CHAPTER I

PRELIMINARIES

A triple $(S,+,\cdot)$ is called a semiring if

(i) $(S,+)$ and $(S,\cdot)$ are semigroups and

(ii) $x\cdot(y+z) = x\cdot y+x\cdot z$ and $(y+z)\cdot x = y\cdot x+z\cdot x$ for all $x,y,z \in S$, and the operations $+$ and $\cdot$ are usually called the addition and the multiplication of the semiring, respectively.

A semiring $(S,+,\cdot)$ is said to be additively commutative [multiplicatively commutative] if $(S,+)$ [$(S,\cdot)$] is a commutative semigroup. A commutative semiring is a semiring which is both additively commutative and multiplicatively commutative.

Let $S = (S,+,\cdot)$ be a semiring. An element 0 of S is called a zero of S if $x+0 = 0+x = x$ and $x\cdot 0 = 0\cdot x = 0$ for every $x \in S$. An element 1 of S is called a multiplicative identity of S if $x\cdot 1 = 1\cdot x = x$ for every $x \in S$. If S has a zero 0 [a multiplicative identity 1] and $x \in S$ is such that $x+y = y+x = 0$ [$x\cdot y = y\cdot x = 1$] for some $y \in S$, then x is said to be additively invertible [multiplicatively invertible] in S. It is easily seen that if S has a zero and $x,y \in S$ are additively invertible in S, then $x+y$, sx and xs are additively invertible in S for all $s \in S$, and if S has a multiplicative identity and $x,y \in S$ are multiplicatively invertible in S, then so is xy.

A semiring S is called an additively idempotent semiring if $x+x = x$ for all $x \in S$ and it is called a Boolean semiring or a multiplicatively idempotent semiring if $x^2 = x$ for all $x \in S$. An

<u>idempotent semiring</u> is a semiring which is both additively idempotent and multiplicatively idempotent.

An element x of a semigroup S is called an <u>inverse</u> of an element y of S if $x = xyx$ and $y = yxy$. A semigroup S is called an <u>inverse semigroup</u> if every element of S has a unique inverse. It is known that a commutative inverse semigroup S is a disjoint union of subgroups of S.

A semiring $(S,+,\cdot)$ is called an <u>additively inverse semiring</u> if $(S,+)$ is an inverse semigroup, and for $x \in S$, the notation x' will denote the inverse of x in the inverse semigroup $(S,+)$. Then an additively inverse semiring which is additively commutative is a semiring with its additive structure a union of groups. Note that an additively idempotent semiring S which is additively commutative is an additively inverse semiring with $x' = x$ for every $x \in S$.

From now on, semirings always mean "commutative semirings with zero 0 and multiplicative identity 1 and $0 \neq 1$".

A semiring $(S,+,\cdot)$ is called a <u>semifield</u> if $(S \setminus \{0\},\cdot)$ is a group.

An $n \times n$ matrix A over a semiring S is said to be <u>invertible</u> over S if there exists an $n \times n$ matrix B over S such that $AB = BA = I_n$ where I_n is the $n \times n$ identity matrix over S. The transpose of a matrix A over a semiring is denoted by A^T. Then for $n \times n$ matrices A and B over a semiring S, $(A^T)^T = A$, $(AB)^T = B^T A^T$, and A is invertible over S if and only if A^T is invertible over S.

If A is an $n \times n$ matrix over a semiring, then for $i,j \in \{1,2,\ldots,n\}$, let A_{ij} denote the element of A in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column.

A square matrix A over a semiring S is called a permutation matrix over S if every element of A is either 0 or 1 and every row and every column of A has exactly one element 1. Then every permutation matrix over a semiring S is invertible over S. To prove this, let A be an $n\times n$ permutation matrix over a semiring S. Claim that $AA^T = I_n$. Let $i,j \in \{1,2,\ldots,n\}$. Then $(AA^T)_{ij} = \sum_{t=1}^{n} A_{it}A_{jt}$. If $i \neq j$, then for each $t \in \{1,2,\ldots,n\}$, $A_{it} = 0$ or $A_{jt} = 0$ since every column of A has exactly one nonzero element, hence $(AA^T)_{ij} = 0$. If $i = j$, then there exists a $t_o \in \{1,2,\ldots,n\}$ such that $A_{it_o} = 1$ and $A_{it} = 0$ for all $t \in \{1,2,\ldots,n\}\setminus\{t_o\}$ since every element of A is either 0 or 1 and every row of A has exactly one element 1, thus $(AA^T)_{ij} = (AA^T)_{ii} = \sum_{t=1}^{n} A_{it}^2 = A_{it_o}^2 = 1$. Hence $AA^T = I_n$. It can be shown similarly that $A^TA = I_n$.

Let A be an $n\times n$ matrix over a semiring S. The positive determinant of A, $\det^+A$, the negative determinant of A, $\det^-A$, and the permanent of A, per(A), are defined respectively by

$$\det{}^+A = \sum_{\sigma\in\mathcal{A}_n}\left(\prod_{k=1}^{n} A_{k\sigma(k)}\right),$$

$$\det{}^-A = \begin{cases} \sum_{\sigma\in\mathcal{B}_n}\left(\prod_{k=1}^{n} A_{k\sigma(k)}\right) & \text{if } n > 1, \\ 0 & \text{if } n = 1 \end{cases}$$

and

$$\operatorname{per}(A) = \sum_{\sigma\in\mathcal{S}_n}\left(\prod_{k=1}^{n} A_{k\sigma(k)}\right)$$

where $\mathcal{S}_n$ is the permutation group (the symmetric group) of degree n, $\mathcal{A}_n$ is the alternating group of degree n (that is, $\mathcal{A}_n = \{\sigma \in \mathcal{S}_n | \sigma$ is an even permutation$\}$) and $\mathcal{B}_n = \mathcal{S}_n \setminus \mathcal{A}_n$ (that is, $\mathcal{B}_n = \{\sigma \in \mathcal{S}_n | \sigma$ is an odd permutation}). Note that the following statements hold :

(1) $per(A) = det^+A + det^-A$,

(2) $det(A) = det^+A - det^-A$ if S is a commutative ring with identity,

(3) $det^+I_n = 1$ and $det^-I_n = 0$ where I_n is the n×n identity matrix over S,

(4) $per(A) = det(A)$ if S is a Boolean ring with identity,

(5) $det^+A = det^+(A^T)$ and $det^-A = det^-(A^T)$,

(6) $per(A) = per(A^T)$.

The statements (1), (2) and (3) are clearly seen. The statement (4) follows from the fact that $x = -x$ for every x in any Boolean ring. The statement(6) follows from the statement(5). A proof of (5) is given as follows : Since $det^+(A^T) = \sum_{\sigma\in\mathcal{A}_n} A_{\sigma(1)1}A_{\sigma(2)2}\cdots A_{\sigma(n)n}$ and $\mathcal{A}_n = \{\sigma^{-1} | \sigma \in \mathcal{A}_n\}$, we have that $det^+(A^T) = \sum_{\sigma\in\mathcal{A}_n} A_{\sigma^{-1}(1)1}A_{\sigma^{-1}(2)2}\cdots A_{\sigma^{-1}(n)n}$.

Because for each $\sigma \in \mathcal{A}_n$, $\{\sigma^{-1}(1),\sigma^{-1}(2),\ldots,\sigma^{-1}(n)\} = \{1,2,\ldots,n\}$, and $k = \sigma(\sigma^{-1}(k))$ for all $k \in \{1,2,\ldots,n\}$, it follows that for each $\sigma \in \mathcal{A}_n$,

$$A_{\sigma^{-1}(1)1}A_{\sigma^{-1}(2)2}\cdots A_{\sigma^{-1}(n)n} = A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)}.$$

Hence $det^+(A^T) = \sum_{\sigma\in\mathcal{A}_n} A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)} = det^+A$. And, $det^-(A^T) = det^-A$ can be proved similarly.

The following known results will be used in this thesis.

<u>Theorem 1.1</u>. If A and B are n×n matrices over a semiring S, then there exists an element r of S such that

$$\det^{+}(AB) = (\det^{+}A)(\det^{+}B)+(\det^{-}A)(\det^{-}B)+r$$

and

$$\det^{-}(AB) = (\det^{+}A)(\det^{-}B)+(\det^{-}A)(\det^{+}B)+r \ [5].$$

<u>Theorem 1.2.</u> Let A and B be $n\times n$ matrices over a semiring S and I_n the $n\times n$ identity matrix over S. If $AB = I_n$, then $BA = I_n$ [5].

<u>Theorem 1.3.</u> If S is an additively inverse semiring, then for all $x,y \in S$,

(1) $(x+y)' = x' + y'$

(2) $(xy)' = x'y = xy'$ and

(3) $x'y' = xy$ [6].

<u>Theorem 1.4.</u> If S is a semifield, then S is a field or 0 is the only additively invertible element of S [7].

<u>Theorem 1.5.</u> Let S be a semiring such that S has no zero divisors (that is, $xy = 0$ in S implies that $x = 0$ or $y = 0$) and 0 is the only additively invertible element of S. Then a square matrix A over S is invertible over S if and only if every row and every column of A has exactly one nonzero element and every nonzero element of A is a multiplicatively invertible element of S [4].

<u>Theorem 1.6.</u> Let $\mathbb{N}$ be the set of all positive integers. Then there exists a binary operation $*$ on $\mathbb{N}\cup\{0\}$ such that $(\mathbb{N}\cup\{0\},*,\cdot)$ is a ring where $\cdot$ is the usual multiplication [8].

In this thesis, we let $\mathbb{N}$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ denote the set of all positive integers, the set of all positive rational numbers and the set of all positive real numbers, respectively.