CHAPTER III
PROBLEM-SOLVING METHOD

3.1 Introduction

There are two major parts for the problem to be solved: a pipe and pump part, and an open channel part. Solution for the first part can be achieved using the Newton-Raphson iterative technique to linearize the simultaneous nonlinear nodal material balance equations, followed by the Gaussian elimination method to solve the system of linear equations. For the second part, the flow rate and depth of every channel can be determined by means of a material balance and nodal elevation changes.

3.1.1 Node

All nodes in the network, that are formulated from a number of n nodes, are identified by an index such as i or j. The different types of node are specified by a node-type vector T_i, with possible values for a representative element T_i shown in Table 3.1.

Table 3.1 Value for elements of node-type vector (T_i)

T_i	Interpretation
0	Pressure unspecified at node i.
1	Pressure specified (fixed) at node i.
2	Injection or withdrawal rate specified at node i.
3	Terminal node i with specified injection rate

3.1.2 Connection elements

The nodal connections are connected directly to other nodes by pipeline, pump or open channel. A connection matrix is established with possible values for a representative element C_{ij} as in Table 3.2.

Table 3.2 Value for elements of connection matrix

C_{ij}	Interpretation
0	No connection between node i and node j.
1	Node i and node j are joined by a pipeline.
2	Centrifugal pump is connecting node i to node j.
3	Node i and node j are connected by an open channel.

3.2 Part I: Liquid flow in pipes and pumps

3.2.1 Pipe

The flow rate from node j to node i is given by:

$$Q_{ji} = \sqrt{\frac{y}{\alpha_{ji} f_{F_{ji}}}} \quad \text{for} \quad y > 0 \tag{3.1}$$

The flow rate from node i to node j is given by:

$$Q_{ij} = -\sqrt{\frac{-y}{\alpha_{ji} f_{F_{ji}}}} \quad \text{for} \quad y < 0 \tag{3.2}$$

where:

$$\alpha_{ji} = \frac{32\rho L_{ji}}{\pi^2 D_{ji}^5}, \quad \beta = \rho g,$$

and $y = p_j - p_i + \beta(z_j - z_i)$.

3.2.2 Pump

From equation (2.15), there are two separate cases to be considered for each of three possibilities, as follows:

1. $Q_{ji} > 0$, for flow across the pump from node j to node i:

$$Q_{ji} = 0 \qquad \text{for } p_i + \beta z_i > p_j + \beta z_j + a_{ji}, \tag{3.3}$$

$$Q_{ji} = \sqrt{a_{ji} / b_{ji}} \qquad \text{for } p_j + \beta z_j > p_i + \beta z_i, \tag{3.4}$$

$$Q_{ji} = \sqrt{\frac{(p_i - p_j + a_{ji} + \beta(z_j - z_i))}{b_{ji}}} \qquad \text{otherwise.} \tag{3.5}$$

2. $Q_{ij} > 0$, for flow across the pump from node j to node i:

$$Q_{ij} = 0 \qquad \text{for } p_j + \beta z_j > p_i + \beta z_i + a_{ij}, \tag{3.6}$$

$$Q_{ij} = -\sqrt{a_{ij} / b_{ij}} \qquad \text{for } p_i + \beta z_i > p_j + \beta z_j, \tag{3.7}$$

$$Q_{ij} = -\sqrt{\frac{(p_i - p_j + a_{ij} + \beta(z_i - z_j))}{b_{ij}}} \qquad \text{otherwise.} \tag{3.8}$$

where: $\beta = \rho g$.

3.2.3 Partial derivatives

The partial derivatives for the Newton-Raphson method with respect to p_j and p_i are given in Table 3.3.

Table 3.3 Partial derivatives of flow rate for flow in a pipe

Partial derivatives	$Q_{ji} = \sqrt{\dfrac{y}{\alpha_{ji} f_{F_{ji}}}}$ $y > 0$	$Q_{ij} = \sqrt{\dfrac{-y}{\alpha_{ji} f_{F_{ji}}}}$ $y < 0$
$\dfrac{\partial Q}{\partial p_j}$	$\dfrac{\partial Q_{ji}}{\partial p_j} = \dfrac{1}{2}\sqrt{\dfrac{1}{\alpha_{ji} f_{F_{ji}} y}}$	$\dfrac{\partial Q_{ij}}{\partial p_j} = \dfrac{1}{2}\sqrt{\dfrac{-1}{\alpha_{ji} f_{F_{ji}} y}}$
$\dfrac{\partial Q}{\partial p_i}$	$\dfrac{\partial Q_{ji}}{\partial p_i} = -\dfrac{1}{2}\sqrt{\dfrac{1}{\alpha_{ji} f_{F_{ji}} y}}$	$\dfrac{\partial Q_{ij}}{\partial p_i} = -\dfrac{1}{2}\sqrt{\dfrac{-1}{\alpha_{ji} f_{F_{ji}} y}}$

where: $y = p_j - p_i + \beta(z_j - z_i)$.

Table 3.4 Partial derivatives of flow rate for flow through a pump

Partial Derivatives	$Q_{ji} = \sqrt{\frac{w_{ji}}{b_{ji}}}$ $w_{ji} = p_j - p_i + a_{ji} + \beta(z_j - z_i)$ in which: $C_{ji} = 2$	$Q_{ij} = -\sqrt{\frac{w_{ij}}{b_{ij}}}$ $w_{ij} = p_i - p_j + a_{ij} + \beta(z_i - z_j)$ in which: $C_{ij} = 2$
$\frac{\partial Q}{\partial p_j}$	$\frac{\partial Q_{ji}}{\partial p_j} = \frac{1}{2}\sqrt{\frac{1}{b_{ji} w_{ji}}}$	$\frac{\partial Q_{ij}}{\partial p_j} = \frac{1}{2}\sqrt{\frac{1}{b_{ij} w_{ij}}}$
$\frac{\partial Q}{\partial p_i}$	$\frac{\partial Q_{ji}}{\partial p_i} = -\frac{1}{2}\sqrt{\frac{1}{b_{ji} w_{ji}}}$	$\frac{\partial Q_{ij}}{\partial p_i} = -\frac{1}{2}\sqrt{\frac{1}{b_{ij} w_{ij}}}$

3.2.4 Units and conversion factors

Table 3.5 British and SI units

Quantity	British units	SI units
pressure, p	psig	kPa
flow rate, Q	gpm	m^3/hr
density, ρ	lb_m/ft^3	kg/m^3
viscosity, μ	centipoise	mPa-s
length, L	ft	m
diameter, D	inch	mm
elevation, z	ft	m
pipe roughness, ε	ft	mm
gravitational acceleration, g	ft/sec^2	m/sec^2
Fanning friction factor, f_F	none	none
Reynolds number, Re	none	none
perimeter, P	ft	m
width, w	ft	m
depth, h	ft	m

Table 3.6 Conversion units for α_{ji}, β, and Re_{ji}

British units	SI units
$\alpha_{ji} = \frac{32\times 12^5 \times \rho L_{ji}}{\pi^2 \times 144 \times 32.2 \times (448.8)^2 \times D_{ji}^5}$	$\alpha_{ij} = \frac{32\times 10^{10} \times \rho L_{ji}}{\pi^2 \times 3600^2 \times 1.01325 \times D_{ji}^5}$
$\beta = \frac{\rho}{144}$	$\beta = \frac{9.81\times\rho}{1.01325\times 10^5}$
$Re_{ji} = \frac{4\times 12\times 10^5 \times \rho Q_{ji}}{7.48\times 60\times 32.2\times 2.089\times \pi\mu D_{ji}}$	$Re_{ji} = \frac{4\times 10^6 \times \rho Q_{ji}}{3600\times \pi\mu D_{ji}}$

3.2.5 Nodal material balance

The nodal material balance equations for all nodes i at which the pressure p_i is not specified ($T_i \neq 1$) can be described as follows.

For steady-state, the net flows into any node i must equal to zero, so:

$$
\begin{aligned}
f_i(\mathbf{P}) &= \text{injection rate (or withdrawal rate)} \\
&\quad + \text{Net flow in/out by pipeline} \\
&\quad + \text{Net flow in/out by pump} \\
&= 0
\end{aligned}
\tag{3.9}
$$

Here:

$f_i(\mathbf{P})$ is the net flow into node i.

$$\mathbf{P} = [p_1, p_2, p_3, \ldots, p_n]^t \tag{3.10}$$

The equation for $f_i(\mathbf{P})$ becomes:

$$
\begin{aligned}
f_i(\mathbf{P}) = \ & V_i \text{ (injection (positive) or withdrawal (negative) rate)} \\
& + \sum_{j, C_{ji}=1} Q_{ji} \quad \text{(two cases for pipeline)} \\
& + \sum_{j, C_{ji}=2} Q_{ji} \quad \text{(three cases for pumping in)} \\
& + \sum_{j, C_{ij}=2} Q_{ji} \quad \text{(three cases for pumping out)} \qquad (3.11)
\end{aligned}
$$

3.2.6 Newton-Raphson and Gaussian elimination method

The simultaneous nonlinear equations in the unknown pressure that are obtained from nodal material balance at all nodes i are solved by the iterative Newton-Raphson method as follows:

1. The estimates of f_{Fij} for all pipe connections (C_{ji} = 1) and the pressures at all nodes are specified.

2. Find the appropriate partial derivatives of the functions $f_i(P)$ with respect to p_j, (i, j = 1, 2,…, n), which are stored as the elements of the left hand side coefficient matrix, ϕ of the simultaneous linear equations.

By first defining $$f_{ij}(p) = \frac{\partial f_i(p)}{\partial p_j} \qquad (3.12)$$

The matrix becomes: $$\phi(\mathbf{P}) = [f_{ij}(P)], \quad 1 \le i \le n, \ 1 \le j \le n \qquad (3.13)$$

Thus, the simultaneous linear equations are

$$\phi(\mathbf{P})\delta(\mathbf{P}) = -\mathbf{f}(\mathbf{P}) \qquad (3.14)$$

where: **f(P)** = [f_1(**P**), f_2(**P**), f_3(**P**), ... , f_n(**P**)] ^{t}

$\delta \mathbf{P} = \mathbf{P}_{new} - \mathbf{P}_{old}$ are the solutions of the simultaneous linear equations.

Then, the partial derivatives of $f_i(\mathbf{P})$ with respect to p_j are given by one of following forms:

$$f_{ij}(\mathbf{P}) = \frac{\partial Q_{ji}}{\partial p_j} = \left\{ \begin{array}{ll} \frac{1}{2}\sqrt{1/\alpha_{ji} f_{F_{ji}} y} & \text{for } y > 0 \\ \frac{1}{2}\sqrt{-1/\alpha_{ji} f_{F_{ji}} y} & \text{for } y < 0 \end{array} \right\} \quad \text{if } C_{ji} = 1$$

$$f_{ij}(\mathbf{P}) = \frac{\partial Q_{ji}}{\partial p_j} = \frac{1}{2}\sqrt{1/(b_{ji} w_{ji})} \qquad \text{if } C_{ji} = 2$$

$$f_{ij}(\mathbf{P}) = \frac{\partial Q_{ji}}{\partial p_j} = \frac{1}{2}\sqrt{1/(b_{ij} w_{ij})} \qquad \text{if } C_{ij} = 2 \tag{3.15}$$

The partial derivatives of $f_i(\mathbf{P})$ with respect to p_i are given by summations as follow:

$$f_{ii}(\mathbf{P}) = \left\{ \begin{array}{ll} \sum\limits_{j, C_{ji}=1} -\frac{1}{2}\sqrt{1/\alpha_{ji} f_{F_{ji}} y} & \text{for } y > 0 \\ \sum\limits_{j, Cji=1} -\frac{1}{2}\sqrt{-1/\alpha_{ji} f_{F_{ji}} y} & \text{for } y < 0 \end{array} \right\}$$

$$+ \sum_{j, C_{ji}=2} -\frac{1}{2}\sqrt{1/(b_{ji} w_{ji})}$$

$$+ \sum_{j, C_{ij}=2} -\frac{1}{2}\sqrt{1/(b_{ij} w_{ij})} \tag{3.16}$$

3. Use LU decomposition of the Gaussian elimination method with column pivoting only to solve the simultaneous linear equations with $\phi(\mathbf{P})$ as the left-hand side coefficient matrix.

4. Substitute the solution from equations (3.15) and (3.16) to find the correction vector $\delta\mathbf{P}$ and stabilize the method of the method at all nodes i by using a damping factor, σ , in the correction as follows:

$$\delta p_i = \sigma \times \delta p_i^* \qquad (3.17)$$

where :

δp_i is the value of the correction actually applied.

δp_i^* is the value of the correction obtained from the Newton-Raphson method

5. Check the convergence after the corrections δp_i have been solved at all nodes i by using a criterion:

$$|\delta p_i| < \varepsilon \qquad i = 1, 2, 3, \dots, n \qquad (3.18)$$

where : ε is the tolerance for the convergence criterion

6. If all of corrections do not satisfy the criterion, the current vector of pressures is modified according to:

$$\mathbf{P}_{k+1} = \mathbf{P}_k + \delta\mathbf{P}_k$$

where: $\mathbf{P}_k$ is the current vector (or matrix) of pressures

$\mathbf{P}_{k+1}$ is the updated matrix of pressures used for the next iteration

$\delta\mathbf{P}_k$ is the matrix of pressure corrections just calculated

k is the iteration number

7. After obtaining these new pressures P_{k+1}, the updated flow rates (Q_{ji}) can be calculated by using equations (3.1) and (3.2) with the old Fanning friction factor, $f_{F_{ji}}^{k}$, for all pipeline segments, and using equations (3.5)-(3.10) to calculate the flow rate for all pumps.

8. The Reynolds numbers Re_{ji}, are computed for all pipelines as follows:

$$Re_{ji} = \frac{4 \times 12 \times 10^5 \times \rho Q_{ji}}{7.48 \times 60 \times 32.2 \times 2.089 \times \pi \mu D_{ji}} \quad \text{for British units}$$

$$Re_{ji} = \frac{4 \times 10^6 \times \rho Q_{ji}}{3600 \times \pi \mu D_{ji}} \quad \text{for SI units}$$

9. The program updates the Fanning friction factors as a function of the Reynolds number and roughness ratio in all pipeline segments as follows:

For turbulent flow ($Re_{ji} > 4000$):

$$f_{F_{ji}} = \left\{ -1.737 \ln \left[0.269 \frac{\varepsilon_{ji}}{D_{ji}} - \frac{2.185}{Re_{ji}} \ln \left(0.269 \frac{\varepsilon_{ji}}{D_{ji}} + \frac{14.5}{Re_{ji}} \right) \right] \right\}^{-2} \tag{3.19}$$

For laminar flow ($Re_{ji} < 2000$):

$$f_{F_{ji}} = \frac{16}{Re_{ji}} \tag{3.20}$$

Equations 3.21 and 3.22 are adapted from *Wilkes* [5]

10. The sequence of steps given above is repeated for sequential iteration k = 1, 2, 3, … until the convergence occurs according to the criterion or until a specified maximum number of iterations (k_{max}) is exceeded.

3.3 Part II: Liquid flow in open channels

3.3.1 The assumptions

The flow in an open channel network must be calculated using the following assumptions:

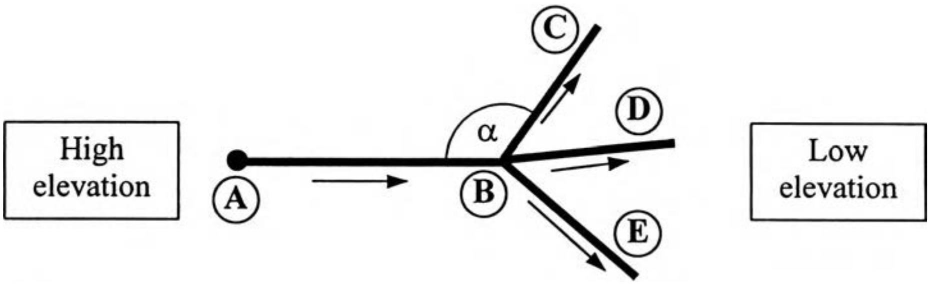

Figure 3.1 Channel branching for consider the assumptions

1. It is always exposed to atmospheric pressure, so its inlet and outlet pressures are always zero (gauge) (P_A, P_B, . . .,$P_E = 0$).

2. The liquid in the channel (presumably always water) can only flow downhill, so the flow is always in one direction.

3. The depth of the water will adjust to whatever input is received into the channel.

4. The flow type in every channel is a steady and uniform.

5. The depth of the liquid that flow out from one channel to the next at the same node must be the same ($h_{BC} = h_{BD} = h_{BE}$).

6. The effects of connected angle (such as α) between any two channels can be neglected.

3.3.2 Calculation steps

1. After the first part is achieved, the flow rates at all nodes are calculated.

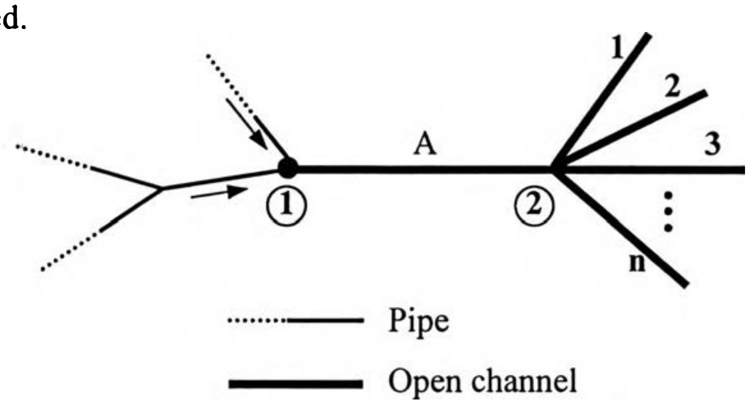

Figure 3.2 A simple channel branching

By considering the node that joins both pipes and channels (node 1), the flow rate in the channel **A** can be calculated from the flow out of the piping system.

2. If node 2 is connected with n channels that slope down further:

From material balance at node 2:

$$Q_A = Q_1 + Q_2 + Q_3 + \ldots + Q_n\,, \quad Q_A \text{ is known}$$

From assumption number 5:

$$h_1 = h_2 = h_3 = \ldots = h_n$$

Using equation (2.22) for every channel, n equations were generated.

$$Q_i = \sqrt{\frac{g\Delta z_i\, 4(w_i h_i)^3}{2f_{F_i} L_i (2h_i + w_i)}}\;; \quad i = 1, 2, 3, \ldots, n \qquad (3.21)$$

3. The depth of the inlet channel (channel **A**) is used as the estimated depth for all outlet channels (channels **1** to **n**). Then calculate the flow rate Q_1 to Q_{n-1} by using equation (3.23)

So, $$Q_n = Q_A - (Q_1 + Q_2 + Q_3 + \ldots + Q_{n-1})$$

4. Calculate the depth at the channel **n** (h_n) from the inverse of equation (3.23)

$$[g\Delta z_n 4(w_n^3)Q_n^2]h_n^3 + [4f_{F_{ni}} L_n Q_n^2]h_n + 2f_{F_n} L_n w_n Q_n^2 = 0$$

5. Update the depth at channel **1** from

$$h_{1,k+1} = \frac{h_{1,k} + h_{2,k} + h_{3,k} + \ldots + h_{n,k}}{n}$$

where: k = iteration number

6. Check the convergence after the new depth ($h_{1,k+1}$) has been updated by using the criterion:

$$\left|h_{i,k+1} - h_{i,k}\right| < \varepsilon \qquad , i = 1, 2, 3, \ldots, n \qquad (3.22)$$

where : ε is the tolerance for the convergence criterion

7. If the criterion is not satisfied, then use the updated depth ($h_{1,k+1}$) to recalculate the flow rates and the depths until the convergence criterion is satisfied.

3.3.3 The adjacent nodes

After the flow rates and the depths at all outlet channels of the current node (node 2) are solved, the program will continue to repeat the calculation steps (3.3.2) for the adjacent nodes according to the elevation, until all nodes in the open channels system are calculated.