CHAPTER III
PROBLEM SOLVING METHOD

3.1 Elliptic Equation

The general form of the elliptic equation in three space dimensions is

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial y^2}+c\frac{\partial^2 u}{\partial z^2}=f(x,y,z) \tag{3.1}$$

3.1.1 <u>One-Dimensional Elliptic Equation</u>

The general form of the one-dimensional elliptic equation is shown in eq.(3.2).

$$a\frac{\partial^2 u}{\partial x^2}=m \tag{3.2}$$

The central finite difference is substituted in eq.(3.2) with a grid shown in fig. 3.1 to get eq.(3.3).

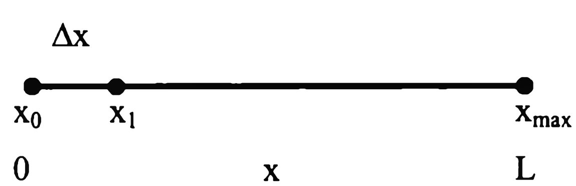

Figure 3.1 Arrangement of grid points for one dimension.

$$a\frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta x)^2}=m \tag{3.3}$$

Here, the spatial variable $0< x <L$ is discretized such that $\Delta x = L/p$ (p = number of increment) and $x_i = i\Delta x$. 55

The eq. (3.3) is rearranged to eq.(3.4).

$$u_i = (u_{i+1} + u_{i-1} - m(\Delta x)^2 / a) / 2 \tag{3.4}$$

By adding a superscript to denote the number of the iteration, eq.(3.4) becomes the recursive form, eq.(3.5).

$$u_i^{n+1} = (u_{i+1}^n + u_{i-1}^n - \frac{m(\Delta x)^2}{a}) / 2 \tag{3.5}$$

The explicit recursive form is calculated repeatedly until the latest relative difference of the values of u and previous values of u is less than the relative tolerance, ε, which is shown in eq.(3.6).

$$\left| \frac{u_i^{n+1} - u_i^n}{u_i^n} \right| < \varepsilon \tag{3.6}$$

3.1.2 Two-Dimensional Elliptic Equation

The general form of the two-dimensional elliptic equation is

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial y^2} = m \tag{3.7}$$

The central finite difference form is used for eq.(3.7) is

$$a \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + b \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = m \tag{3.8}$$

Here, the subscription scheme, corresponds to the two-dimensional space grid shown in fig. 3.2

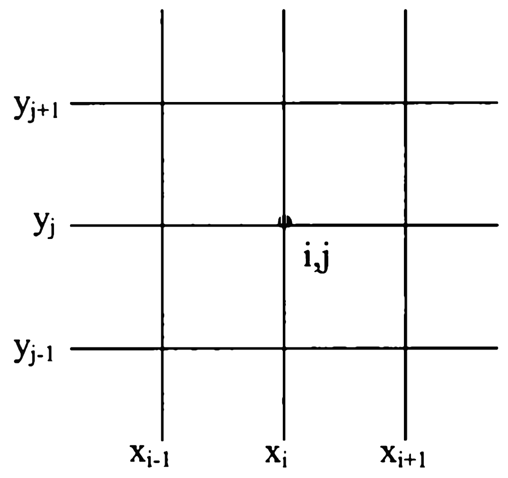

Figure 3.2 Arrangement of grid points for two dimensions.

3.1.2.1 Gauss Seidel Method

The Gauss Seidel method is applied to eq.(3.8), which is rearraged to eq. (3.9).

$$u_{i,j} = \left(\frac{a}{(\Delta x)^2}(u_{i+1,j} + u_{i-1,j}) + \frac{b}{(\Delta y)^2}(u_{i,j+1} + u_{i,j-1}) - m\right) \Big/ \left(\frac{2a}{(\Delta x)^2} + \frac{2b}{(\Delta y)^2}\right) \tag{3.9}$$

The recursive form of eq.(3.9) is shown in eq.(3.10), where the superscript denotes the number of the iteration.

$$u_{i,j}^{n+1} = \left(\frac{a}{(\Delta x)^2}(u_{i+1,j}^{n} + u_{i-1,j}^{n}) + \frac{b}{(\Delta y)^2}(u_{i,j+1}^{n} + u_{i,j-1}^{n}) - m\right) \Big/ \left(\frac{2a}{(\Delta x)^2} + \frac{2b}{(\Delta y)^2}\right) \tag{3.10}$$

The explicit iterative calculation for u continues until the latest relative difference between the new value of u at point ij and previous value of u is less than the relative tolerance, ε as shown in eq.(3.11).

$$\left|\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{u_{i,j}^{n}}\right|<\varepsilon \tag{3.11}$$

The Gauss Seidel method will solve the solution, u by using the components of u on the right hand side of recursive form

3.1.2.2 Alternating Direction Implicit Method

This numerical method will help accelerate convergence to the solution of the finite-difference equations. The eq.(3.8) is rearranged to the two sets of linear equations eq.(3.12) and (3.13) with a tridiagonal coefficient matrix. The eq.(3.12) uses the implicit form only in the x-direction and eq.(3.13) uses it only in y-direction. Eq.(3.12) and eq.(3.13) can be solved by using the Thomas method

$$\begin{aligned}&\frac{a}{(\Delta x)^2}(u_{i-1,j}^{n+1})-\frac{2a}{(\Delta x)^2}u_{i,j}^{n+1}+\frac{a}{(\Delta x)^2}(u_{i+1,j}^{n+1}))=\\&\quad m-\frac{b}{(\Delta y)^2}(u_{i,j-1}^{n})+\frac{2b}{(\Delta y)^2}(u_{i,j}^{n})-\frac{b}{(\Delta y)^2}(u_{i,j+1}^{n})\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\frac{b}{(\Delta y)^2}(u_{i,j-1}^{n+2})-\frac{2b}{(\Delta y)^2}u_{i,j}^{n+2}+\frac{b}{(\Delta y)^2}(u_{i,j+1}^{n+2}))=\\&\quad m-\frac{a}{(\Delta x)^2}(u_{i-1,j}^{n+1})+\frac{2a}{(\Delta x)^2}(u_{i,j}^{n+1})-\frac{a}{(\Delta x)^2}(u_{i+1,j}^{n+1})\end{aligned} \tag{3.13}$$

At the first iteration, all values of $u^{n}_{i,j}$ on the right hand side of eq.(3.12) are assumed for the initial guess. Then $u^{n+1}_{i,j}$ can be solved by the Thomas method. After that, $u^{n+2}_{i,j}$, is calculated by eq.(3.13) which $u^{n+1}_{i,j}$ is known from the previous equation. The iteration continues by solving with eq.(3.12) again. Like the Gauss Seidel method, the iteration continues until the relative error between the new and previous iterative values is less than the relative tolerance, ε.

$$\left|\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{u_{i,j}^{n}}\right| < \varepsilon \tag{3.11}$$

3.1.3 Three-Dimensional Elliptic Equation

The elliptic equation in the x,y and z directions is shown in eq. (3.14).

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 u}{\partial z^2} = m \tag{3.14}$$

The above equation will be transformed to eq.(3.15) by substituting the central differences.

$$a\frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2} + b\frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta y)^2} + c\frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\Delta z)^2} = m \tag{3.15}$$

3.1.3.1 Gauss Seidel Method

Eq.(3.15) is rearraged to eq.(3.16)

$$u_{i,j} = \left(\frac{a}{(\Delta x)^2}(u_{i+1,j,k} + u_{i-1,j,k}) + \frac{b}{(\Delta y)^2}(u_{i,j+1,k} + u_{i,j-1,k}) + \frac{c}{(\Delta z)^2}(u_{i,j,k+1} + u_{i,j,k-1}) - m\right) / \left(\frac{2a}{(\Delta x)^2} + \frac{2b}{(\Delta y)^2} + \frac{2c}{(\Delta z)^2}\right) \tag{3.16}$$

Eq.(3.16) is changed into the iteration form for the Gauss Seidel method, where superscript denotes the number of the iteration

$$u_{i,j,k}^{n+1} = \left(\frac{a}{(\Delta x)^2}(u_{i+1,j,k}^{n} + u_{i-1,j,k}^{n}) + \frac{b}{(\Delta y)^2}(u_{i,j+1,k}^{n} + u_{i,j-1,k}^{n})\right.$$
$$\left. + \frac{c}{(\Delta z)^2}(u_{i,j,k+1}^{n} + u_{i,j,k-1}^{n}) - m\right) / \left(\frac{2a}{(\Delta x)^2} + \frac{2b}{(\Delta y)^2} + \frac{2c}{(\Delta z)^2}\right) \tag{3.17}$$

The iteration continues until the relative error in n, i, j, and k for the new and previous iterative values is less than the tolerance.

$$\left|\frac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n}}{u_{i,j,k}^{n}}\right| < \varepsilon \tag{3.18}$$

3.1.3.2 Alternating Direction Implicit Method

Eq.(3.15) is rearranged to the three sets of linear equations with a tridiagonal coefficient matrix. The implicit form is used only in the x-direction, y-direction, and z-direction in eq.(3.19), eq.(3.20), and eq.(3.21) respectively, where the superscript denotes the number of the iteration.

$$\frac{a}{(\Delta x)^2}(u_{i-1,j,k}^{n+1}) - \frac{2a}{(\Delta x)^2}(u_{i,j,k}^{n+1}) + \frac{a}{(\Delta x)^2}(u_{i+1,j,k}^{n+1}) =$$
$$m - \frac{b}{(\Delta y)^2}(u_{i,j-1,k}^{n}) + \frac{2b}{(\Delta y)^2}(u_{i,j,k}^{n}) - \frac{b}{(\Delta y)^2}(u_{i,j+1,k}^{n})$$
$$- \frac{c}{(\Delta z)^2}(u_{i,j,k-1}^{n}) + \frac{2c}{(\Delta z)^2}(u_{i,j,k}^{n}) - \frac{c}{(\Delta z)^2}(u_{i,j,k+1}^{n}) \tag{3.19}$$

$$\frac{b}{(\Delta y)^2}(u_{i,j-1,k}^{n+2}) - \frac{2b}{(\Delta y)^2}(u_{i,j,k}^{n+2}) + \frac{b}{(\Delta y)^2}(u_{i,j+1,k}^{n+2}) =$$
$$m - \frac{a}{(\Delta x)^2}(u_{i-1,j,k}^{n+1}) + \frac{2a}{(\Delta x)^2}(u_{i,j,k}^{n+1}) - \frac{a}{(\Delta x)^2}(u_{i+1,j,k}^{n+1})$$
$$- \frac{c}{(\Delta z)^2}(u_{i,j,k-1}^{n+1}) + \frac{2c}{(\Delta z)^2}(u_{i,j,k}^{n+1}) - \frac{c}{(\Delta z)^2}(u_{i,j,k+1}^{n+1}) \tag{3.20}$$

$$\frac{c}{(\Delta z)^2}(u_{i,j,k-1}^{n+3}) - \frac{2c}{(\Delta z)^2}(u_{i,j,k}^{n+3}) + \frac{c}{(\Delta z)^2}(u_{i,j,k+1}^{n+3}) =$$
$$m - \frac{a}{(\Delta x)^2}(u_{i-1,j,k}^{n+2}) + \frac{2a}{(\Delta x)^2}(u_{i,j,k}^{n+2}) - \frac{a}{(\Delta x)^2}(u_{i+1,j,k}^{n+2})$$
$$- \frac{b}{(\Delta y)^2}(u_{i,j-1,k}^{n+2}) + \frac{2b}{(\Delta y)^2}(u_{i,j,k}^{n+2}) - \frac{b}{(\Delta y)^2}(u_{i,j+1,k}^{n+2}) \tag{3.21}$$

These three sets of linear equations cannot be expressed explicitly in terms of the known values at the previous time step which are implicit forms. They can be solved by using the Thomas method. Like the two-dimensional problem, the solving algorithm starts with eq.(3.19), eq.(3.20), and then eq.(3.21). The iterative calculation continues until the relative error is less than the relative tolerance, ε ,as show in eq.(3.18).

3.2 Parabolic Equation

The general form of the parabolic PDE is shown below.

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \tag{3.22}$$

3.2.1 <u>One-Dimensional Parabolic Equation</u>

The one-dimensional parabolic equation is shown in eq.(3.23).

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \tag{3.23}$$

3.2.1.1 Method of Lines

The method of lines (MOL) converts the partial differential equations into a set of coupled first order ordinary differential equations by using the finite difference method from eq.(3.23) and the central finite difference approximation of the second derivative .

$$\frac{\partial u}{\partial t} = a\left(\frac{u_{l+1} - 2u_l + u_{l-1}}{(\Delta x)^2}\right) = f(u) \tag{3.24}$$

Eq.(3.24) is solved by using the fourth-order of the Runge-kutta algorithm

$$y_{l+1} = y_l + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3.25}$$

Where

$$k_1 = f(x_l, y_l) \tag{3.26}$$

$$k_2 = f(x_l + \frac{1}{2}h, y_l + \frac{1}{2}hk_1) \tag{3.27}$$

$$k_3 = f(x_l + \frac{1}{2}h, y_l + \frac{1}{2}hk_2) \tag{3.28}$$

$$k_4 = f(x_l + h, y_l + hk_3) \tag{3.29}$$

3.2.1.2 Implicit Method

The one-dimensional parabolic equation is shown below.

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \tag{3.23}$$

A forward finite difference form is substituted for the left hand side term.

$$\frac{\partial u}{\partial t} = \frac{u_{l,n+1} - u_{l,n}}{\Delta t} \tag{3.30}$$

The represent $\partial^2 u/\partial x^2$ by a central finite difference form evaluated at the advance point of time t_{n+1}.

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{l-1,n+1} - 2u_{l,n+1} + u_{l+1,n+1}}{(\Delta x)^2} \tag{3.31}$$

When eq.(3.30) and eq.(3.31) are substituted in eq.(3.23) , the difference equation becomes eq.(3.32).

$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = a\left(\frac{u_{i-1,n+1} - 2u_{i,n+1} + u_{i+1,n+1}}{(\Delta x)^2}\right) \tag{3.32}$$

The eq.(3.32) is rearranged into eq.(3.33)

$$-Fu_{i-1,n+1} + (1 - 2F)u_{i,n+1} - Fu_{i+1,n+1} = u_{i,n} \tag{3.33}$$

A tridiagonal coefficient matrix is solved by using the Thomas method.
F is defined by eq.(3.34)

$$F = \frac{a(\Delta t)}{(\Delta x)^2} \tag{3.34}$$

3.2.1.3 Forward Time Central Space Method

The one-dimensional parabolic equation is shown in eq.(3.23).

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \tag{3.23}$$

The derivative of eq.(3.23) can be replaced by finite difference form.

$$\frac{\partial u}{\partial t} = \frac{u_{i,n+1} - u_{i,n}}{\Delta t} \tag{3.30}$$

Eq.(3.30) is rearraged to eq.(3.35).

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,n} - 2u_{i,n} + u_{i+1,n}}{(\Delta x)^2} \tag{3.35}$$

All the $u_{i,n}$ are known at any time level, t_n. Eq.(3.36) enables $u_{i,n+1}$ to be calculated directly at the time level t_{n+1}.

3.2.1.4 Crank Nicolson Method

Crank Nicolson is one of the implicit methods which is stable for all values of F.

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \tag{3.23}$$

In this method, to march one time step from n to n+1, the one-dimensional heat equation is represented in finite difference form at the half-time step. For the t-derivative, a central difference is used.

$$\left(\frac{\partial u}{\partial t}\right)_i^{n+1/2} = \frac{u_i^{n+1} - u_i^n}{\Delta t} \tag{3.36}$$

The x-derivative is taken as the average of the x-derivatives at the n- and (n+1)-th t-steps which are represented by the central difference.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1/2} = \frac{1}{2}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1} + \left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n}\right] \tag{3.37}$$

Therefore,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1/2} = \frac{1}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{(\Delta x)^2}\right] \tag{3.38}$$

Eq.(3.37) and eq.(3.39) are substituted into eq.(3.23).

$$\frac{u_i^{n+1} - u_i}{\Delta t} = a\left(\frac{1}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{(\Delta x)^2}\right]\right) \tag{3.39}$$

Eq.(3.39) is rearranged to eq.(3.40).

$$-Fu_{i+1}^{n+1} + (2+2F)u_i^{n+1} - Fu_{i-1}^{n+1} = Fu_{i+1}^{n} + (2-2F)u_i^{n} + Fu_{i-1}^{n} \tag{3.40}$$

This is an implicit method, as u^n_{i+1} cannot be expressed explicitly in terms of the known values at the n-th step. A set of linear equations has to be solved for each time step by the Thomas method.

3.2.2 Two-Dimensional Parabolic Equation

The two-dimensional parabolic equation is shown as below.

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} \tag{3.41}$$

3.2.2.1 Alternating Direction Implicit Method

The alternating direction implicit method manages to use a system of equations with a tridiagonal coefficient matrix. The principle is to employ two difference equations which are used in turn over successive time-steps each of a duration of $\Delta t/2$. Eq.(3.42) is implicit only in the x direction, and the eq.(3.43) is implicit only in the y-direction.

$$\frac{u_{i,j,n+1} - u^*_{i,j}}{\Delta t/2} = \delta^2_x u^*_{i,j} + \delta^2_y u_{i,j,n+1} \tag{3.42}$$

$$\frac{u^*_{i,j} - u_{i,j,n}}{\Delta t/2} = \delta^2_x u^*_{i,j} + \delta^2_y u_{i,j,n} \tag{3.43}$$

Here, $u^*_{i,j}$ is an intermediate value at the end of the first time-step and $\delta^2_x u_{i,j}$ and $\delta^2_y u_{i,j}$ are central difference forms

$$\delta^2_x u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} \tag{3.44}$$

and

$$\delta^2_y u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2} \tag{3.45}$$

Eq.(3.44) and eq.(3.45) are written out in full and rearranged to eq.(3.46) and eq.(3.47).

$$F_x u^*_{i-1,j} - (2F_x + 2)u^*_{i,j} + F_x u^*_{i+1,j} = -F_y u_{i,j-1,n} + (2F_y - 2)u_{i,j,n} - F_y u_{i,j+1,n} \tag{3.46}$$

$$F_y u_{i,j-1,n+1} - (2F_y + 2)u_{i,j,n+1} + F_y u_{i,j+1,n+1} = - F_x u^*_{i-1,j} + (2F_x - 2)u^*_{i,j} - F_x u^*_{i+1,j} \tag{3.47}$$

F_x and F_y are difined by eq.(3.48) and eq.(3.49).

$$F_x = \frac{a(\Delta t)}{(\Delta x)^2} \tag{3.48}$$

$$F_y = \frac{b(\Delta t)}{(\Delta y)^2} \tag{3.49}$$

3.2.2.2 Forward Time Central Space Method

The equation of unsteady-state heat conduction is substituted with the forward difference for the t-derivative and the central difference for the x-derivative and the y-derivative to become eq.(3.50).

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a\frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{(\Delta x)^2} + b\frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{(\Delta y)^2} \tag{3.50}$$

The differential equation of eq.(3.50) is rearraged to eq.(3.51).

$$u_i^{n+1} = F_x(u^n_{i+1,j} + u^n_{i-1,j}) + F_y(u^n_{i,j+1} + u^n_{i,j-1}) + (1 - 2F_x - 2F_y)u^n_{i,j} \tag{3.51}$$

3.3 Linear Equations with a Tridiagonal Coefficient Metrix Solved by Thomas Method

This system of equations can be written in the following form.

$$
\begin{aligned}
d_1x_1 + e_1x_2 \qquad\qquad &= b_1 \\
c_2x_1 + d_2x_2 + e_2x_3 \qquad &= b_2 \\
c_3x_2 + d_3x_3 + e_3x_4 &= b_3 \\
&\vdots \\
c_nx_{n-1} + d_nx_n &= b_n
\end{aligned}
\tag{3.52}
$$

where c_i, d_i, and e_i represented the coefficients of x_{i-1}, x_i , and x_{i+1} ,repectively, in the i-th equation.

By application of Gaussian elimination, the solution to such a set of linear equations is as follows:

$$x_n = \gamma_n \tag{3.53}$$

and

$$x_i = \gamma_i - \frac{e_ix_{i+1}}{\beta_i} \qquad (i = n-1, n-2, \ldots, 1) \tag{3.54}$$

where γ_i and β_i are determined by the following recursive relationships :

$$\beta_1 = d_1 \tag{3.55}$$

$$\gamma_1 = \frac{b_1}{\beta_1} \tag{3.57}$$

$$\beta_i = d_i - \frac{c_i e_{i-1}}{\beta_{i-1}} \qquad (i = 2,3,\ldots,n) \qquad (3.57)$$

$$\gamma_i = \frac{b_i - c_i \gamma_{i-1}}{\beta_i} \qquad (i = 2,3,\ldots,n) \qquad (3.58)$$

Therefore, the β_i's and γ_i's are first calculated using eq.(3.57) through eq. (3.60). Then eq.(3.55) and eq.(3.56) are used to determine the values of x_i that solve the set of linear equations. This procedure is known as the Thomas method.