Chapter III

MODELING

3.1 Introduction

The simple equations of the motion of a fin-stabilized short-range rocket have been derived from Newton's law described in the previous chapter. For the further implementation, it is seen that the equations of the motion previously described can be divided into two parts.

(a) The motion along the path of the rocket.

(b) The motion around the path of the rocket.

In this section, three kinds of actual actions are encountered in the equations of the motions. These are pitch, yaw and roll actions.

(a) Pitch

Pitching is the angular motion about center of mass in the vertical plane.

(b) Yaw

Yawing is the angular motion about center of mass in the lateral or horizontal plane if the rocket is travelling horizontally.

(c) Roll

Rolling is the angular motion about the longitudinal axis.

For a fin-stabilized rocket, the motions in the vertical and the horizontal plane are symmetrical due to the shape of fins fixed at the rear of the rocket. According to this symmetrical axis, pitching and yawing can be represented by the same equation. Thus the motion can be divided into two main portions. These motions consist of the trajectory, pitching, yawing and rolling which are described in the next sections.

3.2 Motion During Burning

These are the motions from launching until the powder is completely burnt. The motion comprises only three kinds of actions.

3.2.1 Trajectory

In the case of a normal firing, a good approximation of the actual trajectory during burning can be obtained by the assumption that the stabilizing action of the fins is perfect so that the axis of the rocket always parallels to its direction of the motion. Hence the rocket axis is always tangent to the trajectory and the yaw angle ϕ is always zero.

From eqns. (2.2) and (2.3), the appropriate equations for this case can be written as :

$$\ddot{X} = \frac{T}{m} - \frac{D}{m} - g \operatorname{Sin} \theta \qquad (3.1)$$

or

$$\ddot{X} = \frac{T}{m} - \frac{C_D \rho v^2 d^2}{m} - g \operatorname{Sin} \theta \qquad (3.2)$$

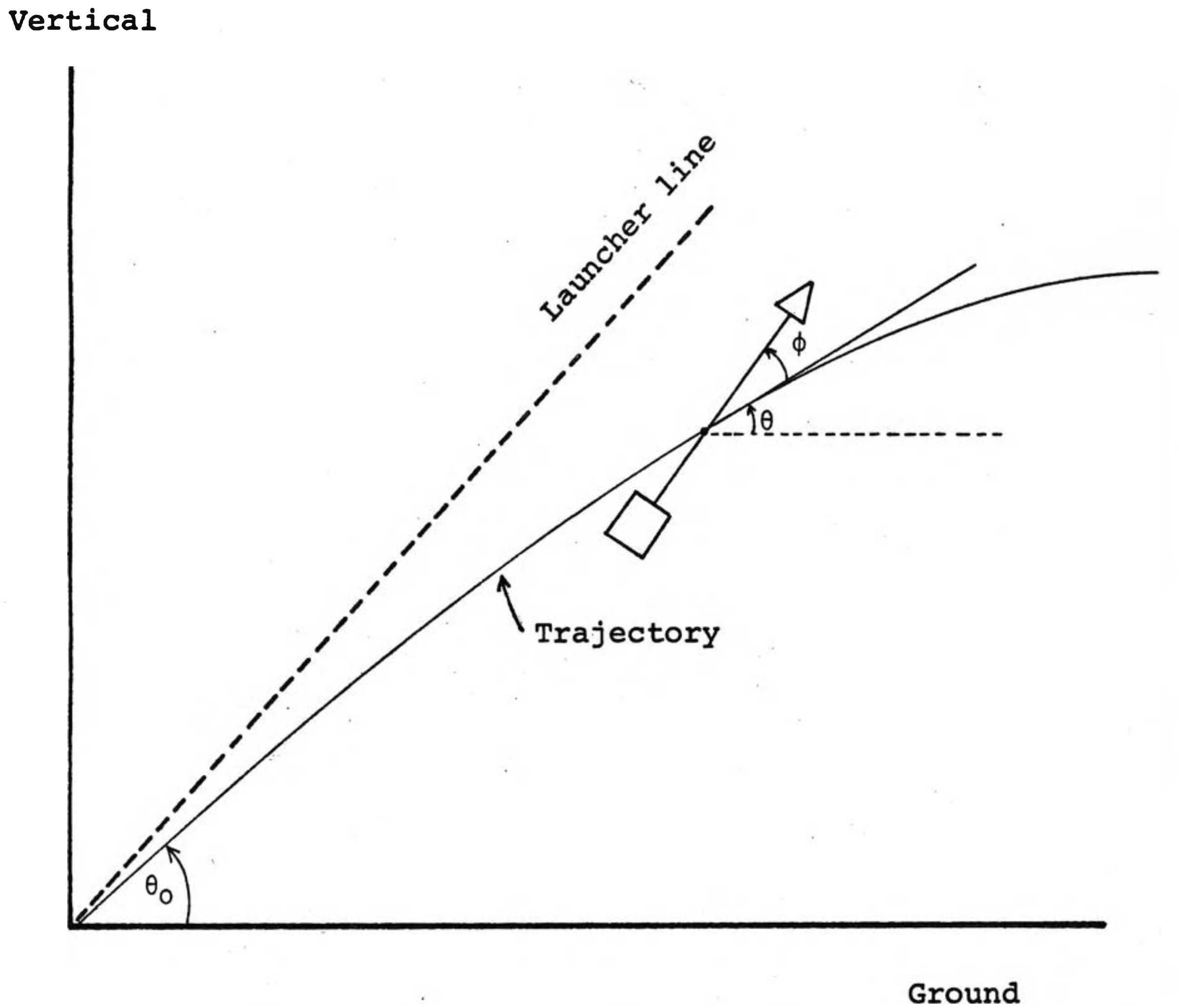

Figure 3.1

A Diagram of Trajectory During Burning.

in which

$$v = \dot{X} \qquad (3.3)$$

and $$v.\frac{v}{r} = g \text{ Cos } \theta$$

$$v.\dot{\theta} = g \text{ Cos } \theta \qquad (3.4)$$

Equns. (3.2) and (3.4) represent the motions of the center of the mass for a fin-stabilized rocket. The trajectory of the rocket occurs only in the vertical plane. For a short-range rocket the firing range is very much greater than the height of the trajectory.

3.2.2 Pitching and Yawing

In this case, pitching and yawing may be considered together. When the pitch or yaw behavior of a varying mass is encountered and the variation of the mass may be constrained by the approximate equation.

$$m = m_o (1 - ct) \qquad (3.5)$$

where m_o is the initial mass

c is the rate of the mass flow

From eqns. (3.2) and (3.5), we obtain

$$m_o (1 - ct). \frac{dv}{dt} = T - \frac{1}{2} C_D \rho d^2 v^2 \qquad (3.6)$$

which is a Riccati equation and the velocity v can be solved as

$$v(t) = \sqrt{\frac{T}{\frac{1}{2} C_D \rho d^2}} \cdot \frac{\{C_o + (1 - ct)^B\}}{\{C_o - (1 - ct)^B\}} \qquad (3.7)$$

where

$$B = \frac{d}{cm_o} \sqrt{2C_D \rho T}$$

$$C_o = \frac{V_o + \frac{1}{d}\sqrt{\frac{2T}{C_D\rho}}}{V_o - \frac{1}{d}\sqrt{\frac{2T}{C_D\rho}}}$$

The equation of a varying mass rocket related to the yaw angle has been derived by W.F. Byrne and S. Raynor[4] as :

$$m_o k^2 (1 - ct)\ddot{\phi} + (r^2 - k^2) cm_o . \dot{\phi} + \tfrac{1}{2} C_M \rho \ell . v^2 d^2 . \phi = 0 \quad (3.8)$$

where ℓ = the rocket length

C_M = aerodynamic restoring moment coefficient

r = the distance from the center of the mass to the nozzle axis.

From eqn. (3.8) we may write

$$\ddot{\phi} + \frac{(r^2 - k^2).cm_o}{m_o k^2 (1 - ct)} . \dot{\phi} + \frac{C_M \rho \ell v^2 d^2}{2m_o k^2 (1-ct)} . \phi = 0 \quad (3.9)$$

Substituting v(t) from eqn. (3.7) into eqn. (3.9), we obtain

$$\ddot{\phi} + \frac{c(r^2 - k^2)}{k^2 (1 - ct)} . \dot{\phi} + \frac{C_M \ell T}{C_D m_o k^2 (1-ct)} \left[\frac{C_o + (1-ct)^B}{C_o - (1-ct)^B} \right] . \phi = 0 \quad (3.10)$$

When a small rocket is concerned, the ratio of the final mass to the initial mass is greater than 0.8. Thus the mass ratio is :

$$\frac{M \text{ final}}{M \text{ initial}} \times 100 > 80\%$$

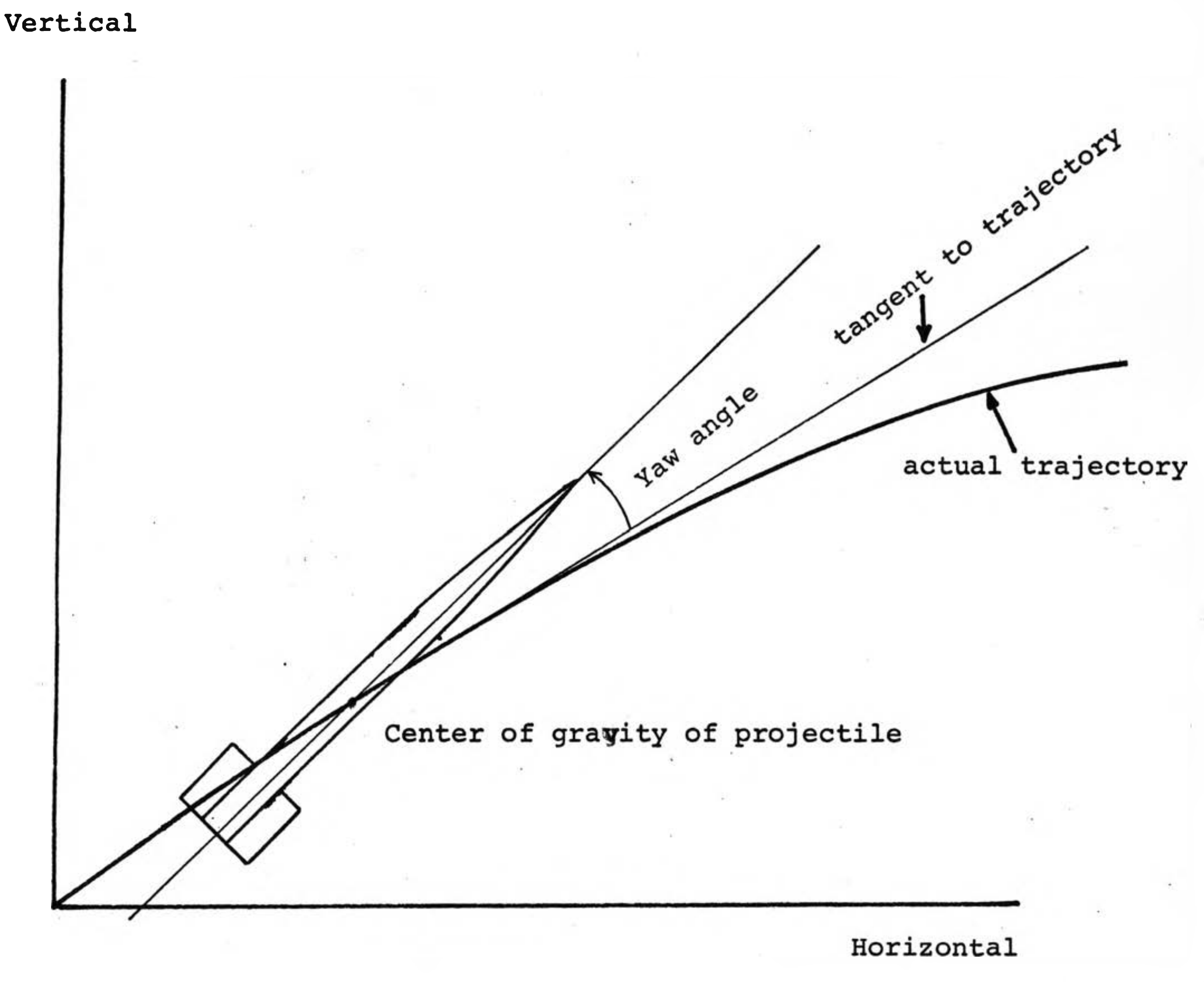

Figure 3.2

A Diagram Showing the Angle of Yaw with Reference to

Projectile Path

For a short-range rocket system, the initial mass is about 2 kg. and the burn out mass remains 0.2 kg. Hence the mass ratio is 90%. The term $(\frac{1}{1 - ct})$ involved in eqn. (3.11) can be expressed as

$$\frac{1}{1 - ct} = 1 + ct + c^2t^2 + c^3t^3 + \ldots\ldots \qquad (3.11)$$

For the mass ratio is greater than 80%, the above equation may be approximated as[4]

$$\frac{1}{1 - ct} \simeq 1 + ct + c^2t^2 \simeq (1 + ct)^2 \qquad (3.12)$$

Since the rate of mass flow is equal to the inverse of the burning time Tb. Hence, eqn. (3.12) becomes

$$\frac{1}{1 - ct} = (1 + \frac{t}{T_b})^2 \qquad (3.13)$$

Assume that the velocity during very short burning time is constant, that is

$$v(t) = V_o \qquad (3.14)$$

From eqns. (3.10), (3.13) and (3.14), the eqns. (3.10) becomes

$$\ddot{\phi} + 2\,K.\,\dot{\phi} + K_1^2V_o^2 \;(1 + \frac{t}{T_b})^2 \;.\phi = 0 \qquad (3.15)$$

where $K = \frac{c(r^2 - k^2)}{2k^2} \;(1 + \frac{t}{T_b})^2$ (3.16)

$$K_1 = \left[\frac{C_M \rho \ell d^2}{2\, m_o k^2}\right]^{\frac{1}{2}} \qquad (3.17)$$

Let $\tau = 1 + \frac{t}{T_b}$ = non-dimension time

$$d\tau = \frac{dt}{T_b}$$

$$\phi' = \frac{d\phi}{d\tau}$$

$$\phi'' = \frac{d^2\phi}{d\tau^2}$$

We have $\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{T_b d\tau} = \frac{1}{T_b}\phi$ (3.18)

$$\ddot{\phi} = \frac{1}{T_b^2} \frac{d^2\phi}{d\tau^2} = \frac{1}{T_b^2} \phi'' \quad (3.19)$$

Therefore, the eqn. (3.15) becomes

$$\frac{1}{T_b^2} \frac{d^2\phi}{d\tau^2} + \frac{2K}{T_b} \frac{d\phi}{d\tau} + K_1^2 V_o^2 \tau^2 \phi = 0$$

Simplifying, we obtain

$$\phi'' + 2KT_b \phi' + K_1^2 V_o^2 T_b^2 \tau^2 \phi = 0 \quad (3.20)$$

Let $\phi(\tau) = e^{-KT_b\tau} \theta(\tau)$

$$\phi'(\tau) = e^{-KT_b\tau} \theta'(\tau) - KT_b e^{-KT_b\tau} \theta(\tau)$$

$$\phi''(\tau) = e^{-KT_b\tau} K^2 T_b^2 \theta(\tau) - 2e^{-KT_b\tau} KT_b\theta'(\tau) + e^{-KT_b\tau}\theta''(\tau)$$

Substituting ϕ'', ϕ' and ϕ into eqn. (3.20), we have finally

$$\theta''(\tau) + K_1^2 V_o^2 T_b^2 \left(\tau^2 - \frac{K^2}{K_1^2 V_o^2}\right) \theta(\tau) = 0 \quad (3.21)$$

For the small size of the rocket is concerned and the mass ratio > 80%, the value of $(\frac{K}{K_1})^2$ is approximated to three[4]. However, the initial velocity V_o is very large comparing to $(\frac{K}{K_1})^2$, the term $(\frac{K}{K_1})^2 \frac{1}{V_o^2}$ may be neglected. Thus the eqn.

(3.21) becomes

$$\theta''(\tau) + K_1 V_o^2 T_b^2 \tau^2 \theta(\tau) = 0 \qquad (3.22)$$

The solution of the above equation can be obtained in the terms of Bessel Functions as

$$\theta(\tau) = A\tau^{\frac{1}{2}} J_{\frac{1}{4}}(\tfrac{1}{2}K_1 V_o T_b \tau^2) + B\tau^{\frac{1}{2}} Y_{\frac{1}{4}}(\tfrac{1}{2}K_1 V_o T_b \tau^2) \qquad (3.23)$$

where A and B are constant and determined from the initial conditions.

It can be seen that the factor $\frac{1}{2}K_1 V_o T_b{}^2$ which is the argument of the Bessel Functions is large.

Finally, the yaw angle ϕ may be expressed as

$$\phi(t) = A_1 \tau^{-\frac{1}{2}} e^{-KT_b \tau} \cos(\tfrac{1}{2}K_1 V_o T_b \tau^2 - B_1) \qquad (3.24)$$

where the constants A_1 and B_1 are determined from the initial conditions.

3.2.3 Rolling

For a symmetrical rocket in a perfect rolling flight, the equation of the rotation of a rigid body for the angle of roll, ψ is given by

$$I.\ddot{\psi} = -\zeta \qquad (3.25)$$

where I = the axial moment of inertia (Slug - ft^2)

ζ = the roll damping moment

The roll damping moment has been derived by Ray E. Bolz and John D. Nicolaides[3] and expressed as

$$\zeta = K_R \mathring{\psi} V \qquad (3.26)$$

where ψ is the angle of roll

K_R is the roll damping coefficient $= \frac{4\pi^2 I}{d^3 v^2 \sigma}$

Thus, eqn. (3.26) becomes

$$I\ddot{\psi} + K_R \dot{\psi} V = 0 \tag{3.27}$$

For a short-range rocket during burning time, we have

$$I\ddot{\psi} + (K_R . at)\, \dot{\psi} = 0$$

or

$$\ddot{\psi} + (K_R \frac{a}{I})\, t\, \dot{\psi} = 0 \tag{3.28}$$

let z be the horizontal distance and $V \approx \frac{dz}{dt}$, eqn. (3.28) becomes

$$\psi'' + \psi' \left(\frac{V'}{V} + \frac{K_R}{I}\right) = 0 \tag{3.29}$$

where the prime denotes differentiation with respect to z.

Now consider the forces in horizontal plane, we may write

$$m\dot{V} = - \{\frac{\rho}{2} d^2 C_D\} V^2 = - C_R V^2 \tag{3.30}$$

where C_R is a constant.

It can be found that

$$\frac{V'}{V} = - \frac{C_R}{m} \tag{3.31}$$

Substituting eqn. (3.31) into eqn. (3.29), we obtain

$$\psi'' + \psi' \left[\frac{K_R}{I} - \frac{C_R}{m}\right] = 0 \tag{3.32}$$

It can be seen that this equation is stable for $\frac{K_R}{I} > \frac{C_R}{m}$.

Therefore the solution of above equation can be obtained as

$$\psi'(z) = C_2 e^{-C_1 z} \tag{3.33}$$

and

$$\psi(z) = -\frac{C_2 e^{-C_1 z}}{C_1} + C_3 \tag{3.34}$$

It is known that the angle of roll is zero when $t = 0$. Hence $C_3 = \frac{C_2}{C_1}$. Finally, we have

$$\psi(z) = \frac{C_2}{C_1}(1 - e^{-C_1 z}) \tag{3.35}$$

and

$$\dot{\psi}(t) = aC_2 t e^{-C_1 a\frac{t^2}{2}}$$

Therefore

$$\psi(t) = \frac{C_2}{C_1}(1 - e^{-C_1 a\frac{t^2}{2}}) \tag{3.36}$$

where

$$C_1 = \frac{K_R}{I} - \frac{C_R}{m}$$

$$C_2 = \text{constant}$$

For simplicity, we may write

$$\psi(t) = \psi_o (1 - e^{-kt^2}) \tag{3.37}$$

3.3 Motion After Burning

As mentioned in Chapter 1, the motions of the rocket after the propellant completely burnt are similar to those of an artillery projectile. However, the analysis of these motions will be discussed in more details in the following sections.

3.3.1 Trajectory After Burn Out

The forces acting on the rocket can be divided into the tangential and normal components as follows.

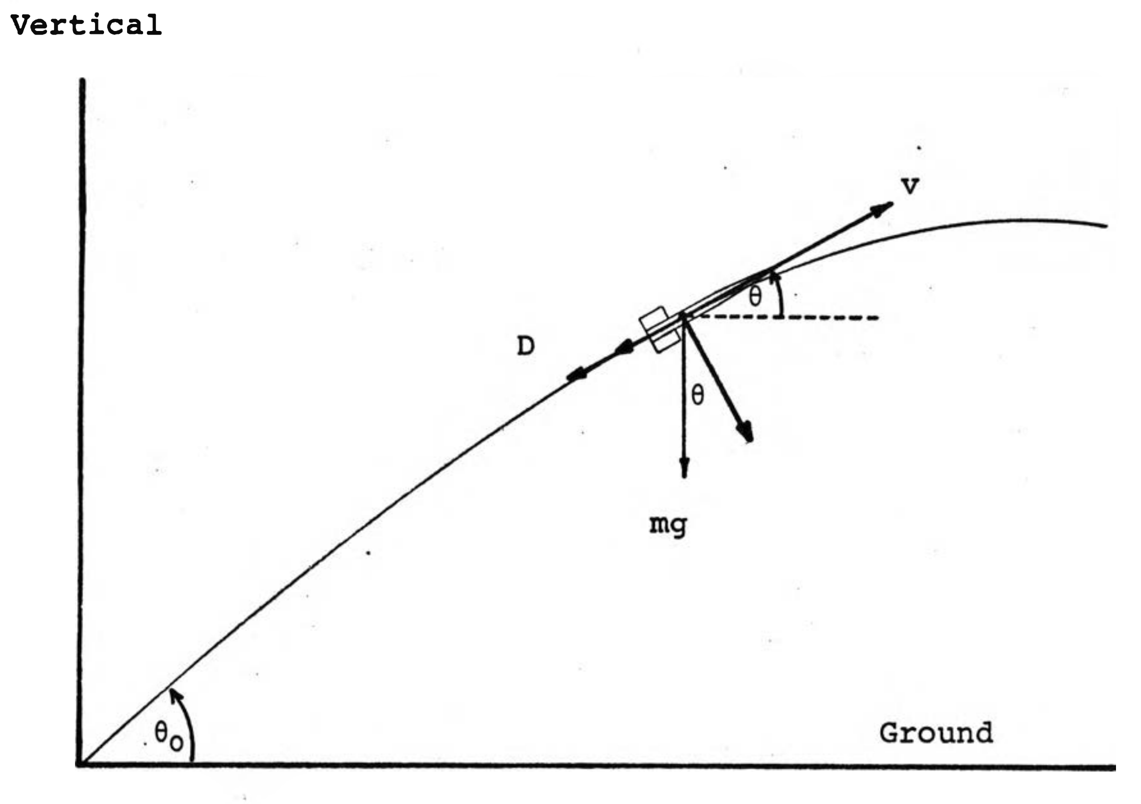

Figure 3.3

A Diagram of Forces on the Rocket after Burn Out.

(a) Tangential Component

The trajectory along the tangential direction is

$$\frac{dV}{dt} = -\frac{C_R}{m} V^2 - g \sin\theta \qquad (3.38)$$

where the drag $\frac{C_R}{m} V^2$ is defined in the previous section.

(b) Normal Component

The centrifugal force as shown in Fig. (3.3) is

$$m\frac{V^2}{r} = -mg \cos\theta \qquad (3.39)$$

From the radius of curvature, we have

$$\frac{1}{r} = \frac{-d^2y/dx^2}{1 + (\frac{dy}{dx})^2 \; \frac{3}{2}}$$

$$\frac{1}{r} = -(d^2y/dx^2) \cos^3\theta$$

we obtain

$$\frac{d^2y}{dx^2} = \frac{-1}{r \cos^3\theta} \qquad (3.40)$$

Substituting eqn. (3.39) into the above equation, we obtain

$$\frac{d^2y}{dx^2} = \frac{-g}{V^2 \cos^3\theta} \qquad (3.41)$$

From eqn. (3.39), we may write

$$V\frac{d\theta}{dt} = -g \cos\theta \qquad (3.42)$$

From eqn. (3.38), we have

$$\frac{dV}{dt} = -C f(v) - g \sin\theta$$

where $C = \frac{C_R}{m}$

For a short-range rocket with speed about 60 mph to 1 Mach[17], f(v) is expressed as v^2. The equation in the horizontal line is

$$\frac{dV}{dt} \text{ Cos } \theta = -C \text{ f(v) Cos}\theta$$

Multiplying both sides of this equation by dθ , we obtain

$$d\ (V\text{Cos}\theta)\ \frac{d\theta}{dt} = -C\ f(v)\ \text{Cos}\theta d\theta$$

Replacing $\frac{d\theta}{dt}$ by the value given in eqn. (3.42), we obtain

$$g\ d(v\ \text{Cos}\theta) = VC\ f(v)\ d\theta$$

$$= CV^3\ d\theta$$

Since the velocity in the horizontal line is

$$V_x = \frac{dx}{dt} = V\ \text{Cos}\theta = \dot{X}$$

Therefore

$$g\ d(\dot{X}) = CV^3\ d\theta$$

$$d\dot{X} = \frac{CV^3 d\theta}{g\dot{X}}$$

$$= \frac{CV^3 d\ \theta}{gV\ \text{Cos}\theta}$$

$$= \frac{CV^2\ d\ \theta}{g\ \text{Cos}\theta}$$

$$\frac{d\dot{X}}{\dot{X}} = -C\ d\ s$$

which gives

$$\dot{X} = \dot{X}_o e^{-Cs} \qquad (3.43)$$

Substituting eqn. (3.43) into eqn. (3.41), we have

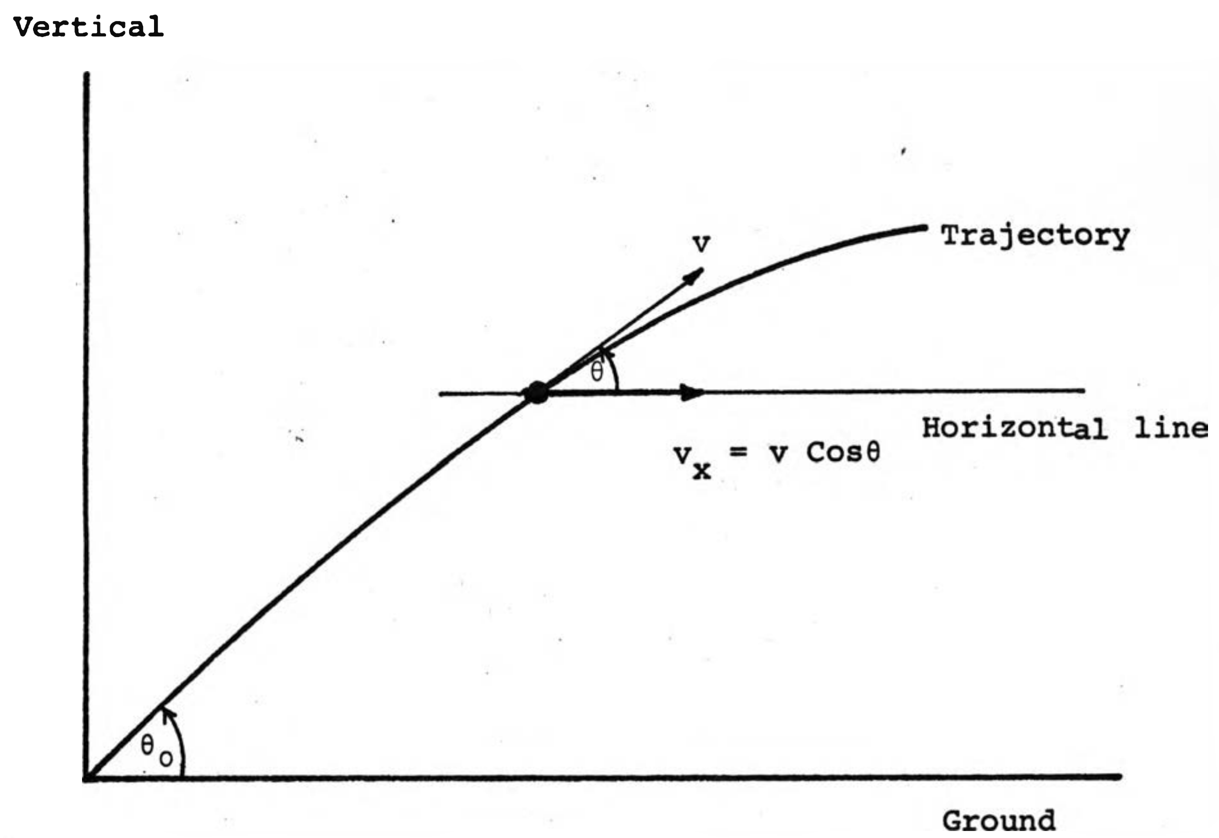

Figure 3.4

Horizontal Component.

$$\frac{d^2y}{dx^2} = \frac{-g}{(\dot{X}_o e^{-Cs})^2}$$

For a flat trajectory, the trajectory path s is about the distance x or s $\simeq$ x, and for high elevation s $\simeq \frac{x}{\cos\theta}$

Thus $$\frac{d^2y}{dx^2} = \frac{-g}{(\dot{X}_o)^2} e^{2Cx}$$

Integrating, we have

$$\frac{dy}{dx} = \frac{-ge^{2Cx}}{2C(\dot{X}_o)^2} + C_1 \qquad (3.44)$$

$$y = \frac{-ge^{2Cx}}{2C^2(\dot{X}_o)^2} + C_1x + C_2$$

To find C_1 and C_2, consider Fig. (3.5), when $X_o = 0$

we have $$C_1 = \tan\theta + \frac{g}{2C(\dot{X}_o)^2} \qquad (3.45)$$

$$C_2 = \frac{g}{4C^2(\dot{X}_o)^2} \qquad (3.46)$$

The complete solution is

$$y = x\tan\theta - \frac{ge^{2Cx}}{4C^2(\dot{X}_o)^2} + \frac{gx}{2C(\dot{X}_o)^2} + \frac{g}{4C^2(\dot{X}_o)^2} \qquad (3.47)$$

where θ = the launcher angle

$\dot{X}_o$ = the velocity at initial of after burning (see Chapter 4)

$C = \frac{C_R}{m}$

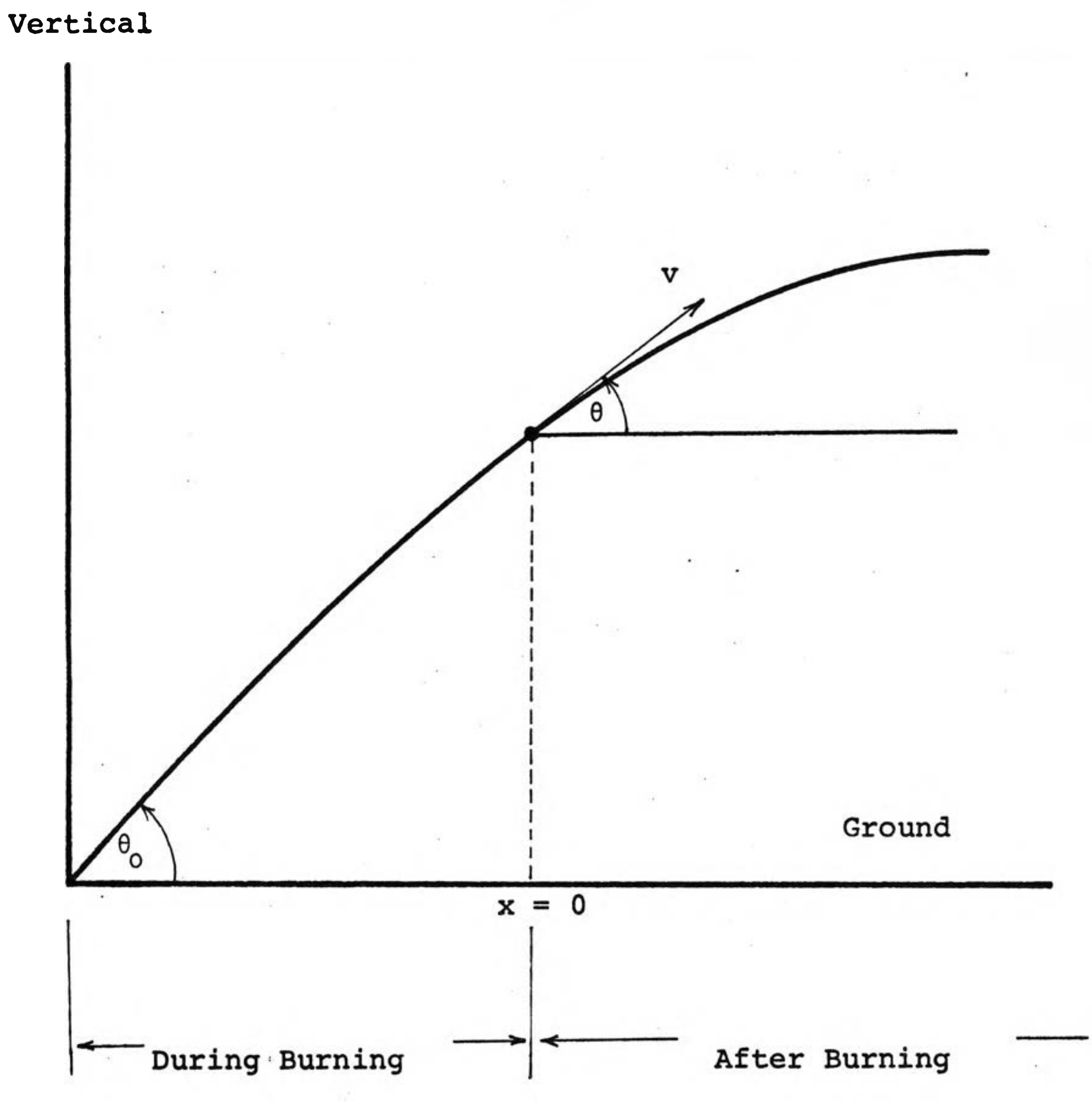

Figure 3.5

Initial Condition Consideration

3.3.2 Pure Roll after Burn Out

Refering to the section (3.2.3), we have the equation of roll in the form

$$I\ddot{\psi} + (K_R at)\dot{\psi} = 0 \qquad (3.48)$$

$$\ddot{\psi} + \frac{K_R V}{I}\dot{\psi} = 0 \qquad (3.49)$$

Assume that the velocity after the powder is completely burnt is decreased as an exponential form.

$$V(t) = V_m e^{-bt} \qquad (3.50)$$

This equation may be approximated as

$$V(t) \cong \frac{V_m}{1 + bt} \qquad (3.51)$$

where V_m is the maximum initial velocity after burning

b is a constant which can be determined from an experiment

Substituting eqn. (3.51) into eqn. (3.49), we have

$$\ddot{\psi} + KV_m e^{-bt}\dot{\psi} = 0 \qquad (3.52)$$

The horizontal distance can be directly evaluated by simple integrating from eqn. (3.50)

Hence $$z = z_o + \frac{V_m}{b}(1 - e^{-bt}) \qquad (3.53)$$

From eqn. (3.52), the yaw motion may be expressed as

$$\ddot{\psi} + \frac{KV_m \dot{\psi}}{1 + bt} = 0$$

let $p = \frac{d\psi}{dt}$, $A = KV_m = \text{constant}$

Then $$\frac{dp}{dt} + \frac{A}{(1 + bt)} \cdot p = 0 \qquad (3.54)$$

The solution of eqn. (3.54) can be simply determined as

$$p = (1 + bt)^{-A/b} \qquad (3.55)$$

Therefore $$\psi = \int p\,dt$$

$$= \int (1 + bt)^{-A/b}\,dt$$

Thus $$\psi(t) = \frac{1}{b}\,(1 + bt)^{1 + (A/b)} \qquad (3.56)$$