CHAPTER IV

HIGHER CURVATURES OF CURVES IN EUCLIDEAN N-SPACE

The main purpose of this chapter is to characterize the Frenet frame and the curvatures of curves in Euclidean n-space. These results first reported in [4] .

1. The Frenet frame and the curvatures.

Let $I = [a,b]$ be an interval in R and $F : I \longrightarrow R^n$ a C^k-parametrization by arc length. This means that the arc length along the curve from $F(s_1)$ to $F(s_2)$ is $|s_1 - s_2|$ and by Theorem 2.3.5, $|F'(s)| = 1$ for all $s \in I$. Suppose that for each $s \in I$, the vectors,

$$F'(s),\ F''(s), \ldots,\ F^{(r)}(s) \qquad\qquad r < k$$

are linear independent. A pplying the Gram-Schmidt orthonormalization process to these vectors, we obtains an orthonormal r-tuple of vectors,

$$(V_1(s),\ V_2(s), \ldots,\ V_r(s)) \ ,$$

called the Frenet r-frame associated with the curve at the point $F(s)$.

The Gram-Schmidt process is actually carried out as follows :

Let

$$E_1(s) = F'(s) \quad \text{and} \quad V_1(s) = \frac{E_1(s)}{|E_1(s)|}$$

If $V_1(s),\ V_2(s), \ldots,\ V_{i-1}(s)$ have already been determined.

Let

$$E_i(s) = F^{(i)}(s) - \sum_{j<i} \left[F^{(i)}(s) \cdot V_j(s)\right] V_j(s)$$

and

$$V_i(s) = \frac{E_i(s)}{|E_i(s)|} .$$

This works for $i = 1,2,\ldots, r$.

4.1.1 Remark. By our assumption that for each $s \in I$, the vectors $F'(s)$, $F''(s),\ldots, F^{(r)}(s)$ are linearly independent, we can easily show that $F^{(i)}(s)$, $E_i(s)$, and $V_i(s)$ can not be equal to zero vector for all $s \in I$ and for $i = 1,2,\ldots, r$.

4.1.2 Lemma. Every vector $V_i(s)$ is a linear combination

$$V_i(s) = a_{i1}(s)F'(s) + a_{i2}F''(s) + \ldots + a_{ii}(s)F^{(i)}(s)$$

of the vectors $F'(s)$, $F''(s),\ldots, F^{(i)}(s)$, where $s \in I$ and a_{ij} are C^{k-i}- real-valued functions defined on I, for $j = 1,2,\ldots,i$, $i = 1,2,\ldots, r$.

Proof. When $i = 1$, by the Gram-Schmidt process

$$V_1(s) = \frac{1}{|F'(s)|} F'(s) .$$

By letting $a_{11}(s) = \frac{1}{|F'(s)|} = 1$, we are done.

Assume that the lemma holds for all value of $j < i \leq n$. Then from the induction hypothesis,

$$V_j(s) = a_{j1}(s)F'(s) + a_{j2}(s)F''(s) + \ldots + a_{jj}(s)F^{(j)}(s), \ldots (1)$$

where $a_{j1}, a_{j2}, \ldots, a_{jj}$ are of class C^{k-j}, $j = 1,2,\ldots, i-1$.

Obviously a_{jm}, $m = 1,2,\ldots,j$, $j = 1,2,\ldots, i-1$ are at least of class C^{k-i}, thus $V_1, V_2, \ldots, V_{i-1}$ are at least of class C^{k-i}.

Let's consider,

$$E_i(s) = F^{(i)}(s) - \sum_{j<i} \left[F^{(i)}(s).V_j(s)\right] V_j(s). \quad \ldots\ldots\ldots (2)$$

Using Equ.(1) and Equ.(2), we can express $E_i(s)$ in term of $F'(s)$, $F''(s), \ldots, F^{(i)}(s)$:

$$E_i(s) = F^{(i)}(s) - \sum_{j<i} \left(\left[F^{(i)}(s).V_j(s)\right] \left(\sum_{m=1}^{j} a_{jm}(s)F^{(m)}(s)\right)\right)$$

$$= F^{(i)}(s) - \sum_{j<i} \left(\sum_{m=1}^{j} a_{jm}(s)\left[F^{(i)}(s).V_j(s)\right] F^{(m)}(s)\right)$$

$$= F^{(i)}(s) - b_{ii-1}(s)F^{(i-1)}(s) - \ldots - b_{i1}(s)F(s),$$

where $b_{ij}(s) = \sum_{m=j}^{i-1} a_{mj}(s)\left[F^{(i)}(s).V_m(s)\right]$, $j = 1,2,\ldots, i-1$.

By the induction hypothesis and theorem 2.2.15, we can easily verify that the functions b_{ij} are of class C^{k-i} for $j = 1,2,\ldots., i-1$. Therefore E_i is of class C^{k-i}, but $E_i(s)$ is never zero for all $s \in I$ thus $|E_i(s)|$ is also of class C^{k-i}.

Now

$$V_i(s) = \frac{E_i(s)}{|E_i(s)|}$$

$$= \frac{1}{|E_i(s)|} F^{(i)}(s) - \frac{b_{ii-1}(s)}{|E_i(s)|} F^{(i-1)}(s) - \dots - \frac{b_{i1}(s)}{|E_i(s)|} F'(s).$$

Let

$$a_{ii}(s) = \frac{1}{|E_i(s)|} \text{ and } a_{ij}(s) = -\frac{b_{ij}(s)}{|E_i(s)|},$$

for $j = 1,2,\dots, i-1$.

Hence we can write

$$V_i(s) = a_{ii}(s)F^{(i)}(s) + \dots + a_{i1}(s)F'(s),$$

and a_{ii}, $a_{ii-1}, \dots, a_{i1}$ are of class C^{k-i} by virtue of Theorem 2.2.15.

The proof now follows by Mathematical induction.

4.1.3 Corollary. V_i is of class C^{k-i} for $i = 1,2,\dots, r < k$.

Proof. Using the fact that $F^{(i)}$ is of class C^{k-i} and applying Lemma 4.1.2, finishes the proof.

4.1.4 Theorem Every vector $V_i'(s)$ is a linear combination

$$V_i'(s) = c_{i1}(s)V_1(s) + c_{i2}(s)V_2(s) + \dots + c_{ii+1}(s)V_{i+1}(s)$$

of the vectors $V_1(s)$, $V_2(s), \dots$ $V_{i+1}(s)$, where c_{ij} is of class C^{k-i-1} for $j = 1,2,\dots, i+1$, $i = 1,2,\dots, r-1$.

Proof. By Lemma 4.1.2, we can write

$$V_i(s) = a_{i1}(s)F'(s)+a_{i2}(s)F''(s)+\ldots+a_{ii}(s)F^{(i)}(s), \ldots\ldots\ldots(3)$$

where a_{ij} are of class C^{k-i}, $j = 1,2,\ldots, i$. Differentiating Equ.(3) yields

$$V_i'(s) = a_{i1}(s)F''(s)+a_{i1}'(s)F'(s)+a_{i2}(s)F'''(s)+a_{i2}'(s)F''(s)$$
$$+\ldots+a_{ii}(s)F^{(i+1)}(s)+a_{ii}'(s)F^{(i)}(s)$$
$$= a_{i1}'(s)F'(s)+(a_{i1}(s)+a_{i2}'(s))F''(s)+\ldots+$$
$$(a_{ii-1}(s)+a_{ii}'(s))F^{(i)}(s)+a_{ii}(s)F^{(i+1)}(s).$$
$$= d_{i1}(s)F'(s)+d_{i2}(s)F''(s)+\ldots+d_{ii+1}(s)F^{(i+1)}(s), \ldots\ldots\ldots(4)$$

where $d_{i1}(s) = a_{i1}'(s)$

$d_{im}(s) = a_{im-1}(s)+a_{im}'(s)$, $m = 2,3,\ldots i$ and

$d_{ii+1}(s) = a_{ii}(s)$.

obviously $d_{i1}, d_{i2},\ldots, d_{ii+1}$ are of class C^{k-i-1} .

$$E_i(s) = F^{(i)}(s)-\left[F^{(i)}(s)\cdot V_1(s)\right]V_1(s)-\ldots-\left[F^{(i)}(s)\cdot V_{i-1}(s)\right]V_{i-1}(s).$$

Thus

$$F^{(i)}(s) = \left[F^{(i)}(s)\cdot V_1(s)\right]V_1(s)+\ldots+\left[F^{(i)}(s)\cdot V_{i-1}(s)\right]V_{i-1}(s)+E_i(s)$$
$$= \left[F^{(i)}(s).V_1(s)\right]V_1(s)+\ldots+\left[F^{(i)}(s).V_{i-1}(s)\right]V_{i-1}(s)+\left|E_i(s)\right|V_i(s), \ldots\ldots(5)$$

for $i = 2,3,\ldots,r$ and $F'(s) = V_1(s)$.

Replacing $F^{(i)}(s)$ by Equ.(5) in Equ.(4), we have

$$V_i'(s) = d_{i1}(s)V_1(s)+d_{i2}(s)(\left[F^{(2)}(s).V_1(s)\right]V_1(s)+\left|E_2(s)\right|V_2(s))+\ldots$$
$$+d_{ii+1}(s)(\left[F^{(i+1)}(s).V_1(s)\right]V_1(s)+\ldots+$$
$$+\left[F^{(i+1)}(s).V_i(s)\right]V_i(s) \quad \ldots+\left|E_{i+1}(s)\right|V_{i+1}(s))$$
$$= c_{i1}(s)V_1(s)+c_{i2}(s)V_2(s)+\ldots+c_{ii+1}(s)V_{i+1}(s),$$

where

$$c_{ij}(s) = \sum_{m=j+1}^{i+1} d_{im}(s)\left[F^{(m)}(s).V_j(s)\right] + d_{ij}(s)\left|E_j(s)\right| ,$$

$j = 1,2,\ldots, i$ and $c_{ii+1}(s) = d_{ii+1}(s)\left|E_{i+1}(s)\right|$.

Therefore $V_i'(s)$ is a linear combination of $V_1(s),V_2(s),\ldots,V_{i+1}(s)$. Since d_{ij}, $F^{(j)}$, V_j, and $\left|E_j\right|$ are at least of class C^{k-i-1}, $j = 1,2,\ldots, i+1$, then by virtue of Theorem 2.2.15, one easily sees that the c_{ij} are of class C^{k-i-1}, $j = 1,2,\ldots, i+1$. The theorem is proved.

4.1.5 <u>Theorem</u>. For $i = 1,2,\ldots, r-1$, $j = 1,2,\ldots, r$,

$$V_i'(s) \cdot V_j(s) \equiv 0,$$

except possibly for $j = i-1$ and $j = i+1$.

<u>Proof</u>. Let i,j be two numbers such that $i \in \{1,2,\ldots, r-1\}$, $j \in \{1,2,\ldots, r\}$, and assume that $j \neq i-1$ and $j \neq i+1$.

By assumption, $j \neq i-1$ and $j \neq i+1$ thus we have three cases, i.e., either $j < i-1$ or $j = i$ or $j > i+1$. We shall prove the theorem for each case separately.

Case 1. If $j < i-1$, then $i \neq j$. Thus

$$V_i(s) \cdot V_j(s) \equiv 0,$$

which gives

$$V_i'(s) \cdot V_j(s) = - V_j'(s) \cdot V_i(s) . \qquad \ldots\ldots\ldots\ldots(6)$$

By Theorem 4.1.4,

$$V_j'(s) = c_{j1}(s)V_1(s)+c_{j2}(s)V_2(s)+\ldots+c_{jj+1}(s)V_{j+1}(s). \qquad \ldots\ldots(7)$$

Replacing $V_j'(s)$ by $c_{j1}(s)V_1(s)+\ldots c_{jj+1}(s)V_{j+1}(s)$ in Eq .(6), we obtain

$$V_i(s) \cdot V_j'(s) = -(c_{j1}(s)V_1(s)+c_{j2}(s)V_2(s)+\ldots+c_{jj+1}(s)V_{j+1}(s)) \cdot V_i(s).$$

Since $j < i-1$ the right hand side of the last equation is equal to zero, hence

$$V_i'(s) \cdot V_j(s) \equiv 0.$$

Case 2. If $j = i$.

Since $V_i(s) \cdot V_i(s) \equiv 1$, therefore

$$2V_i'(s) \cdot V_i(s) \equiv 0 .$$

Thus

$$V_i'(s) \cdot V_i(s) \equiv 0 .$$

<u>Case 3</u>. If $j > i+1$, then from Equ.(7) we get

$$V_i'(s) \cdot V_j(s) = (c_{i1}(s)V_1(s)+c_{i2}(s)V_2(s)+\ldots+c_{ii+1}(s)V_{i+1}(s)) \cdot V_j(s).$$

But $j > i+1$ then the right hand side of the last equation is equal to zero, therefore

$$V_i'(s) \cdot V_j(s) \equiv 0 .$$

Combining all three cases proves the theorem.

4.1.6 <u>Theorem</u>. The derivative formula of V_i can be written as follows :

$$V_1'(s) = k_1(s)V_2(s)$$

$$V_i'(s) = -k_{i-1}(s)V_{i-1}(s)+k_i(s)V_{i+1}(s) \quad (i = 2,3,\ldots,r-1),$$

where the k_i are real-valued and of class C^{k-i-1} inside the given interval I.

<u>Proof</u>. First, we shall find the derivative formula of V_1. By Theorem 4.1.4, the derivative formula of V_1 is

$$V_1'(s) = c_{11}(s)V_1(s)+c_{12}(s)V_2(s) .$$

But by Theorem 4.1.5,

$$V_1'(s) \cdot V_1(s) \equiv 0 = c_{11}(s) .$$

Hence $$V_1'(s) = c_{12}(s)\, V_2(s) .$$

By letting $k_1(s) = c_{12}(s)$, therefore

$$V_1'(s) = k_1(s)\, V_2(s).$$

Now we shall find the derivative formula of V_i, $i = 2,3,\ldots,r-1$.

For $i = 2$, by Theorem 4.1.4,

$$V_2'(s) = c_{21}(s)V_1(s)+c_{22}(s)V_2(s)+c_{23}(s)V_3(s).$$

But
$$\begin{aligned} c_{21}(s) &= V_2'(s) \cdot V_1(s) \\ &= -V_1'(s) \cdot V_2(s) \\ &= -(k_1(s)\,V_2(s)) \cdot V_2(s) \\ &= -k_1(s) \end{aligned}$$

and $c_{22}(s) = V_2'(s) \cdot V_2(s) \equiv 0$; therefore

$$V_2'(s) = -k_1(s)\,V_1(s) + c_{23}(s)V_3(s).$$

By letting $k_2(s) = c_{23}(s)$, we get

$$V_2'(s) = -k_1(s)V_1(s) + k_2(s)V_3(s).$$

Now assume that $V_j'(s) = -k_{j-1}(s)V_{j-1}(s)+k_j(s)V_{j+1}(s)$,

for all $j < i \le r-1$. By Theorem 4.1.4, we obtain

$$V_i'(s) = c_{i1}(s)V_1(s)+c_{i2}(s)V_2(s)+\ldots+c_{ii+1}(s)V_{i+1}(s) .$$

From the induction hypothesis and Theorem 4.1.5,

$$\begin{aligned} c_{im}(s) &= V_i'(s) \cdot V_m(s) \\ &\equiv 0 \qquad (m = 1,2,\ldots, i-2), \\ c_{ii-1}(s) &= V_i'(s) \cdot V_{i-1}(s) \\ &= -V_{i-1}'(s) \cdot V_i(s) \end{aligned}$$

$$= -(-k_{i-2}(s)V_{i-2}(s)+k_{i-1}(s)V_i(s)) \cdot V_i(s)$$

$$= -\ k_{i-1}(s),$$

and $c_{ii}(s) = V_i'(s) \cdot V_i(s) \equiv 0$; hence

$$V_i'(s) = -\ k_{i-1}(s)V_{i-1}(s)+c_{ii+1}(s)\ V_{i+1}(s)\ .$$

Let $k_i(s) = c_{ii+1}(s)$, we get

$$V_i'(s) = -\ k_{i-1}(s)\ V_{i-1}(s)+\ k_i(s)V_{i+1}(s)\ .$$

Moreover, by Theorem 4.1.4, the function k_i are of class C^{k-i-1} . Thus by induction on i, the theorem is proved.

4.1.7 Remark. There is a bit of a problem with $V_r'(s)$, since there may be no $V_{r+1}(s)$. Given $s_0 \in I$, if $F^{(r+1)}(s_0)$ is linearly independent with respect to $F'(s_0), F''(s_0) \ldots,\ F^{(r)}(s_0)$, then by the continuity of $F^{(r+1)}(s)$ this will also be true in some neighborhood of s_0 in I. (See Appendix C). For s in such a neighborhood, $V_{r+1}(s)$ can be defined as above and we will have

$$V_r'(s) = -\ k_{r-1}(s)V_{r-1}(s)+\ k_r(s)V_{r+1}(s)\ .$$

If $F^{(r+1)}(s_0)$ happens to be linearly dependent upon $F'(s_0),\ F''(s_0), \ldots,\ F^{(r)}(s_0)$, then

$$V_r'(s_0) = \alpha_1 V_1(s_0)+\alpha_2 V_2(s_0)+\ldots+\ \alpha_r V_r(s_0)$$

where α_i are real constants, $i = 1,2,\ldots, r$.

Obviously,

$$\alpha_i = V_r'(s_0) \cdot V_i(s_0)\ ,\quad i = 1,2,\ldots, r\ .$$

By Theorem 4.1.5, we thus have

$$\alpha_i = 0 , \quad i = 1,2,\ldots, r-2 .$$

If $i \geq r-2$,

$$\begin{aligned}\alpha_{r-1} &= V_r'(s_0)\cdot V_{r-1}(s_0)\\ &= -V_{r-1}'(s_0)\cdot V_r(s_0)\\ &= -(-k_{r-2}(s_0)V_{r-2}(s_0)+k_{r-1}(s_0)V_r(s_0)).V_r(s_0)\\ &= -k_{r-1}(s_0),\end{aligned}$$

and

$$\begin{aligned}\alpha_r &= V_r'(s_0)\cdot V_r(s_0)\\ &= 0.\end{aligned}$$

Hence $$V_r'(s_0) = -k_{r-1}(s_0)V_{r-1}(s_0) .$$

4.1.8 Definition. The coefficient appearing above, $k_1(s)$, $k_2(s)$, ..., $k_{r-1}(s)$, are called the curvatures associated with the given curve at the point $F(s)$. The r th curvature $k_r(s)$ may be defined similarly when $F^{(r+1)}(s)$ is independent of $F'(s)$, $F''(s),\ldots, F^{(r)}(s)$, and to be zero in the dependent case.

It turns out that $k_i(s) > 0$ for $i = 1,2,\ldots, r-1$, and $k_r(s) \geq 0$. By virtue of Theorem 4.1.6, the functions k_i are of class C^{k-i-1}, $i = 1,2,\ldots, r-1$ and k_r is of class C^{k-r-1} when it does not vanish, but over all can only be guaranteed to be continuous.

2. The algorithm for parametrization by arc length.

The algorithm for computing the curvatures derives from the following :

4.2.1 Theorem. $k_i(s) = \dfrac{|E_{i+1}(s)|}{|E_i(s)|}$ for $i = 1,2,\ldots, r$.

Proof. First assume $i < r$. Then by Theorem 4.1.6,

$$\begin{aligned} k_i(s) &= V_i'(s)\cdot V_{i+1}(s) \\ &= \left(\frac{E_i(s)}{|E_i(s)|}\right)'\cdot V_{i+1}(s) \\ &= \left(\frac{|E_i(s)|E_i'(s) - E_i(s)(|E_i(s)|)'}{|E_i(s)|^2}\right)\cdot V_{i+1} \\ &= \frac{E_i'(s)\cdot V_{i+1}(s)}{|E_i(s)|} - \left(\frac{(|E_i(s)|)'}{|E_i(s)|^2}\right)E_i(s)\cdot V_{i+1}(s) \\ &= \frac{E_i'(s).V_{i+1}(s)}{|E_i(s)|} + \left(\frac{1}{|E_i(s)|}\right)' E_i(s).V_{i+1}(s) . \end{aligned}$$

Now $E_i(s)$ and $V_{i+1}(s)$ are orthogonal, so the second term on the right above is zero. Hence

$$k_i(s) = \frac{E_i'(s)\cdot V_{i+1}(s)}{|E_i(s)|} .$$

To verify the theorem, we must show that $E_i'(s)\cdot V_{i+1}(s) = |E_{i+1}(s)|$.

Differentiating the equation $E_i(s) = F^{(i)}(s) - \sum_{j<i}\left[F^{(i)}(s).V_j(s)\right]V_j(s)$

yields

$$E_i'(s) = F^{(i+1)}(s) - \sum_{j<i}\left[F^{(i)}(s).V_j(s)\right]' V_j(s) - \sum_{j<i}\left[F^{(i)}(s).V_j(s)\right]V_j'(s).$$

Every vector on the right hand side of this last equation, except for $F^{(i+1)}(s)$, is a linear combination of $V_1(s), V_2(s), \ldots, V_i(s)$, and these are all orthogonal to $V_{i+1}(s)$. Therefore

$$E_i'(s) \cdot V_{i+1}(s) = F^{(i+1)}(s) \cdot V_{i+1}(s) .$$

But $E_{i+1}(s) = F^{(i+1)}(s) - \sum_{j < i+1} \left[F^{(i+1)}(s) \cdot V_j(s)\right] V_j(s)$, so

$$F^{(i+1)}(s) \cdot V_{i+1}(s) = E_{i+1}(s) \cdot V_{i+1}(s) = |E_{i+1}(s)| ,$$

completing the proof in the case $i < r$.

If $i = r$ and $E_{i+1}(s) \neq \theta$ (θ = zero vector), the same proof works. If $E_{i+1}(s) = \theta$, so is $k_i(s) = 0$, and the theorem is proved.

4.2.2 Example. Let a and b be two real number such that $a > 0$, $b \neq 0$, and $a^2 + b^2 = 1$. Consider the curve given by $F : [0, 2\pi] \longrightarrow R^3$, where

$$F(s) = (a \cos s, a \sin s, bs).$$

This curve is called a circular helix, since $|F'(s)| \equiv 1$ then by Theorem 2.3.5, the curve F is parametrization by arc length. The determinant of the three vector $F'(s)$, $F''(s)$, $F'''(s)$ is never zero for all $s \in [0, 2\pi]$, since

$$(F'(s), F''(s), F'''(s)) = \begin{vmatrix} -a \sin s & a \cos s & b \\ -a \cos s & -a \sin s & 0 \\ a \sin s & -a \cos s & 0 \end{vmatrix} = a^2 b \neq 0.$$

Hence for each $s \in [0, 2\pi]$ the vectors $F'(s)$, $F''(s)$, $F'''(s)$ are linearly independent.

Thus the various curvatures may be calculated according to the algorithm given by Theorem 4.2.1, as follow :

$$F'(s) = (-\ a \sin s,\ a \cos s,\ b)$$

$$F''(s) = (-\ a \cos s,\ -\ a \sin s,\ 0)$$

$$F'''(s) = (a \sin s,\ -\ a \cos s,\ 0) \quad .$$

Beginning the Gram-Schmidt process, we get

$$E_1(s) = F'(s) = (-\ a \sin s,\ a \cos s,\ b)$$

$$|E_1(s)| = 1$$

$$V_1(s) = \frac{E_1(s)}{|E_1(s)|} = (-\ a \sin s,\ a \cos s,\ b).$$

Next we have

$$E_2(s) = F''(s)-[F''(s)\cdot V_1(s)]\,V_1(s) = F''(s)=(-a \cos s,-a \sin s,\ 0)$$

$$|E_2(s)| = a$$

$$V_2(s) = (-\cos s,\ -\sin s,\ 0).$$

Finally we have

$$E_3(s) = F'''(s)-[(F'''(s)\cdot V_1(s)]\,V_1(s)-[F'''(s)\cdot V_2(s)]\,V_2(s)$$

$$= ((a-a^3)\sin s,\ (a^3-\ a) \cos s,\ -ba^2)$$

$$|E_3(s)| = \sqrt{(a-a^3)^2+\ b^2a^4} \quad .$$

The curvature are then given by :

$$k_1(s) = \frac{|E_2(s)|}{|E_1(s)|} = a$$

$$k_2(s) = \frac{|E_3(s)|}{|E_2(s)|} = b \quad .$$

Classically this is the curvature and torsion of curves in R^3.