delta(I)$. Since $a\gamma b\beta j_2 \in I$, $a\gamma b \notin I$ and $b\beta j_2 \notin \delta(I)$, we have $a\beta j_2 \in \delta(I)$. Since $a\gamma b\beta(j_1 + j_2) \in I$ and $a\gamma b \notin I$, we have $a\beta(j_1 + j_2) \in \delta(I)$ or $b\beta(j_1 + j_2) \in \delta(I)$.

Case 1. $a\beta(j_1 + j_2) \in \delta(I)$. Since $\delta(I)$ is a k-ideal in R and $a\beta j_2 \in \delta(I)$, it follows that $a\beta j_1 \in \delta(I)$, which is a contradiction.

Case 2. $b\beta(j_1 + j_2) \in \delta(I)$. Since $\delta(I)$ is a k-ideal in R and $b\beta j_1 \in \delta(I)$, it follows that $b\beta j_2 \in \delta(I)$, which is a contradiction.

Therefore, $a\beta J \subseteq \delta(I)$ or $b\beta J \subseteq \delta(I)$. □

Theorem 6.8. *Let R be a commutative Γ-semiring with zero, δ be an ideal expansion of R, I be a proper k-ideal in R and $\delta(I)$ be a k-ideal in R. Then I is a 2-absorbing δ-primary ideal in R if and only if whenever ideals I_1, I_2, I_3 in R and $\gamma, \beta \in \Gamma$ with $I_1\gamma I_2\beta I_3 \subseteq I$, then $I_1\gamma I_2 \subseteq I$ or $I_1\beta I_3 \subseteq \delta(I)$ or $I_2\beta I_3 \subseteq \delta(I)$.*

Proof. First, suppose that I is a 2-absorbing δ-primary ideal in R and let I_1, I_2 and I_3 be ideals in R and $\gamma, \beta \in \Gamma$ such that $I_1\gamma I_2\beta I_3 \subseteq I$. Suppose to the contrary that $I_1\gamma I_2 \nsubseteq I$ and $I_1\beta I_3 \nsubseteq \delta(I)$ and $I_2\beta I_3 \nsubseteq \delta(I)$. Then, there exist $a, q_1 \in I_1$ and $b, q_2 \in I_2$ such that $a\gamma b \notin I$ and $q_1\beta I_3 \nsubseteq \delta(I)$ and $q_2\beta I_3 \nsubseteq \delta(I)$. Since $q_1\gamma q_2\beta I_3 \subseteq I$ and $q_1\beta I_3 \nsubseteq \delta(I)$ and $q_2\beta I_3 \nsubseteq \delta(I)$, by Lemma 6.7, we have $q_1\gamma q_2 \in I$. Since $a\gamma b\beta I_3 \subseteq I$ and $a\gamma b \notin I$, by Lemma 6.7, we have $a\beta I_3 \subseteq \delta(I)$ or $b\beta I_3 \subseteq \delta(I)$.

Case 1. $a\beta I_3 \subseteq \delta(I)$ and $b\beta I_3 \nsubseteq \delta(I)$. Since $q_1\gamma b\beta I_3 \subseteq I$ and $b\beta I_3 \nsubseteq \delta(I)$ and $q_1\beta I_3 \nsubseteq \delta(I)$, by Lemma 6.7, we have $q_1\gamma b \in I$. Since $\delta(I)$ is a k-ideal and $a\beta I_3 \subseteq \delta(I)$ and $q_1\beta I_3 \nsubseteq \delta(I)$, we have $(a + q_1)\beta I_3 \nsubseteq \delta(I)$. Since $(a + q_1)\gamma b\beta I_3 \subseteq I$, $(a + q_1)\beta I_3 \nsubseteq \delta(I)$ and $b\beta I_3 \nsubseteq \delta(I)$, by Lemma 6.7, we have $(a + q_1)\gamma b \in I$. Since I is a k-ideal and $q_1\gamma b \in I$, we have $a\gamma b \in I$, which is a contradiction.

Case 2. $a\beta I_3 \nsubseteq \delta(I)$ and $b\beta I_3 \subseteq \delta(I)$. This case is not possible similarly to Case 1.

Case 3. $a\beta I_3 \subseteq \delta(I)$ and $b\beta I_3 \subseteq \delta(I)$. Since $\delta(I)$ is a k-ideal, $b\beta I_3 \subseteq \delta(I)$ and $q_2\beta I_3 \nsubseteq \delta(I)$, we have $(b + q_2)\beta I_3 \nsubseteq \delta(I)$. Since $\delta(I)$ is a k-ideal, $a\beta I_3 \subseteq \delta(I)$ and $q_1\beta I_3 \nsubseteq \delta(I)$, we have $(a + q_1)\beta I_3 \nsubseteq \delta(I)$. Since $q_1\gamma(b + q_2)\beta I_3 \subseteq I$, $q_1\beta I_3 \nsubseteq \delta(I)$ and $(b + q_2)\beta I_3 \nsubseteq \delta(I)$, by Lemma 6.7, we have $q_1\gamma(b + q_2) \in I$. Since I is a k-ideal and $q_1\gamma q_2 \in I$, we have $q_1\gamma b \in I$. Since $(a + q_1)\gamma q_2\beta I_3 \subseteq I$, $q_2\beta I_3 \nsubseteq \delta(I)$ and $(a + q_1)\beta I_3 \nsubseteq \delta(I)$, by Lemma 6.7, we have $(a + q_1)\gamma q_2 \in I$. Since I is a k-ideal and $q_1\gamma q_2 \in I$, we have $a\gamma q_2 \in I$. Since $(a + q_1)\gamma(b + q_2)\beta I_3 \subseteq I$, $(a + q_1)\beta I_3 \nsubseteq \delta(I)$ and

$(b+q_2)\beta I_3 \nsubseteq \delta(I)$, by Lemma 6.7, we have $(a+q_1)\gamma(b+q_2) \in I$. Since I is a k-ideal, $q_1\gamma q_2 \in I$, $a\gamma q_2 \in I$ and $q_1\gamma b \in I$, we have $a\gamma b \in I$, which is a contradiction.
Hence, $I_1\gamma I_2 \subseteq I$ or $I_1\beta I_3 \subseteq \delta(I)$ or $I_2\beta I_3 \subseteq \delta(I)$.

On the other hand, suppose that whenever ideals I_1, I_2, I_3 in R and $\gamma, \beta \in \Gamma$ with $I_1\gamma I_2\beta I_3 \subseteq I$, then $I_1\gamma I_2 \subseteq I$ or $I_1\beta I_3 \subseteq \delta(I)$ or $I_2\beta I_3 \subseteq \delta(I)$. Let $x, y, z \in I$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in I$. Then, $\langle x\rangle\gamma\langle y\rangle\beta\langle z\rangle \subseteq I$. By assumption, $x\gamma y \in \langle x\rangle\gamma\langle y\rangle \subseteq I$ or $x\beta z \in \langle x\rangle\beta\langle z\rangle \subseteq \delta(I)$ or $y\beta z \in \langle y\rangle\beta\langle z\rangle \subseteq \delta(I)$. So, I is a 2-absorbing δ-primary ideal in R. □

Finally, we present results involving homomorphisms.

Theorem 6.9. *Let R_1 and R_2 be commutative Γ-semirings, δ_i be an ideal expansion of R_i for all $i \in \{1, 2\}$ such that (δ_1, δ_2) is global and $g : R_1 \to R_2$ be a homomorphism. If I is a 2-absorbing δ_2-primary ideal in R_2 such that $g^{-1}(I) \neq R_1$, then $g^{-1}(I)$ is a 2-absorbing δ_1-primary in R_1.*

Proof. Suppose that I is a 2-absorbing δ_2-primary ideal in R_2 such that $g^{-1}(I) \neq R_1$. Then, $g^{-1}(I)$ is a proper ideal in R_1. Let $x, y, z \in R$ and $\beta, \gamma \in \Gamma$ be such that $x\beta y\gamma z \in g^{-1}(I)$. Then, $g(x)\beta g(y)\gamma g(z) = g(x\beta y\gamma z) \in I$. Since I is a 2-absorbing δ_2-primary ideal in R_2, we have $g(x\beta y) = g(x)\beta g(y) \in I$ or $g(x\gamma z) = g(x)\gamma g(z) \in \delta_2(I)$ or $g(y\gamma z) = g(y)\gamma g(z) \in \delta_2(I)$. Hence, $x\beta y \in g^{-1}(I)$ or $x\gamma z \in g^{-1}(\delta_2(I)) = \delta_1(g^{-1}(I))$ or $y\gamma z \in g^{-1}(\delta_2(I)) = \delta_1(g^{-1}(I))$.

Therefore, $g^{-1}(I)$ is a 2-absorbing δ_1-primary ideal in R_1. □

Unlike the previous theorem, $g(\delta(I)) = \delta(g(I))$ holds provided that g must also be surjective and I has to be a k-ideal.

Theorem 6.10. *Let R_1 and R_2 be commutative Γ-semirings, δ_i be an ideal expansion of R_i for all $i \in \{1, 2\}$ such that (δ_1, δ_2) is global, $g : R_1 \to R_2$ be an epimorphism and I be a k-ideal in R_1. If I is a 2-absorbing δ_1-primary ideal in R_1 such that $g(I) \neq R_2$ and $\{ x \in R_1 \mid \exists\, a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$, then $g(I)$ is a 2-absorbing δ_2-primary ideal in R_2.*

Proof. Suppose that I is a 2-absorbing δ_1-primary ideal in R_1 such that $g(I) \neq R_2$ and $\{ x \in R_1 \mid \exists\, a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$. Then, $g(I)$ is a proper ideal in R_2. Let $x, y, z \in R_2$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in g(I)$. Then, there exists $t \in I$ such that $x\gamma y\beta z = g(t)$. Since g is surjective, there exist $p, q, r \in R$ such that $g(p) = x, g(q) = y$ and $g(r) = z$. Hence, $g(p\gamma q\beta r) = g(p)\gamma g(q)\beta g(r) = x\gamma y\beta z = g(t)$. So, $p\gamma q\beta r + t \in \{ x \in R_1 \mid \exists\ a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$. Since $t \in I$ and I is a k-ideal, $p\gamma q\beta r \in I$. Since I is a 2-absorbing δ_1-primary ideal, $p\gamma q \in I$ or $p\beta r \in \delta_1(I)$ or $q\beta r \in \delta_1(I)$. Hence, $x\gamma y = g(p)\gamma g(q) = g(p\gamma q) \in g(I)$ or $x\beta z = g(p)\beta g(r) = g(p\beta r) \in g(\delta_1(I)) = \delta_2(g(I))$ or $y\beta z = g(p)\beta g(r) = g(q\beta r) \in g(\delta_1(I)) = \delta_2(g(I))$.

Therefore, $g(I)$ is a 2-absorbing δ_2-primary ideal in R_2. □

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Appendix

กลุ่ม MATH12 สอบวันอังคารที่ 20 พฤศจิกายน 2561 เวลา 15:00 น. ห้อง 608/6

เอกสารนี้ได้รับการอนุมัติจากอาจารย์ที่ปรึกษาโครงงานแล้ว

ลงชื่อ..

(วันที่...)

The Project Proposal of Course 2301399 Project Proposal

First Semester, Academic Year 2018

Project Title (Thai)	การวางนัยทั่วไปของไอดีลเฉพาะในแกมมากึ่งริง
Project Title (English)	Generalization of Prime Ideals in Γ-semirings
Project Advisor	Associate Professor Dr. Sajee Pianskool
By	Mr. Nuttawut Sangjaer ID 5833527023 Mathematics Program, Department of Mathematics and Computer Science Faculty of Science, Chulalongkorn University

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Background and Rationale

As a generalization of rings, semirings were introduced by H.S. Vandiver [7] in 1934. The notion of Γ-rings was introduced by N. Nobusawa [4] in 1964. Also, as a generalization of semirings and Γ-rings, the notion of Γ-semirings was introduced by M.K. Rao [5] in 1995 that is for any commutative semigroups $(R, +)$ and $(\Gamma, +)$, R is said to be a **Γ-semiring** if there exists a function $\cdot$, called a **Γ-operation**, from $R \times \Gamma \times R$ into R, where $\cdot(x, \gamma, y)$ is denote by $x\gamma y$ for all $x, y \in R$ and $\gamma \in \Gamma$, satisfying the following properties: for all $x, y \in R$ and $\gamma, \beta \in \Gamma$

1. $x\gamma(y + z) = x\gamma y + x\gamma z$ and $(x + y)\gamma z = x\gamma z + y\gamma z$;
2. $x(\gamma + \beta)y = x\gamma y + x\beta y$; and
3. $(x\gamma y)\beta z = x\gamma(y\beta z)$.

For example, let R be the additive commutative semigroup containing all $m \times n$ matrices over the set of all non-negative integers and let Γ be the additive commutative semigroup containing all $n \times m$ matrices over the same set. Then we can verify that R is a Γ-semiring, where $a\gamma b$ is the usual matrix product for any $a, b \in R$ and $\gamma \in \Gamma$.

Ideals and k-ideals in Γ-semirings were extensively studied by S. Kyuno [3] in 1978. A subsemigroup I of a Γ-semiring R is called an **ideal** in R if $I\Gamma R \subseteq I$ and $R\Gamma I \subseteq I$ where $I\Gamma R = \{x\gamma r \mid x \in I,\ r \in R \text{ and } \gamma \in \Gamma\}$ and $R\Gamma I$ is defined similarly. An ideal I in a Γ-semiring R is called a ***k*-ideal** in R if for all $x, y \in R$, $x + y \in I$ and $x \in I$ implies $y \in I$. For example, let Γ be the additive semigroup of natural numbers. Then $(\mathbb{Z}_0^+, +)$ is a Γ-semiring, where $x\gamma y$ is the usual multiplication of integers, so $3\mathbb{Z}_0^+$ is a k-ideal in $\mathbb{Z}_0^+$ and $3\mathbb{Z}_0^+ - \{3\}$ is an ideal in $\mathbb{Z}_0^+$ but it is not a k-ideal in $\mathbb{Z}_0^+$ because $6 + 3 \in 3\mathbb{Z}_0^+ - \{3\}$, $6 \in 3\mathbb{Z}_0^+ - \{3\}$ but $3 \notin 3\mathbb{Z}_0^+ - \{3\}$.

T.K. Dutta and S.K. Sardar [1] in 2001 gave the definition of prime ideals which states that in a Γ-semiring R, a proper ideal I in R is said to be a **prime ideal** if for any two ideals H and K in R, $H\Gamma K \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$. For example, let Γ be the additive semigroup $p\mathbb{Z}_0^+$ where p is a prime number. Then $(\mathbb{Z}_0^+, +)$ is a Γ-semiring, where $x\gamma y$ is the usual multiplication of integers, so $q\mathbb{Z}_0^+$ is a prime ideal in $\mathbb{Z}_0^+$ for all prime numbers q with $p \neq q$.

M.K. Rao and B. Venkateswarlu [6] in 2017 studied properties of primary ideals. In a Γ-semiring R, a proper ideal I in R is said to be a **primary ideal** if for all $x, y \in R, \gamma \in \Gamma, x\gamma y \in I$ and $x \notin I$ implies $y \in \sqrt{I}$ where $\sqrt{I} = \{x \in R \mid \text{ there exists } n \in \mathbb{N} \text{ such that } (x\gamma)^{n-1}x \in I \text{ for all } \gamma \in \Gamma\}$, $(x\gamma)^0 x = x$ and $(x\gamma)^n x = (x\gamma)^{n-1}x\gamma x$ for all $x \in R$, $\gamma \in \Gamma$ and $n \in \mathbb{N}$. For example, let Γ be the additive semigroup of natural numbers. Then $(\mathbb{Z}_0^+, +)$ is a Γ-semiring, where $x\gamma y$ is the usual multiplication of integers. Let $I = \{x \in \mathbb{Z}_0^+ \mid x \geq c\}$ for some positive integer c. So I is a primary ideal in $\mathbb{Z}_0^+$.

In a commutative ring, D. Zhao [8] defined a mapping δ to substitute $\sqrt{I}$ by $\delta(I)$ in the definition of primary ideals, called δ-primary ideals in commutative rings, which are more general than primary ideals. Let $\mathsf{J}(M)$ be the set of all ideals of a commutative ring M. An **expansion of ideals** in M (or an ideal expansion of M) is defined to be a function $\delta : \mathsf{J}(M) \to \mathsf{J}(M)$ such that:

1. for all $I \in \mathsf{J}(M)$, $I \subseteq \delta(I)$; and
2. for all $I, J \in \mathsf{J}(M), I \subseteq J$ implies $\delta(I) \subseteq \delta(J)$.

For example, the constant function $c : \mathsf{J}(M) \to \mathsf{J}(M)$ defined by $I \mapsto M$ is an ideal expansion of a commutative ring M. Given an ideal expansion δ of a commutative ring M, an ideal $I \in \mathsf{J}(M)$ is said to be a **δ-primary ideal** if for all $x, y \in M, xy \in I$ and $x \notin I$ implies $y \in \delta(I)$.

In 2016, M.Y. Elkettani and A. Kasem [2] also extended a concept of prime ideals and primary ideals in Γ-rings to 2-absorbing ideals and 2-absorbing primary ideals which were unified to 2-absorbing δ-primary ideals. They gave the definitions of 2-absorbing ideals, 2-absorbing primary ideals and 2-absorbing δ-primary ideals in Γ-rings. A proper ideal I in a Γ-ring R is called a **2-absorbing ideal** in R if whenever $x, y, z \in R,\ \gamma, \beta \in \Gamma$ and $x\gamma y\beta z \in I$, then $x\gamma y \in I$ or $x\beta z \in I$ or $y\beta z \in I$. A proper ideal I in a Γ-ring R is called a **2-absorbing primary ideal** in R if whenever $x, y, z \in R,\ \gamma, \beta \in \Gamma$ and $x\gamma y\beta z \in I$, then $x\gamma y \in I$ or $x\beta z \in \sqrt{I}$ or $y\beta z \in \sqrt{I}$. Given an ideal expansion δ of a Γ-ring R, an ideal $I \in \mathsf{J}(R)$ is said to be a **2-absorbing δ-primary ideal** in R if for all $x, y, z \in R, \gamma, \beta \in \Gamma$ and $x\gamma y\beta z \in I$ implies $x\gamma y \in I$ or $x\beta z \in \delta(I)$ or $y\beta z \in \delta(I)$. Note that an ideal expansion δ of a Γ-ring R is defined in the same way as an ideal expansion of a commutative ring M by replacing M by R.

Our main goal is to extend the concepts of 2-absorbing primary ideals and 2-absorbing δ-primary ideals in Γ-ring to those in Γ-semiring.

Objectives

To study some properties of 2-absorbing primary ideals and 2-absorbing δ-primary ideals in Γ-semirings.

Scope

In this project, some algebraic properties of 2-absorbing primary ideals and 2-absorbing δ-primary ideals which are some generalized prime ideals in Γ-semirings are studied.

Project Activities

1. Literature reviews on Γ-semirings.
2. Study properties of 2-absorbing primary ideals in Γ-semirings.
3. Study properties of 2-absorbing δ-primary ideals in Γ-semirings.
4. Write a report.

Activities Table

Project Activity	August 2018 - April 2019								
	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
1. Literature reviews on Γ-semirings.									
2. Study properties of 2-absorbing primary ideals in Γ-semirings.									
3. Study properties of 2-absorbing δ-primary ideals in Γ-semirings.									
4. Write a report.									

Benefits

Obtain some properties and results of 2-absorbing primary ideals and 2-absorbing δ-primary ideals in Γ-semirings.

Equipment

1. Computer
2. Printer

Budget

1. Two reams of A4 paper	200	Bahts
2. Magic Macbook Mouse	2,300	Bahts
Total	**2,500**	**Bahts**

Reference

[1] T.K. Dutta and S.K. Sardar, On prime ideals and prime radicals of Γ- semirings, *An. Stiint. Univ. Al. I Cuza Iasi Mat.*, Vol.46 (2001), pp. 319–-329.

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