# CHAPTER III

## INVERTIBLE MATRICES OVER AN ADDITIVELY INVERSE SEMIRING

We have mentioned in Chapter II that invertible matrices over a commutative ring with identity are characterized in [1] as follows : A square matrix A over a commutative ring R with identity is invertible over R if and only if det(A) is multiplicatively invertible in R.

Additively inverse semirings are a generalization of commutative rings with identity. In this chapter, we are concerned with invertible matrices over an additively inverse semiring and show that the result in [1] mentioned above is in fact a corollary of our main theorem.

Some examples of additively inverse semirings which are not rings are as follows :

(1) Any idempotent semiring containing more than one element,

(2) $([0,1],\max,\cdot)$, $(\mathbb{N}\cup\{0\},\max,\cdot)$ and $(\mathbb{R}^+\cup\{0\},\max,\cdot)$ where $\cdot$ is the usual multiplication of real numbers,

(3) $S\times R$ where S is an idempotent semiring containing more than one element, R is a commutative ring with identity and the operations on $S\times R$ are the componentwise operations.

Recall that in any additively inverse semiring $S = (S,+,\cdot)$, let $x'$ denote the unique inverse of the element x in the inverse semigroup (S,+). It is easily seen that in any additively inverse semiring S, x+y = 0 implies that $y = x'$, hence if x is an additively invertible element of S, then $x+x' = 0$.

The following theorem is the main result of this chapter.

Theorem 3.1. Let S be an additively inverse semiring and A an n×n matrix over S. Then the matrix A is invertible over S if and only if

(i) $\det^{+}A + (\det^{-}A)'$ is multiplicatively invertible in S and

(ii) $A_{ij}A_{ik}$ is additively invertible in S for all i,j,k in {1,2,...,n}, $j \neq k$.

Proof : Assume that there is an n×n matrix B over S such that $AB = BA = I_n$. By Theorem 1.1, there exists an element r of S such that

$$\det^{+}(AB) = (\det^{+}A)(\det^{+}B)+(\det^{-}A)(\det^{-}B)+r$$

and

$$\det^{-}(AB) = (\det^{+}A)(\det^{-}B)+(\det^{-}A)(\det^{+}B)+r.$$

But since $\det^{+}(AB) = \det^{+}I_n = 1$ and $\det^{-}(AB) = \det^{-}I_n = 0$, it is immediate that

$$1 = (\det^{+}A)(\det^{+}B)+(\det^{-}A)(\det^{-}B)+r \quad \ldots\ldots. (1)$$

and

$$0 = (\det^{+}A)(\det^{-}B)+(\det^{-}A)(\det^{+}B)+r. \quad \ldots\ldots. (2)$$

From (2), we have that $r = [(\det^{+}A)(\det^{-}B)+(\det^{-}A)(\det^{+}B)]'$. By Theorem 1.3(1) and Theorem 1.3(2),

$$r = [(\det^{+}A)(\det^{-}B)]'+[(\det^{-}A)(\det^{+}B)]'$$

$$= (\det^{+}A)(\det^{-}B)'+(\det^{-}A)'(\det^{+}B).$$

Substitute r by $(\det^+A)(\det^-B)'+(\det^-A)'(\det^+B)$ in (1), so we have that

$$
\begin{aligned}
1 &= (\det^+A)(\det^+B)+(\det^-A)(\det^-B)+(\det^+A)(\det^-B)'+ \\
&\quad (\det^-A)'(\det^+B) \\
&= \det^+A[\det^+B+(\det^-B)']+(\det^-A)'(\det^+B)+ \\
&\quad (\det^-A)(\det^-B).
\end{aligned}
$$

Since $(\det^-A)(\det^-B) = (\det^-A)'(\det^-B)'$ (by Theorem 1.3(3)), it follows that

$$
\begin{aligned}
1 &= \det^+A[\det^+B+(\det^-B)']+(\det^-A)'(\det^+B)+ \\
&\quad (\det^-A)'(\det^-B)' \\
&= \det^+A[\det^+B+(\det^-B)']+(\det^-A)'[\det^+B+(\det^-B)'] \\
&= [\det^+A+(\det^-A)'][\det^+B+(\det^-B)'].
\end{aligned}
$$

Hence $\det^+A+(\det^-A)'$ is multiplicatively invertible in S, so we obtain (i).

The condition (ii) follows from Proposition 2.1.

For the converse, assume that (i) and (ii) hold. Then by (i), $x[\det^+A+(\det^-A)'] = 1$ for some $x \in S$, and by (ii), $A_{ij}A_{ik}+(A_{ij}A_{ik})' = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$. If $n = 1$, then A is an $1\times1$ matrix whose element is multiplicatively invertible in S, hence A is invertible over S. Assume that $n > 1$ and let B be the $n\times n$ matrix over S defined by

$$
B_{ij} = x\Big[ \sum_{\substack{\sigma\in\mathcal{A}_n \\ \sigma(j)=i}} \Big( \prod_{\substack{k=1 \\ k\neq j}}^{n} A_{k\sigma(k)}\Big) + \Big(\sum_{\substack{\sigma\in\mathcal{B}_n \\ \sigma(j)=i}} \Big( \prod_{\substack{k=1 \\ k\neq j}}^{n} A_{k\sigma(k)}\Big)\Big)' \Big]
$$

for all $i,j \in \{1,2,\ldots,n\}$ where for $t \in \{1,2,\ldots,n\}$, we let

$\sum_{\substack{\sigma\in\mathcal{A}_n \\ \sigma(j)=t}} \Big( \prod_{\substack{k=1 \\ k\neq j}}^{n} A_{k\sigma(k)}\Big) = 0$ if there is no $\sigma \in \mathcal{A}_n$ such that $\sigma(j)=t$ and

$$\sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=t}} \Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)' = 0 \text{ if there is no } \sigma\in\mathcal{B}_n \text{ such that } \sigma(j)=t.$$

Claim that $AB = BA = I_n$. Let $i,j \in \{1,2,\ldots,n\}$. Then

$$(AB)_{ij} = \sum_{t=1}^{n} A_{it}B_{tj}$$

$$= \sum_{t=1}^{n} A_{it}x\Big[\sum_{\substack{\sigma\in\mathcal{A}_n\\ \sigma(j)=t}} \Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big) + \sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=t}} \Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'\Big]$$

$$= x\Big[\sum_{t=1}^{n}\Big(\sum_{\substack{\sigma\in\mathcal{A}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)\Big) + \sum_{t=1}^{n}\Big(\sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'\Big)\Big]$$

where for $t \in \{1,2,\ldots,n\}$, we let $\sum_{\substack{\sigma\in\mathcal{A}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big) = 0$ if there is no $\sigma \in \mathcal{A}_n$ such that $\sigma(j) = t$ and $\sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)' = 0$ if there is no $\sigma \in \mathcal{B}_n$ such that $\sigma(j) = t$. Since $\mathcal{A}_n = \bigcup_{t=1}^{n}\{\sigma\in\mathcal{A}_n \mid \sigma(j)=t\}$ is a disjoint union, we have that

$$\sum_{t=1}^{n}\Big(\sum_{\substack{\sigma\in\mathcal{A}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)\Big) = \sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big).$$

Also, since $\mathcal{B}_n = \bigcup_{t=1}^{n}\{\sigma\in\mathcal{B}_n \mid \sigma(j)=t\}$ is a disjoint union,

$$\sum_{t=1}^{n}\Big(\sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'\Big) = \sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'.$$

Hence

$$(AB)_{ij} = x\Big[\sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big) + \sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'\Big]. \qquad \ldots\ldots (*)$$

If i = j, then

$$\sum_{\sigma\varepsilon\mathcal{A}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{n} A_{k\sigma(k)}\Big) = \sum_{\sigma\varepsilon\mathcal{A}_n} A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i}}^{n} A_{k\sigma(k)}\Big)$$

$$= \sum_{\sigma\varepsilon\mathcal{A}_n}\Big(\prod_{k=1}^{n} A_{k\sigma(k)}\Big)$$

$$= \det{}^{+}A \qquad \ldots\ldots\ldots\ldots \text{(I)}$$

and

$$\sum_{\sigma\varepsilon\mathcal{B}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{n} A_{k\sigma(k)}\Big)' = \sum_{\sigma\varepsilon\mathcal{B}_n} A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i}}^{n} A_{k\sigma(k)}\Big)'$$

$$= \sum_{\sigma\varepsilon\mathcal{B}_n}\Big(\prod_{k=1}^{n} A_{k\sigma(k)}\Big)' \quad \text{(by Theorem 1.3(2))}$$

$$= \Big(\sum_{\sigma\varepsilon\mathcal{B}_n}\Big(\prod_{k=1}^{n} A_{k\sigma(k)}\Big)\Big)' \quad \text{(by Theorem 1.3(1))}$$

$$= (\det{}^{-}A)'. \qquad \ldots\ldots\ldots \text{(II)}$$

Therefore, if i = j, it follows from (*), (I) and (II) that

$$(AB)_{ij} = (AB)_{ii} = x[\det{}^{+}A+(\det{}^{-}A)'] = 1.$$

Next, assume that $i \neq j$. If n = 2, then

$$(AB)_{ij} = x(A_{ij}A_{ii}+A_{ii}A'_{ij}) \qquad \text{(from (*))}$$

$$= x[A_{ij}A_{ii}+(A_{ij}A_{ii})'] \qquad \text{(by Theorem 1.3(2))}$$

$$= 0 \qquad \text{(by the assumption (ii)).}$$

Assume further that n > 2. Then

$$\sum_{\sigma\varepsilon\mathcal{A}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{n} A_{k\sigma(k)}\Big) = \sum_{\sigma\varepsilon\mathcal{A}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big) \quad \ldots\ldots (I')$$

and

$$\sum_{\sigma\varepsilon\mathcal{B}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{n} A_{k\sigma(k)}\Big)' = \sum_{\sigma\varepsilon\mathcal{B}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big)' \text{(by Theorem 1.3(2))}$$

$$= \sum_{\sigma\in\mathcal{B}_n}(A_{i\sigma(j)}A_{i\sigma(i)})'(\prod_{\substack{k=1\\k\neq i,j}}^{n}A_{k\sigma(k)})\text{(by Theorem 1.3(2)).}$$

......... (II')

Hence it follows from (*), (I') and (II') that

$$(AB)_{ij} = x[\sum_{\sigma\in\mathcal{A}_n}A_{i\sigma(j)}A_{i\sigma(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n}A_{k\sigma(k)})+\sum_{\sigma\in\mathcal{B}_n}(A_{i\sigma(j)}A_{i\sigma(i)})'(\prod_{\substack{k=1\\k\neq i,j}}^{n}A_{k\sigma(k)})].$$

Since the map $\sigma\mapsto\bar{\sigma}$ where $\bar{\sigma} = (\sigma(i),\sigma(j))\sigma$ is a 1-1 map from $\mathcal{A}_n$ onto $\mathcal{B}_n$ (Lemma 2.4), we get

$$(AB)_{ij} = x[\sum_{\sigma\in\mathcal{A}_n}A_{i\sigma(j)}A_{i\sigma(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n}A_{k\sigma(k)})+\sum_{\sigma\in\mathcal{A}_n}(A_{i\bar{\sigma}(j)}A_{i\bar{\sigma}(i)})'(\prod_{\substack{k=1\\k\neq i,j}}^{n}A_{k\bar{\sigma}(k)})].$$

For $\sigma\in\mathcal{A}_n$, $\bar{\sigma}(i) = \sigma(j)$, $\bar{\sigma}(j) = \sigma(i)$ and $\bar{\sigma}(k) = \sigma(k)$ for all $k\in\{1,2,\ldots,n\}\setminus\{i,j\}$ (see the proof of Theorem 2.5). These imply that for each $\sigma\in\mathcal{A}_n$, $A_{i\sigma(j)}A_{i\sigma(i)} = A_{i\bar{\sigma}(j)}A_{i\bar{\sigma}(i)}$ and $\prod_{\substack{k=1\\k\neq i,j}}^{n}A_{k\sigma(k)} = \prod_{\substack{k=1\\k\neq i,j}}^{n}A_{k\bar{\sigma}(k)}$.

Thus

$$(AB)_{ij} = x[\sum_{\sigma\in\mathcal{A}_n}(A_{i\sigma(j)}A_{i\sigma(i)}+(A_{i\sigma(j)}A_{i\sigma(i)})')(\prod_{\substack{k=1\\k\neq i,j}}^{n}A_{k\sigma(k)})].$$

Since $i\neq j$, $\sigma(i)\neq\sigma(j)$ for all $\sigma\in\mathcal{A}_n$, so

$A_{i\sigma(j)}A_{i\sigma(i)}+(A_{i\sigma(j)}A_{i\sigma(i)})' = 0$ for all $\sigma\in\mathcal{A}_n$ (by the assumption (ii)). Thus $(AB)_{ij} = 0$.

Hence $AB = I_n$. By Theorem 1.2, $AB = BA = I_n$. #

<u>Corollary 3.2.</u> Let S be an additively inverse semiring and A an nxn matrix over S. Then the matrix A is invertible over S if and only if

(i) $\det^+A + (\det^-A)'$ is multiplicatively invertible in S and

(ii) $A_{ij}A_{kj}$ is additively invertible in S for all i,j,k in $\{1,2,\ldots,n\}$, $i\neq k$.

Proof : It follows easily from Theorem 3.1 and the facts that $\det^+ A = \det^+(A^T)$, $\det^- A = \det^-(A^T)$ and A is invertible over S if and only if $A^T$ is invertible over S. #

If S is a commutative ring with identity, then S is an additively inverse semiring with every element additively invertible and $x' = -x$ for every $x \in S$, hence for a square matrix A over S, $\det^+ A + (\det^- A)' = \det^+ A - \det^- A = \det(A)$. Then the following known result is an immediate consequence of Theorem 3.1.

Corollary 3.3. Let R be a commutative ring with identity and A a square matrix over R. Then the matrix A is invertible over R if and only if det(A) is multiplicatively invertible in R.

Every additively idempotent semiring is an additively inverse semiring. The four following corollaries characterize invertible matrices over an additively idempotent semiring.

Corollary 3.4. Let S be an additively idempotent semiring and A an nxn matrix over S. Then the matrix A is invertible over S if and only if

(i) per(A) is multiplicatively invertible in S and

(ii) $A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$.

Proof : Since S is an additively idempotent semiring, $x' = x$ for every $x \in S$. Then $\det^+ A + (\det^- A)' = \det^+ A + \det^- A = \text{per}(A)$. For $x \in S$, if x is additively invertible in S, then $x + x' = 0$ which implies that $x = x + x = x + x' = 0$. Hence 0 is the only additively invertible element of S. Therefore by Theorem 3.1, the corollary is proved. #

<u>Corollary 3.5.</u> Let S be an additively idempotent semiring and A an nxn matrix over S. Then the matrix A is invertible over S if and only if

(i) per(A) is multiplicatively invertible in S and

(ii) $A_{ij}A_{kj} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $i \neq k$.

<u>Proof</u> : This follows from Corollary 3.4 and the facts that $\text{per}(A) = \text{per}(A^T)$ and A is invertible over S if and only if $A^T$ is invertible over S. #

<u>Corollary 3.6.</u> Let S be an additively idempotent semiring and A an nxn matrix over S. Then the matrix A is invertible over S if and only if

(i) $\sum_{i=1}^{n} A_{ij}$ is multiplicatively invertible in S for all $j \in \{1,2,\ldots,n\}$ and

(ii) $A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$.

<u>Proof</u> : First, we note that $\sum_{i=1}^{n} A_{ij}$ is multiplicatively invertible in S for all $j \in \{1,2,\ldots,n\}$ if and only if $(\sum_{i=1}^{n} A_{i1})(\sum_{i=1}^{n} A_{i2})\ldots(\sum_{i=1}^{n} A_{in})$ is multiplicatively invertible in S since S is commutative. To prove the corollary, by Corollary 3.4, it suffices to show that if $A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$, then $\text{per}(A) = (\sum_{i=1}^{n} A_{i1})(\sum_{i=1}^{n} A_{i2})\ldots(\sum_{i=1}^{n} A_{in})$.

Assume that $A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$. Then

$$\left(\sum_{i=1}^{n} A_{i1}\right)\left(\sum_{i=1}^{n} A_{i2}\right)\cdots\left(\sum_{i=1}^{n} A_{in}\right) = \sum_{t_1,t_2,\ldots,t_n \in \{1,2,\ldots,n\}} (A_{t_1 1}A_{t_2 2}\cdots A_{t_n n}).$$

Since $A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$, it follows that $A_{t_1 1}A_{t_2 2}\cdots A_{t_n n} = 0$ if $t_1,t_2,\ldots,t_n \in \{1,2,\ldots,n\}$ and $t_i = t_j$ for some $i,j \in \{1,2,\ldots,n\}$, $i \neq j$. Then

$$\sum_{t_1,t_2,\ldots,t_n \in \{1,2,\ldots,n\}} (A_{t_1 1}A_{t_2 2}\cdots A_{t_n n})$$

$$= \sum_{\substack{t_1,t_2,\ldots,t_n \in \{1,2,\ldots,n\} \\ t_i \neq t_j \text{ if } i \neq j}} (A_{t_1 1}A_{t_2 2}\cdots A_{t_n n})$$

$$= \sum_{\sigma \in \mathcal{S}_n} (A_{\sigma(1)1}A_{\sigma(2)2}\cdots A_{\sigma(n)n})$$

since $|\{(t_1,t_2,\ldots,t_n) | t_1,t_2,\ldots,t_n \in \{1,2,\ldots,n\},\ t_i \neq t_j \text{ if } i \neq j\}|$ $= |\{(\sigma(1),\sigma(2),\ldots,\sigma(n)) | \sigma \in \mathcal{S}_n\}|$. Hence

$$\left(\sum_{i=1}^{n} A_{i1}\right)\left(\sum_{i=1}^{n} A_{i2}\right)\cdots\left(\sum_{i=1}^{n} A_{in}\right) = \sum_{\sigma \in \mathcal{S}_n} (A_{\sigma(1)1}A_{\sigma(2)2}\cdots A_{\sigma(n)n})$$

$$= \operatorname{per}(A^T)$$

$$= \operatorname{per}(A). \qquad \#$$

Corollary 3.7. Let S be an additively idempotent semiring and A an n×n matrix over S. Then the matrix A is invertible over S if and only if

(i) $\sum_{j=1}^{n} A_{ij}$ is multiplicatively invertible in S for all $i \in \{1,2,\ldots,n\}$ and

(ii) $A_{ij}A_{kj} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $i \neq k$.

Proof : It is clearly obtained from Corollary 3.6 and the fact that A is invertible over S if and only if $A^T$ is invertible over S. #