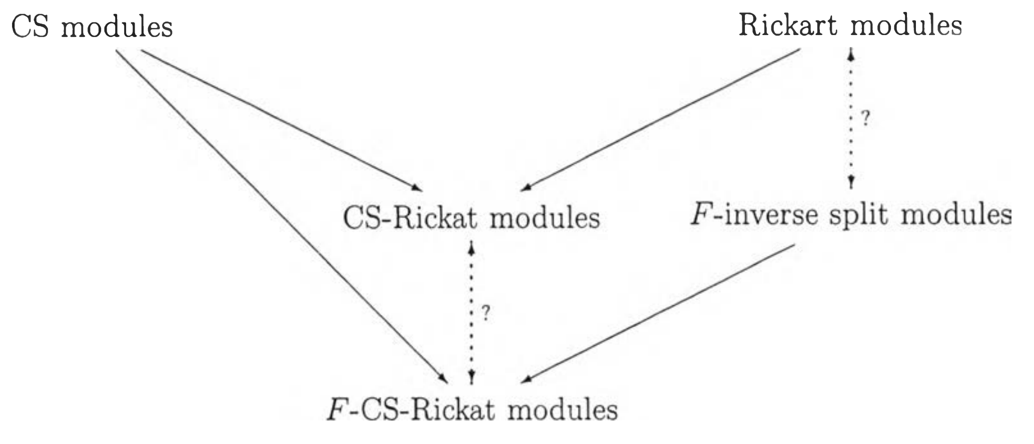
CHAPTER I
INTRODUCTION

Throughout this dissertation, all rings are rings with identity and all modules are unitary right modules. Let R be a ring with identity, M be a right R-module and $\mathrm{End}(M)$ be the set of all endomorphisms of M. Recall that a submodule N of M is a *fully invariant submodule* of M if $f(N) \subseteq N$ for any $f \in \mathrm{End}(M)$. Note that N is always a submodule of $f^{-1}(N)$ for any fully invariant submodule N of M. In addition, a submodule N of M is a *direct summand* of M if there is a submodule K of M such that $N + K = M$ and $N \cap K = 0$ (the zero module); moreover, N is an *essential submodule* of M if $N \cap K \neq 0$ for any nonzero submodule K of M.

A module M is a *CS module* or *extending module*, given by Dung et al., given in [8] in 1994, if for any submodule N of M, N is an essential submodule of a direct summand of M. Furthermore, in 2010, Lee, Rizvi and Roman introduced, in [11], the notion of Rickart modules. They proposed that a module M is a *Rickart module* if $\ker f$ is a direct summand of M for any $f \in \mathrm{End}(M)$. Next, Abyzov and Nhan [1] in 2014, combined the concepts of CS modules and Rickart modules to CS-Rickart modules which are a generalization of these two. A module M is a *CS-Rickart module* if for each $f \in \mathrm{End}(M)$, $\ker f$ is an essential submodule of a direct summand of M.

In addition, Unger, Halicioglu and Harmanci, given in [17] in 2016, generalized the notion of Rickart modules to F-inverse split modules for a given fully invariant submodule F of M. Let F be a fully invariant submodule of M. A module M is an *F-inverse split* module if $f^{-1}(F)$ is a direct summand of M for any $f \in \mathrm{End}(M)$. In this dissertation, we merge the concepts of CS modules and F-inverse split modules similarly to CS-Rickart modules. We define that M is an *F-CS-Rickart* module if for any $f \in \mathrm{End}(M)$ there is a direct summand M' of M such that $f^{-1}(F) \leq_{ess} M'$. Observe that $f^{-1}(F)$ is a submodule of M containing F. So we can conclude that M is an F-CS-Rickart module if and only if any submodule of M containing F is essential in some direct summand of M. Clearly, CS modules and F-inverse split modules are F-CS-Rickart modules; moreover, M is a CS-Rickart module if and only if M is a 0-CS-Rickart module. One of our aims in this research is to investigate relationships between F-CS-Rickart modules and CS-Rickart modules as well as relationships between F-CS-Rickart modules and F-inverse split modules.

The following diagram shows that both CS-Rickart modules and F-CS-Rickart modules are extended from CS modules. Besides, CS-Rickart modules and F-CS-Rickart modules are generalizations of Rickart modules and F-inverse split modules, respectively.

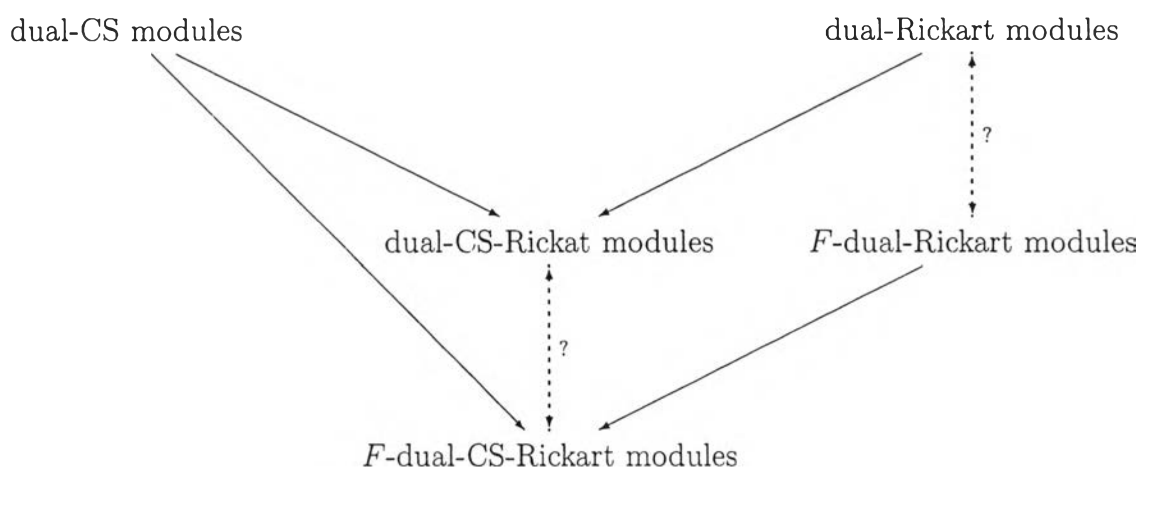


Moreover, we provide various properties and characterizations of F-CS-Rickart modules. Furthermore, we consider F-CS-Rickart modules where F is $Z(M)$, $Z_2(M)$ and $Z^*(M)$. We also pay attention when M is both a projective module and an F-CS-Rickart module. In addition, we define a right I-CS-Rickart ring for some ideal I of R by considering R as the module over itself. Then we obtain that for given positive integer n, the free R-module $\bigoplus_{n \text{ copies}} R$ is an $\left(\bigoplus_{n \text{ copies}} I\right)$-CS-Rickart module if and only if $M_n(R)$ is a right $M_n(I)$-CS-Rickart ring.

It is known that the dual notion of essential submodules is small submodules. A submodule N of M is a *small submodule* of M if $N + K = M$ implies $K = M$ for any submodule K of M; in addition, a submodule N of M *lies above a direct summand* of M if there is a direct summand M' of M and a submodule K of M such that $M' \subseteq N$, $M = M' + K$ and K is a small submodule of M. A module M

is a *dual-CS module* or *lifting module* which was introduced by Clark et al. in [7], if for any submodule N of M, N lies above a direct summand of M. In 2011, Lee, Rizvi and Roman introduced in [12] the notion of dual-Rickart modules. They proposed that a module M is a *dual-Rickart module* if $f(M)$ is a direct summand of M for any $f \in \text{End}(M)$. Later, the concept of dual-CS modules and dual-Rickart module was integrated by Abyzov and Nhan [1] in 2014 by using the idea of lying above some direct summands from CS modules and the idea of being a direct summand of $f(M)$ for all $f \in \text{End}(M)$ from dual-Rickart modules. A module M is a *dual-CS-Rickart module* if for each $f \in \text{End}(M)$, $f(M)$ lies above a direct summand of M. These lead us to study so-called F-dual-CS-Rickart modules when F is a given fully invariant submodule of M. A module M is an *F-dual-CS-Rickart module* if $f(F)$ lies above a direct summand of M for any $f \in \text{End}(M)$. As any direct summands of M lies above itself, we are interested in considering when $f(F)$ is a direct summand of M for any $f \in \text{End}(M)$, and name this module as *F-dual-Rickart modules.* Therefore, relationships between F-dual-CS-Rickart modules and dual-CS Rickart modules, likewise, relationships between F-dual-CS-Rickart modules and F-dual-Rickart modules are examined.

The next diagram describes that both dual-CS Rickart modules and F-dual-CS-Rickart modules are generalized from dual-CS modules. Moreover, both dual-CS-Rickart modules and F-dual-CS-Rickart modules are extended from dual-Rickart modules and F-dual Rickart modules, respectively.



Several properties and characterizations of F-dual-CS-Rickart modules are investigated. We study when submodules of F-dual-CS-Rickart modules still F'-dual-CS-Rickart modules where F' is a fully invariant submodule of that submodule.

This dissertation is organized as follows:

In Chapter II, preliminaries related to direct summands, fully invariant submodules, essential submodules, small submodules and projective modules are provided.

In Chapter III, we introduce the definition of an F-CS-Rickart module for a given fully invariant submodule F. Moreover, several properties and characterizations of F-CS-Rickart modules are examined. We prove when a submodule of an F-CS-Rickart module is also an F'-CS-Rickart module where F' is a fully invariant submodule of that submodule. Next, we provide the definition of relatively F-CS-Rickart modules which extend from F-CS-Rickart modules. Furthermore, we emphasize fixed fully invariant submodules F of M where $F = Z(M), Z_2(M)$ and $Z^*(M)$. For all $f \in \text{End}(M)$, an image of f and a principal right ideal of $\text{End}(M)$ generated by f can be written as the sum of two direct summands satisfying projective modules when M is an F-CS-Rickart module and a projective module. Finally, the notion of right I-CS-Rickart rings for a given ideal I of R are defined; in addition, the n by n matrix ring over R with some ideal is investigated.

In Chapter IV, the definitions of an F-dual-Rickart module and an F-dual-CS-Rickart module are provided for a given fully invariant submodule F. We investigate various properties and generalize F-dual-CS-Rickart modules. We show that when a submodule of an F-dual-CS-Rickart module is also an F'-dual-CS-Rickart module where F' is a fully invariant submodule of that submodule. In addition, relatively F-dual-CS-Rickart modules are provided and these are extended from F-dual-CS-Rickart modules.