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Solutions of interval linear programming

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โดยทั่วไป กำหนดการเชิงเส้นต้องการพารามิเตอร์ในรูปแบบของค่าคงที่ แต่ในโลกแห่งความเป็นจริงการที่เราจะสามารถทราบค่าที่แท้จริงของพารามิเตอร์นั้นเป็นไปได้ยากมากหรือไม่สามารถทราบได้เลย นั่นคือค่าของพารามิเตอร์ที่ได้มาส่วนใหญ่นั้นมาจากการประมาณค่า ดังนั้นกำหนดการเชิงเส้นแบบช่วงจึงเป็นหนึ่งในเครื่องมือที่จะช่วยในการจัดการกับความไม่แน่นอนของปัญหาทางคณิตศาสตร์ในโลกแห่งความเป็นจริง จากการที่เราไม่สามารถทราบค่าที่แท้จริงของพารามิเตอร์ได้ ทำให้เราไม่สามารถทราบค่าที่แท้จริงของมูลค่าจุดประสงค์ได้เช่นกัน ดังนั้นเราจึงได้นำวิธีการพิจารณา 3 วิธีการ ซึ่งได้แก่ วิธีการพิจารณาแบบดีที่สุด วิธีการพิจารณาแบบแย่ที่สุดและวิธีการพิจารณาค่าความสูญเสียแบบที่ต่ำที่สุดในบรรดาค่าความสูญเสียที่มากที่สุด มาใช้ในการอธิบายและช่วยในการตัดสินใจในการเลือกคำตอบที่ดีที่สุดของกำหนดการเชิงเส้นแบบช่วงนี้ โดยวิธีการไหนจะเหมาะสมที่สุดนั้นขึ้นกับแต่ละบุคคลว่ามีความประสงค์แบบใด นอกเหนือจากที่ได้กล่าวมาแล้วนั้น ในบางครั้งการลดขนาดของแต่ละช่วงก็อาจจะเป็นสิ่งที่จำเป็นสำหรับบางปัญหาของกำหนดการเชิงเส้นแบบช่วง เมื่อโจทย์ปัญหาดั้งเดิมของปัญหากำหนดการเชิงเส้นแบบช่วงนี้ไม่มีคำตอบ มากไปกว่านั้นโปรเจคนี้ได้จัดเตรียมโปรแกรมคอมพิวเตอร์ของผลเฉลยของปัญหาของกำหนดการเชิงเส้นแบบช่วงเพื่ออำนวยความสะดวกให้กับผู้ใช้งาน

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In general, linear programming requires constant parameters. But in many real-life situations, knowing the real value of parameters is fiendishly challenging or impossible. Since most of parameters in linear programming are estimated, therefor interval linear programming is utilized as one of the tools to handle the uncertainty of the mathematical problems in the real-world. As we could not know the exact value of some parameters in a linear programming problem, we could not perceive its real objective value. As a consequence, optimistic, pessimistic and minimax regret approaches are employed in this project to help people describing and choosing their own best solutions that depend on their decisions and purposes. Moreover, sometimes reducing the size of each interval entry may become as a neccesary thing for some interval linear programming problem when its original problem is infeasible. In addition, this project provides computer programs of solutions of the interval linear programming problem to facilitate users.

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Contents

	Page
Abstract(Thai)	**iv**
Abstract(English)	**v**
Acknowledgements	**vi**
Contents	**vii**
List of Figures	**ix**
List of Tables	**x**
1 Introduction	**1**
1.1 Background and Rationale	1
1.2 Objectives	2
1.3 Scopes	2
1.4 Project Activities	2
1.5 Benefits	3
2 Background knowledge	**4**
2.1 Linear programming (LP)	4
2.2 Interval linear programming (IvLP)	4
2.3 Strong solvable/feasibility of inequations	6
2.4 Optimistic and pessimistic approaches	7
2.4.1 An optimistic objective value function of minimization interval LP problem	8
2.4.2 A pessimistic objective value function of minimization interval LP problem	9
2.5 Minimax regret model	9

2.6 A reduced interval linear programming 10

3 Algorithms and results **13**

3.1 Python version 3.7.9 13

3.1.1 PuLP 13

3.1.2 NumPy 14

3.1.3 Pandas 14

3.2 Algorithms 14

3.3 Example 18

3.4 Results 22

4 Applications **26**

4.1 Industrial applications 26

4.2 Miscellaneous application 30

5 Summary and Conclusion **34**

5.1 Summary 34

5.2 Conclusion 34

5.3 Suggestions 35

References **37**

Appendix **39**

Appendix A: Code for Python 39

Appendix B: Project Proposal 40

Biography **51**

List of Figures

3.1 Flowchart of Python algorithms 17
3.2 the optimistic IvLP of Example 22
3.3 the 4^{th} optimistic reduced IvLP of Example 23
3.4 the pessimistic IvLP of Example 23
3.5 the 4^{th} pessimistic reduced IvLP of Example 24
3.6 the optimistic, the 4^{th} optimistic and the 4^{th} pessimistic reduced IvLP of Example 24

4.1 Solutions of Healthy Pet Food problem 29
4.2 Solutions to the diet problem 32

5.1 QR code of Python's code 39

List of Tables

4.1 Healthy Pet Food Data 26
4.2 The profit per package for each dog food 27
4.3 Cost and nutrition values 30

Chapter 1
Introduction

1.1 Background and Rationale

In the real life, uncertainty is the only certain thing. As we can see in our daily life, when handling many real-world optimization problems by means of a mathematical program, it is often necessary to treat inexact or uncertain input data due to various measurement errors or estimations. The parameters are not often known exactly and most of them have to be estimated. Unfortunately, the linear programming system requires constant parameters.

There are several tools to handle uncertainty and inexactness in a mathematical programming problem, such as stochastic approach and fuzzy set approach, each bearing their own advantages and disadvantages. An interval linear program is also one of the tools for solving the real-world optimization problems under interval-valued uncertainty. It is a linear programming where its coefficients and parameters are in the pattern of intervals. Instead of approximating or estimating crisp input data, the coefficients of an interval program may perturb independently within the given lower and upper bounds.

This project provides computer programs of many types of solutions of an interval linear program by using knowledge in the course of Operations Research II and three approaches which are optimistic, pessimistic and minimax regret approaches for finding the solutions. The most appropriate solution relies on the purpose of each decision maker. When the width of interval is too large, some of these three approaches may not have a solution. We then introduce an algorithm to reduce the width of the interval parameters. We apply these algorithms to some small example of real-world problem.

1.2 Objectives

Obtain an algorithm to solve interval linear programming problems that have interval uncertain data by using optimistic, pessimistic and minimax regret approaches to describe the solutions of the problems.

1.3 Scopes

1. Explain the general idea of optimistic, pessimistic and minimax regret approaches.
2. Apply optimistic, pessimistic and minimax regret approaches in a linear programming problem with interval data.
3. Provide an application of linear programming problem with interval data.

1.4 Project Activities

1. Study linear programming problems.
2. Study the interval linear programming system.
3. Study optimistic and pessimistic approaches and apply them to an interval linear programming problem.
4. Study minimax regret approach and apply it to an interval linear programming problem.
5. Study strong solvability/feasibility of system of interval linear inequality.
6. Study relationship between regret and strong solution of an interval linear programming.
7. Use the knowledge about optimistic, pessimistic and regret appreaches to explain the solutions of an interval linear programming.

8. Build an algorithm for solving the interval linear programming problem.
9. Recheck and modify the algorithm.
10. Concludes the results and write a report.

1.5 Benefits

1. The benefits of the project owner.
   (a) Apply knowledge from Operations Research II course to solve linear programming problems and built on this knowledge to solve interval linear programming problems .
   (b) Use Python version 3.7.9 to solve the problems.
2. The benefits of project users.
   (a) Increase convenience and efficiency in each decision making, especially someone who is such a risk aversion.
   (b) Increase alternative ways for desicion making which users capable of choosing solution depend on their satisfaction.
   (c) Can use the interval linear programming method in finding many types solutions and can apply this method to many interval linear programming applications.

This project is structured as follows: Chapter 2 explains background knowledge and liturature review. In chapter 3, we review the algorithms and demonstrate how it works. Chapter 4 contains Application and Result. Lastly, Chapter 5 summarizes and concludes the project.

Chapter 2
Background knowledge

In this chapter, we present related knowledge of our project including the interval linear programing concept and literature review on listed constraints of an uncertainty problem in each research paper.

2.1 Linear programming (LP)

A linear programming problem can be formulated as follows:

$$\min z := c^t x \tag{2.1}$$

$$\text{s.t. } Ax < \cdot > \; b, \tag{2.2}$$

$$x \geq 0, \tag{2.3}$$

where c and x are n dimentional vectors, b is an m dimentional vector, A is an $m \times n$ matrix, and the relation $< \cdot >$ can be $\geq, \leq$ and $=$.

2.2 Interval linear programming (IvLP)

There are several ways to express the uncertainty of the data. One of them, which has particularly good properties from the point of view of the uses, employs the so-called interval matrices defined in this section.

If $\underline{A}, \overline{A}$ are two matrices in $M_{m\times n}(\mathbb{R})$ satisfying $\underline{A} \leq \overline{A}$, we define an $interval\ matrix$ as the set

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \{A \in M_{m\times n}(\mathbb{R}) \mid \underline{A} \leq A \leq \overline{A}\}.$$

The matrices $\underline{A}, \overline{A}$ are called the *lower* and *upper bound* of $\boldsymbol{A}$, respectively. Hence, if $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$, then $\boldsymbol{A}$ is the set of all matrices $A = (a_{ij})$ satisfying

$$\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij} \tag{2.4}$$

for $i = 1, ..., m,\ j = 1, ..., n$. It is worth nothing that each coefficient may attain any value in its interval (2.4) independently of the values taken on by other coefficients.

As shown later, in many cases it is more advantageous to express the interval matrix $\boldsymbol{A}$ in terms of the center matrix

$$A_c = \tfrac{1}{2}(\underline{A} + \overline{A}) \tag{2.5}$$

and of the radius matrix

$$\Delta = \tfrac{1}{2}(\overline{A} - \underline{A}), \tag{2.6}$$

which is always nonnegative. From (2.5), (2.6) we easily obtain

$$\underline{A} = A_c - \Delta,$$
$$\overline{A} = A_c + \Delta,$$

so that $\mathbf{A}$ can be given either as $[\underline{A}, \overline{A}]$, or as $[A_c - \Delta, A_c + \Delta]$.

An interval linear programming problem is formulated as:

$$\text{min } z := c^t x \tag{2.7}$$

$$\text{s.t. } \boldsymbol{A}x < \cdot > \ \boldsymbol{b}, \tag{2.8}$$

$$x \geq 0, \tag{2.9}$$

where x is the vector of decision variables, c is the n dimentional vector, $\boldsymbol{A}$ and $\boldsymbol{b}$ are interval matrices where all their entries are intervals, and the relation $< \cdot >$ can be $\geq, \leq$ and $=$, see more details in [1], [7] and [10]. However, in this project, we will only study the case that the relation $< \cdot >$ is $\geq$. Note that the dimensions of these vectors and matrices must be mathematically reasonable in order to perform mathematical (multiplication) operations.

A special case of an interval matrix is an interval vector which is a one column interval matrix $\boldsymbol{b}$ where

$$\boldsymbol{b} = \{b \mid \underline{b} \leq b \leq \overline{b}\} \text{ for some } \underline{b}, \overline{b} \in \mathbb{R}^m.$$

We again use the center vector

$$b_c = \tfrac{1}{2}(\underline{b} + \overline{b})$$

and the nonnegative radius vector

$$\delta = \tfrac{1}{2}(\overline{b} - \underline{b}).$$

We employ both forms $\boldsymbol{b} = [\underline{b}, \overline{b}] = [b_c - \delta, b_c + \delta]$. Notice that interval matrices and vectors are typeset in boldface letters.

2.3 Strong solvable/feasibility of inequations

For $A \in \boldsymbol{A} = [\underline{A}, \overline{A}],\ b \in \boldsymbol{b} = [\underline{b}, \overline{b}]$, a system

$$Ax \geq b$$

is called a subsystem of an interval linear programming system $\boldsymbol{A} \geq \boldsymbol{b}$. This subsystem is called *feasible* if it has a nonnegative solution or *infeasible* if it does not have any nonnegative solution. An interval linear system

$$\boldsymbol{A}x \geq \boldsymbol{b}$$

is called *strong feasibility* if each subsystem is feasible.

Theorem 2.1. A system $\boldsymbol{A}x \geq \boldsymbol{b}$ is strongly feasible if and only if the subsystem

$$\underline{A}x \geq \overline{b} \tag{2.10}$$

is feasible.

Proof. If $\boldsymbol{A}x \geq \boldsymbol{b}$ is strongly feasible, then (2.10) is feasible. Conversely, if (2.10) has a solution $x \geq 0$, then for each $A \in \boldsymbol{A}, b \in \boldsymbol{b}$ we have

$$Ax \geq \underline{A}x \geq \overline{b} \geq b;$$

hence $\boldsymbol{A}x \geq \boldsymbol{b}$ is strongly feasible. □

Theorem 2.2. A system $\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly feasible if and only if the subsystem

$$\overline{A}x \leq \underline{b} \tag{2.11}$$

is feasible.

Proof. If $\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly feasible, then (2.11) is feasible. Conversely, if (2.11) has a solution $x \geq 0$, then for each $A \in \boldsymbol{A}, b \in \boldsymbol{b}$, we have

$$Ax \leq \overline{A}x \leq \underline{b} \leq b;$$

hence $\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly feasible. □

Definition 2.3. A nonnegative solution x is called *a strong solution* if it satisfies all constrains.

Definition 2.4. We called x^* is *a strongly optimal solution* if it is an optimal solution for all constrains.

2.4 Optimistic and pessimistic approaches

Minimization interval LP problem

Let $A \in \boldsymbol{A} = [\underline{A}, \overline{A}]$ and $b \in \boldsymbol{b} = [\underline{b}, \overline{b}]$. We define $z(A, b)$ as the optimal value of Problem I.

$$\begin{aligned} \text{Problem I}: \ \min\ & c^t x \\ \text{s.t.}\ & Ax \geq b, \\ & x \geq 0. \end{aligned}$$

For any given $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$, $A = \alpha\underline{A} + (1-\alpha)\overline{A}$ for some $\alpha \in [0, 1]$ and $b = \beta\underline{b} + (1-\beta)\overline{b}$ for some $\beta \in [0, 1]$. If there exists $x \geq 0$ such that $Ax \geq b$ then

$$\overline{A}x \geq Ax \geq b \geq \underline{b}.$$

This implies that

$$\{x \mid Ax \geq b\} \subseteq \{x \mid \overline{A}x \geq \underline{b}\}, \text{ for any } A \in \boldsymbol{A} \text{ and } b \in \boldsymbol{b}. \tag{2.12}$$

From (2.12), we perceive that the set $\{x \mid \overline{A}x \geq \underline{b}\}$ is the largest feasible set. Therefore, $z(\overline{A}, \underline{b}) \leq z(A, b)$.

On the other hand, if there exists $x \geq 0$ such that $\underline{A}x \geq \overline{b}$, then

$$Ax \geq \underline{A}x \geq \overline{b} \geq b, \text{ for any } A \in \boldsymbol{A}, b \in \boldsymbol{b}.$$

This implies that

$$\{x \mid \underline{A}x \geq \overline{b}\} \subseteq \{x \mid Ax \geq b\}, \text{ for any } A \in \boldsymbol{A}, b \in \boldsymbol{b}. \tag{2.13}$$

From (2.13), the set $\{x \mid \underline{A}x \geq \overline{b}\}$ is the smallest feasible set. Thereby, in this case $z(\underline{A}, \overline{b}) \geq z(A, b)$.

Theorem 2.5. For any given $A \in \boldsymbol{A}, b \in \boldsymbol{b}$ if the subsystem $\overline{A}x \geq \underline{b}$ is infeasible for all $x \geq 0$ then the subsystem $Ax \geq b$ and $\underline{A}x \geq \overline{b}$ are infeasible for all $x \geq 0$.

Proof. Suppose the subsystem $\overline{A}x \geq \underline{b}$ is infeasible for all $x \geq 0$ then $\overline{A}x < \underline{b}$ for all $x \geq 0$. Since $\overline{A} \geq A \geq \underline{A}$ then $\overline{A}x \geq Ax \geq \underline{A}x$ for all $x \geq 0$. Since $\underline{b} \leq b \leq \overline{b}$ and $\overline{A}x < \underline{b}$ then, for all $x \geq 0$,

$$\underline{A}x \leq Ax \leq \overline{A}x < \underline{b} \leq b \leq \overline{b}.$$

So, $Ax < b$ and $\underline{A}x < \overline{b}$.
Hence, $Ax \geq b$ and $\underline{A}x \geq \overline{b}$ are infeasible for all $x \geq 0$. □

2.4.1 An optimistic objective value function of minimization interval LP problem

The optimistic objective value of the interval linear programming Problem I is the best of all optimal values $z(A, b)$ for all $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Therefore, the optimistic objective value is

$$z(\overline{A}, \underline{b}) := \min\{z(A, b), A \in \boldsymbol{A}, b \in \boldsymbol{b}\},$$

if for each $A \in \boldsymbol{A}, b \in \boldsymbol{b}, \{x \mid Ax \geq b\} \neq \varnothing$.

2.4.2 A pessimistic objective value function of minimization interval LP problem

The pessimistic objective value of the interval linear programming Problem I is the worst of all optimal values $z(A,b)$ for all $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. As a result, the pessimistic objective value is

$$z(\underline{A},\overline{b}) := \max\{z(A,b), A \in \boldsymbol{A}, b \in \boldsymbol{b}\},$$

if there exists some $x \geq 0$ such that $\underline{A}x \geq \overline{b}$.

Maximization interval LP problem

In a linear programing problem, the objective of the problem can be either to maximize or to minimize. Since we start with a minimization problem which is min $c^t x$ subject to $x \in \{x \mid Ax \geq b, x \geq 0, A \in \boldsymbol{A}, b \in \boldsymbol{b}\}$, an equivalent maximization problem is max $-c^t x$ subject to $x \in \{x \mid Ax \geq b, x \geq 0, A \in \boldsymbol{A}, b \in \boldsymbol{b}\}$. Therefore, minimizing $-c^t x$ is the same as maximizing $c^t x$ and any solution to the maximization problem will be a solution to the minimization problem.

In conclusion, for changing a minimization problem to a maximization problem, we just multiply the objective function by -1.

2.5 Minimax regret model

Let $\mathcal{S} = \{x \mid Ax \geq b, x \geq 0, A \in \boldsymbol{A}, b \in \boldsymbol{b}\}$. The regret function $r(x)$ shows how the optimal value $c^t x$ with respect to a candidate solution[†] $x \in \mathcal{S}$ differ from the true objective value $z(A,b)$, i.e.,

$$r(x) = c^t x - z(A,b).$$

When the true objective is undiscovered, the worst (maximum) regret among all regret $r(x)$ can be defined by

[†] A candidate solution x is a member in a set of possible solutions to a given problem. A candidate solution does not have to be a likely or reasonable solution to the problem – it is simply in the set that satisfies all constraints.

$$R(x) = \max_{x \in \mathcal{S}} r(x)$$

(see [2] and [8] for an overview of the minimax regret). Since a candidate solution x has to satisfy the condition $\underline{A}x \geq \overline{b}$, a general minimax regret model of minimization interval LP problem is

$$\begin{aligned} \min\ & R \\ \text{s.t. } R &\geq c^t x - z(A, b), \\ \underline{A}x &\geq \overline{b}, \\ x &\geq 0. \end{aligned}$$

Since $z(\overline{A}, \underline{b}) := \min\{z(A, b) \mid Ax \geq b, A \in \boldsymbol{A}, b \in \boldsymbol{b}\}$, if each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $\{x \mid Ax \geq b\} \neq \varnothing$ and $R(x) \geq c^t x - z(A, b)$, then the minimum possible $R(x)$ is $c^t x - z(\overline{A}, \underline{b})$. As a result, a concisely general minimax regret model of minimization interval LP problem is

$$\min\ R \tag{2.14}$$

$$\text{s.t. } R \geq c^t x - z(\overline{A}, \underline{b}), \tag{2.15}$$

$$\underline{A}x \geq \overline{b}, \tag{2.16}$$

$$x \geq 0. \tag{2.17}$$

Since one of the constraints of the minimax regret model of minimization IvLP problem (2.16) is a duplicate of the constraint of the pessimistic model of minimization IvLP problem, it follows that the minimax regret IvLP problem has to be infeasible if the pessimistic IvLP problem is infeasible.

2.6 A reduced interval linear programming

As explained earlier, the minimax regret IvLP problem will be infeasible if the pessimistic IvLP problem is infeasible. Thus, reducing the size of interval matrix may result in the feasibility of the pessimistic IvLP problem as follows:

Let i and n be the natural numbers where $i \leq n$.

Define $\boldsymbol{A}_i = [\underline{A}_i, \overline{A}_i],\ \underline{A}_i = \underline{A} + (\Delta \times \frac{i}{n})$ and $\overline{A}_i = \overline{A} - (\Delta \times \frac{i}{n})$. It is obvious that when $i = n,\ \underline{A}_n = \underline{A} + \Delta = A_c$ and $\overline{A}_n = \overline{A} - \Delta = A_c$.

Define $\boldsymbol{b}_i = [\underline{b}_i, \overline{b}_i],\ \underline{b}_i = \underline{b} + (\delta \times \frac{i}{n})$ and $\overline{b}_i = \overline{b} - (\delta \times \frac{i}{n})$. When $i = n,\ \underline{b}_n = \underline{b} + \delta = b_c$ and $\overline{b}_n = \overline{b} - \delta = b_c$.

Reducing the size of the interval is that each interval is divided into n portions. After that, each interval will be simultaneously increased value of the lower bound and decreased value of the upper bound with $\frac{i}{n}$. In this project, the interval will be reduced not more than 10 times ($n = 10$). Hence, we have the reduced IvLP as shown below:

$$\begin{aligned} \min\ z &:= c^t x \\ \text{s.t. } \boldsymbol{A}_i x &\geq \boldsymbol{b}_i, \\ x &\geq 0, \end{aligned}$$

where x is the vector of decision variables, c is n dimentional vector, $\boldsymbol{A}_i$ and $\boldsymbol{b}_i$ are i^{th} reduced interval matrices where all their entries are intervals. In addition, when $i = 10$, the 10^{th} reduced IvLP problem will be become as a LP problem.

As described earlier, if there exists some $x \geq 0$ such that $\underline{A}x \geq \overline{b}$ then the set $\{x \mid \underline{A}x \geq \overline{b}\}$ is the smallest feasible set. On the contrary, if this set is inadequate then the increment of the smallest feasible set becomes a main concept of the reduced IvLP.

Assume there exists some $x \geq 0$ such that $\underline{A}_i x \geq \overline{b}_i$, for some $\underline{A}_i \in \boldsymbol{A}_i$ and $\overline{b}_i \in \boldsymbol{b}_i$, where $i \leq n$ and $n = 10$. Since $Ax \geq \underline{A}_i x \geq \overline{b}_i \geq b$ for any $A \in \boldsymbol{A}_i, b \in \boldsymbol{b}_i$, $\{x \mid \underline{A}_i x \geq \overline{b}_i\} \subseteq \{x \mid Ax \geq b\}$. This implies that $\{x \mid \underline{A}_i x \geq \overline{b}_i\}$ is the smallest feasible set of i^{th} reduced IvLP and $z(\underline{A}_i, \overline{b}_i) \geq z(A, b)$.

Since the minimax regret IvLP problem will be infeasible if the pessimistic IvLP problem is infeasible, the concept of reduced IvLP is used for solving the problem when its original is infeasible. The pessimistic objective value function and the minimax regret model of i^{th} reduced IvLP Problem I are

$$z(\underline{A}_i, \overline{b}_i) := \max\{z(A, b) \mid A \in \boldsymbol{A}_i, b \in \boldsymbol{b}_i\} \text{ where } i \leq n \text{ and } n = 10,$$

if there exists some $x \geq 0$ such that $\underline{A}_i x \geq \overline{b}_i$ and

$$\begin{aligned} \text{min } R & & \\ \text{s.t. } R &\geq c^t x - z(\overline{A}_i, \underline{b}_i), \\ \underline{A}_i x &\geq \overline{b}_i, \\ x &\geq 0, \end{aligned}$$

respectively. We can conclude that the Problem I does not have minimax regret if its pessimistic or minimax regret of reduced IvLP is still infeasible while $i = n$.

Chapter 3
Algorithms and results

In the previous chapter, we show the inteval linear programming. Next, we will show how to use Python version 3.7.9 to solve the interval linear programming problems and results from these algorithms.

3.1 Python version 3.7.9

Python is an interpreted, object-oriented, high-level programming language with dynamic semantics. Its high-level built in data structures, combined with dynamic typing and dynamic binding, make it very attractive for Rapid Application Development, as well as for use as a scripting or glue language to connect existing components together. Python's simple, easy to learn syntax emphasizes readability and therefore reduces the cost of program maintenance. Python supports modules and packages, which encourages program modularity and code reuse. The Python interpreter and the extensive standard library are available in source or binary form without charge for all major platforms, and can be freely distributed, see more details in [4].

3.1.1 PuLP

PuLP is a library for the Python scripting language that enables users to describe mathematical programs. PuLP works entirely within the syntax and natural idioms of the Python language by providing Python objects that represent optimization problems and decision variables, and allowing constraints to be expressed in a way that is very similar to the original mathematical expression.

PuLP can easily be deployed on any system that has a Python interpreter, as it has no dependencies on any other software packages. It supports a wide range of both commercial and open-source solvers, and can be easily extended to support additional solvers. Finally, it is available under a permissive open-source license that encourages and facilitates the use of PuLP inside other projects that need linear optimization capabilities, see more details in [3].

3.1.2 NumPy

NumPy is the fundamental package for scientific computing in Python. It is a Python library that provides a multidimensional array object, various derived objects (such as masked arrays and matrices), and an assortment of routines for fast operations on arrays, including mathematical, logical, shape manipulation, sorting, selecting, I/O, discrete Fourier transforms, basic linear algebra, basic statistical operations, random simulation and much more, see more details in [6].

3.1.3 Pandas

Pandas is a high-level data manipulation tool developed by Wes McKinney. It is built on the Numpy package and its key data structure is called the DataFrame. DataFrames allow you to store and manipulate tabular data in rows of observations and columns of variables, see more details in [5].

3.2 Algorithms

Task 1: Write a program that asks the user for a type of problem.

$1 =$ minimization problem

$2 =$ maximization problem

Output message displaying 1 or 2

Task 2: Write programs that asks the user for the dimensions of an interval matrix $\boldsymbol{A}$.

$M =$ the number of rows of an interval matrix $\boldsymbol{A}$

$N =$ the number of columns of an interval matrix $\boldsymbol{A}$

Output message displaying the numbers of rows and columns of $\boldsymbol{A}$

Task 3: Write programs that ask the user for the interval coefficients of a known interval matrix $\boldsymbol{A}$, an interval vector $\boldsymbol{b}$ and the coefficients of a known vector c, respectively.

$a_l, a_u =$ lower bound, upper bound of each interval in interval matrix $\boldsymbol{A}$

Print the arrays of $\boldsymbol{A}$, center matrix of $\boldsymbol{A}$, lower bound matrix of $\boldsymbol{A}$, upper bound matrix of $\boldsymbol{A}$ and radius matrix of $\boldsymbol{A}$

$b_l, b_u =$ lower bound, upper bound of each interval in interval matrix $\boldsymbol{b}$

Print the arrays of $\boldsymbol{b}$, center matrix of $\boldsymbol{b}$, lower bound matrix of $\boldsymbol{b}$, upper bound matrix of $\boldsymbol{b}$ and radius matrix of $\boldsymbol{b}$

$c =$ each entry of matrix c

Print the array of matrix c

Task 4: Write programs that find an optimistic solution.

if the optimistic solution does not exist

stop finding

Print "This interval linear programming problem is infeasible."

else

Print the optimistic objective value and its solutions

Continue finding a pessimistic solution

Task 5: Write programs that find a pessimistic solution.

if the pessimistic solution exists

Continue finding a minimax regret solution

if the minimax regret exists

Print the minimax regret and its solution

else

Print "This interval linear programming problem does not have minimax regret solution.”

else

Print “This interval linear programming problem does not have pessimistic and minimax regret solutions.”

Reduce the size of each interval entry o $\boldsymbol{A}$ and $\boldsymbol{b}$

Find a pessimistic solution of the reduced IvLP

if the pessimistic solution of the reduced IvLP exists

Continue findind a minimax regret solution of the reduced IvLP

if the minimax regret of the reduced IvLP exists

Print the minimax regret of the reduced IvLP and its solution

else

Print “This interval linear programming problem does not have pessimistic and minimax regret solutions.”

else

The reduced pessimistic problem still infeasible while the entries of the interval matrices $\boldsymbol{A}$ and $\boldsymbol{b}$ are the center matrices which are A_c and b_c, respectively.

Print “This interval linear programming problem does not have pessimistic and minimax regret solutions.”

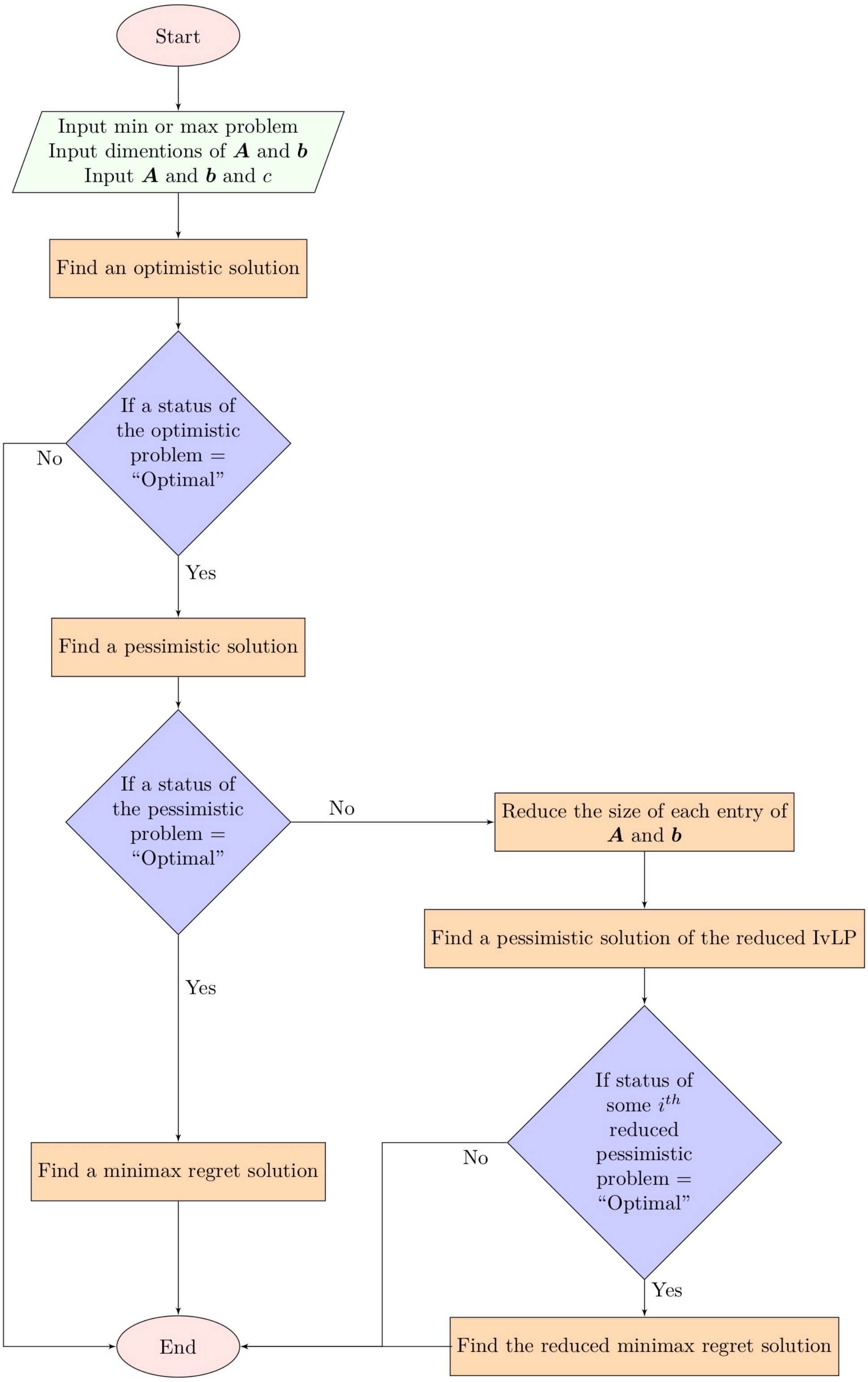

Figure 3.1: Flowchart of Python algorithms

3.3 Example

Consider the following interval linear programming problem:

$$\begin{aligned} &\text{max } x_1 + 3x_2 \\ \text{s.t. } [4,6]x_1 + [-8,-6]x_2 \leq &\ [-10,-8], \\ [9,11]x_1 + [11,13]x_2 \leq &\ [19,21], \\ x_1 \text{ and } x_2 &\geq 0. \end{aligned}$$

$$\boldsymbol{A} = \begin{pmatrix} [4,6] & [-8,-6] \\ [9,11] & [11,13] \end{pmatrix}_{2\times 2}$$

$$\boldsymbol{b} = \begin{pmatrix} [-10,-8] \\ [19,21] \end{pmatrix}_{2\times 1} \text{ and } c = \begin{pmatrix} 1 \\ 3 \end{pmatrix}_{2\times 1}$$

Step 1: Choose either minimization IvLP or maximization IvLP.

```
 Minimization problem -> Input 1
 Maximization problem -> Input 2

Min or Max: 2
```

Step 2: Input the dimensions of an interval matrix $\boldsymbol{A}$.

```
Input number of rows of matrix A: 2

Input number of columns of matrix A: 2
```

Step 3: Input the interval coefficients of a known interval matrix $\boldsymbol{A}$, an interval vector $\boldsymbol{b}$ and the coefficients of a known vector c, respectively.

```
Please, input each interval in a matrix A with a comma in between a lower
bound and an upper bound of each interval.
 Ex. Assume [a b] is an interval in matrix A. User has to input "a,b".

Enter 2 numbers (with a comma in between): 4,6

Enter 2 numbers (with a comma in between): -8,-6

Enter 2 numbers (with a comma in between): 9,11

Enter 2 numbers (with a comma in between): 11,13
Matrix A is [[[4.0, 6.0], [-8.0, -6.0]], [[9.0, 11.0], [11.0, 13.0]]]
Center matrix of A = [[5.0, -7.0], [10.0, 12.0]]
Lower bound matrix of A = [[4.0, -8.0], [9.0, 11.0]]
Upper bound matrix of A = [[6.0, -6.0], [11.0, 13.0]]
Radius matrix of A = [[1.0, 1.0], [1.0, 1.0]]
```

```
Please, input each interval in a matrix B with a comma in between a lower
bound and an upper bound of each interval.
 Ex. Assume [a b] is an interval in matrix B. User has to input "a,b".

Enter 2 numbers (with a comma in between): -10,-8

Enter 2 numbers (with a comma in between): 19,21
Matrix B is [[-10.0, -8.0], [19.0, 21.0]]
Center matrix of B = [-9.0, 20.0]
Lower bound matrix of B = [-10.0, 19.0]
Upper bound matrix of B = [-8.0, 21.0]
Radius matrix of B = [1.0, 1.0]
```

```
Please, input data in matrix C

Enter number: 1

Enter number: 3
Matrix C = [1.0, 3.0]
```

Step 4: Find an optimistic solution.

I Stop finding the rest of the solutions if the optimistic solution does not exist.

II Continue finding a pesstimistic solution if the optimistic solution of the original IvLP exists.

```
Optimistic obj. value fn of maximization interval LP:
MAXIMIZE
1.0*X_1 + 3.0*X_2 + 0.0
SUBJECT TO
_C1: 4 X_1 - 8 X_2 <= -8

_C2: 9 X_1 + 11 X_2 <= 21

VARIABLES
X_1 Continuous
X_2 Continuous

Status Optimal
Optimistic obj. value fn of this maximization interval LP is: 5.7272727

X_1 = 0.0
X_2 = 1.9090909
```

i. Find a minimax regret solution if the pessimistic solution of the original IvLP exists.

ii. If the pessimistic solution of the original IvLP does not exist then reduce the size of each entry of interval matrices $\boldsymbol{A}$ and $\boldsymbol{b}$ to find a pessimistic solution of the reduced IvLP.

```
Pessimistic obj. value fn of maximization interval LP:
MAXIMIZE
1.0*X_1 + 3.0*X_2 + 0.0
SUBJECT TO
_C1: 6 X_1 - 6 X_2 <= -10

_C2: 11 X_1 + 13 X_2 <= 19

VARIABLES
X_1 Continuous
X_2 Continuous

Status Infeasible
```

A. If the entries of the interval matrices $\boldsymbol{A}$ and $\boldsymbol{b}$ are the center matrices (which are A_c and b_c) and the reduced pessimistic problem is still infeasible then this problem does not have the minimax regret solution.

B. If the solution of the reduced pessimistic problems exists then continue finding the reduced minimax regret solution.

```
This interval linear programming does not have minimax regret.
Notice: We are restricting the bounds of interval matrix A and
searching for the solution of new approximated minimax regret
model solution.

(1st)

Optimistic obj. value fn of the new maximization interval LP:
MAXIMIZE
1.0*X_1 + 3.0*X_2 + 0.0
SUBJECT TO
_C1: 4.1 X_1 - 7.9 X_2 <= -8.1

_C2: 9.1 X_1 + 11.1 X_2 <= 20.9

VARIABLES
X_1 Continuous
X_2 Continuous

Status:  Optimal

New optimistic obj. value fn of maximization interval LP is:
5.6486487

X_1 = 0.0
X_2 = 1.8828829
Pessimistic obj. value fn of the new maximization interval LP:
MAXIMIZE
1.0*X_1 + 3.0*X_2 + 0.0
SUBJECT TO
_C1: 5.9 X_1 - 6.1 X_2 <= -9.9

_C2: 10.9 X_1 + 12.9 X_2 <= 19.1

VARIABLES
X_1 Continuous
X_2 Continuous

Status:  Infeasible
```
⋮

```
(4th)

Optimistic obj. value fn of the new maximization interval LP:
MAXIMIZE
1.0*X_1 + 3.0*X_2 + 0.0
SUBJECT TO
_C1: 4.4 X_1 - 7.6 X_2 <= -8.4

_C2: 9.4 X_1 + 11.4 X_2 <= 20.6

VARIABLES
X_1 Continuous
X_2 Continuous

Status:  Optimal

New optimistic obj. value fn of maximization interval LP is:
5.4210525

X_1 = 0.0
X_2 = 1.8070175
Pessimistic obj. value fn of the new maximization interval LP:
MAXIMIZE
1.0*X_1 + 3.0*X_2 + 0.0
SUBJECT TO
_C1: 5.6 X_1 - 6.4 X_2 <= -9.6

_C2: 10.6 X_1 + 12.6 X_2 <= 19.4

VARIABLES
X_1 Continuous
X_2 Continuous

Status:  Optimal
New pessimistic obj. value fn of maximization interval LP is:
4.619047500000001

X_1 = 0.0
X_2 = 1.5396825
```

```
Minimax regret of new maximization interval linear programming problem:
MINIMIZE
1*R_1 + 0
SUBJECT TO
_C1: 5.6 X_1 - 6.4 X_2 <= -9.6

_C2: 10.6 X_1 + 12.6 X_2 <= 19.4

_C3: - R_1 - X_1 - 3 X_2 <= -5.4210525

VARIABLES
R_1 Continuous
X_1 Continuous
X_2 Continuous

Optimal
Minimax regret is: 0.80200488
R_1 = 0.80200488
X_1 = 0.0
X_2 = 1.5396825
```

Step 5: Conclude the results.

3.4 Results

From the above example, after reducing the size of each interval 4 times, the 4^{th} pessimistic IvLP is feasible. Thus, we continue to find the minimax regret of the 4^{th} reduced IvLP and it exists. We can notice that if the minimax regret exists then the result is the difference between pessimistic objective value and optimistic objective value as follows:

the optimistic objective value of the 4^{th} reduced IvLP is 5.4210525

the pessimistic objective value of the 4^{th} reduced IvLP is 4.6190475

the minimax regret of the 4^{th} reduced IvLP is 0.802005

the optimistic objective value of the 4^{th} reduced IvLP $-$ the pessimistic objective value of the 4^{th} reduced IvLP is $5.4210525 - 4.61904750 = 0.802005$.

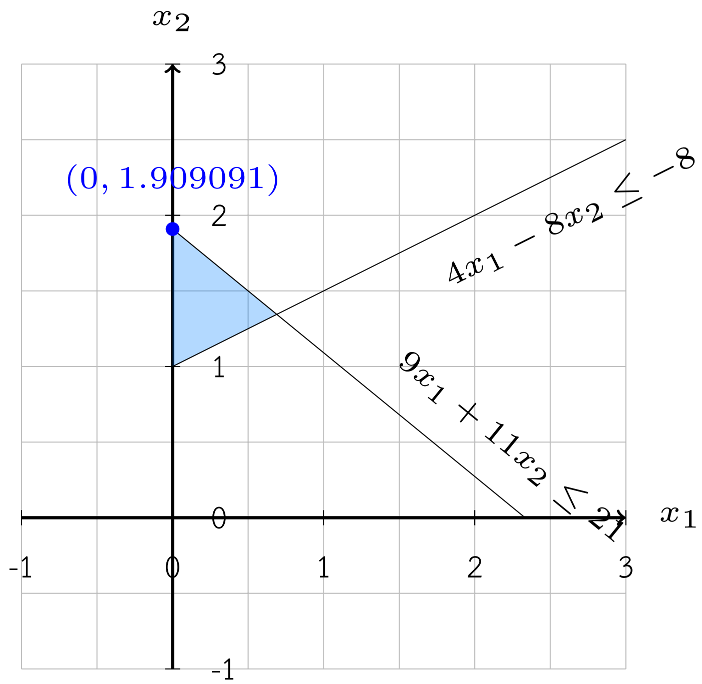

Figure 3.2: the optimistic IvLP of Example

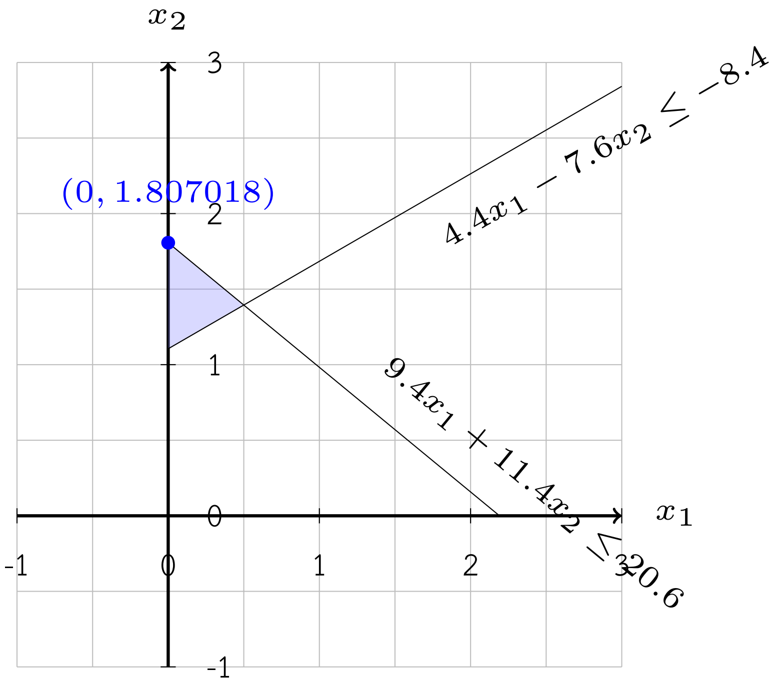

Figure 3.3: the 4^{th} optimistic reduced IvLP of Example

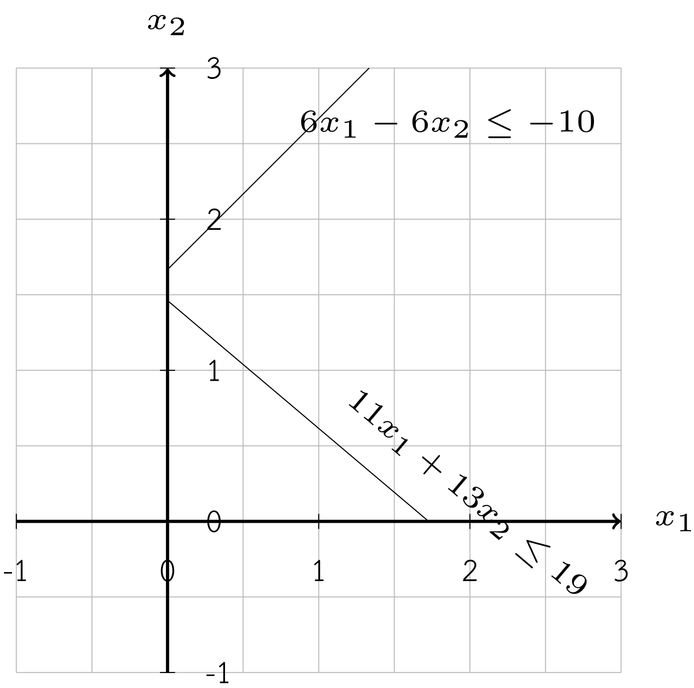

Figure 3.4: the pessimistic IvLP of Example

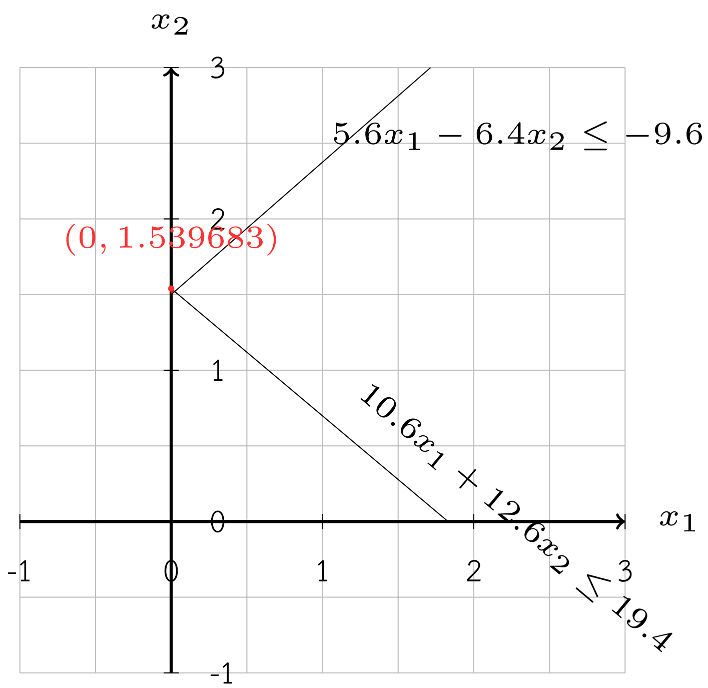

Figure 3.5: the 4^{th} pessimistic reduced IvLP of Example

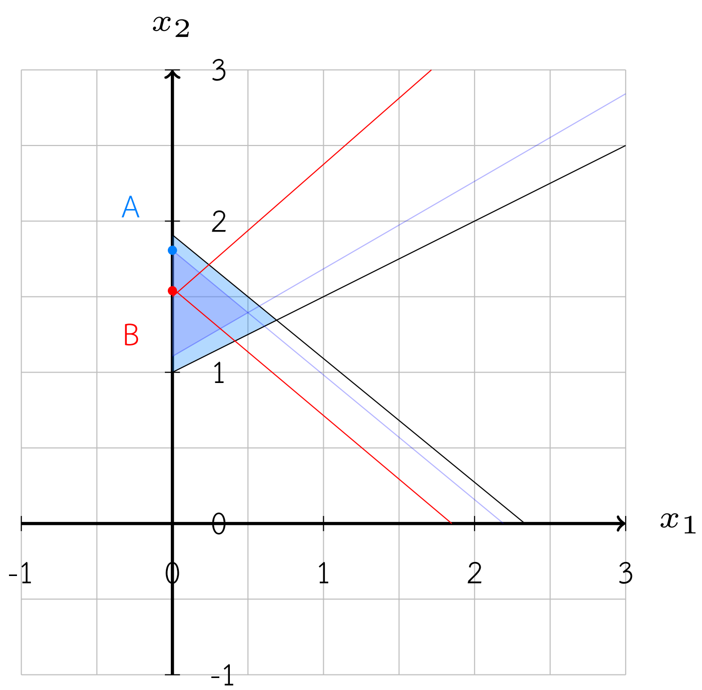

Figure 3.6: the optimistic, the 4^{th} optimistic and the 4^{th} pessimistic reduced IvLP of Example

From those above graphs, the distance between points A and B is minimax regret (min R) of 4^{th} reduced IvLP, if the 4^{th} pessimistic reduced IvLP exists. Moreover, those graphs reveal that reducing the size of each interval in interval matrices $\boldsymbol{A}$ and $\boldsymbol{b}$ result in the bigger feasible region of the pessimistic IvLP.

Chapter 4
Applications

4.1 Industrial applications

Product mix problem

The Healthy Pet Food Company manufactures two types of dog food: Meaties and Yummies. Each package of Meaties contains 1.5-2 pounds of cereal and 2-3 pounds of meat; each package of Yummies contains 2-3 pounds of cereal and 1-1.5 pounds of meat. The company would like to make its monthly profit as much as possible and it believes that it can sell as much of each dog food as it can make. Meaties sell for $2.80 per package and Yummies sell for $2.00 per package.

	Meaties	Yummies
Sales price per package	$2.80	$2.00
Raw materials per package		
Cereal	1.5-2.0 lb.	2.0-3.0 lb.
Meat	2.0-3.0 lb.	1.0-1.5 lb.
Variable cost-blending and packing	$0.25 package	$0.20 package
Resources		
Production capacity for Meaties	90,000 packages per month	
Cereal available per month	400,000-450,000 lb.	
Meat available per month	300,000-320,000 lb.	

Table 4.1: Healthy Pet Food Data

Healthy's production is limited in several ways. First, the company can buy up to 400,000-450,000 pounds of cereal each month at $0.20 per pound. It can buy up to 300,000-320,000 pounds of meat per month at $0.50 per pound. In addition, a special piece of machinery is required to make Meaties, and this machine has a capacity of 90,000 packages per month. The variable cost of blending and packing the dog food is $0.25 per package for Meaties and $0.20 per package for Yummies.

Solution:

In this problem, we have direct control over two quantities: the number of packages of Meaties to make each month, and the number of packages of Yummies to make each month. Thus, we designate the decision variables by the symbols x_1 and x_2 as follows:

x_1 = the number of packages of Meaties to make each month

x_2 = the number of packages of Yummies to make each month.

Objective function

The profit earned by this company is a direct function of the amount of each dog food made, sold and the decision variables. Monthly profit, designated as z, is written as follows:

$z :=$ (profit per package of Meaties) $\times$ (number of packages of Meaties made and sold monthly) $+$ (profit per package of Yummies) $\times$ (number of packages of Yummies made and sold monthly)

The profit per package for each dog food is the difference between selling price and expenses as shown in Table 4.2.

	Selling price	**Expenses**			**Profit per package**
		Meat	Cereal	Blending	
Meaties	2.80	1.50	0.40	0.25	**0.65**
Yummies	2.00	0.75	0.60	0.20	**0.45**

Table 4.2: The profit per package for each dog food

We write the monthly profit as $z := 0.65x_1 + 0.45x_2$.

Constraints

Let's begin with the availability of cereal constraint:

$$\begin{aligned} \text{(The number of lb. of cereal used in production each month)} &\leq [400000, 450000] \text{ lb.,} \\ [1.5, 2.0]x_1 + [2.0, 3.0]x_2 &\leq [400000, 450000] \text{ lb.} \end{aligned}$$

Using similar reasoning, the restriction on the availability of meat is expressed as

$$\begin{aligned} \text{(The number of lb. of meat used in production each month)} &\leq [300000, 320000] \text{ lb.,} \\ [2.0, 3.0]x_1 + [1.0, 1.5]x_2 &\leq [300000, 320000] \text{ lb.} \end{aligned}$$

In addition to these constraints,

$$\begin{aligned} \text{(The number of packages of Meaties produced each month)} &\leq 90000 \text{ packages,} \\ x_1 &\leq 90000 \text{ packages.} \end{aligned}$$

Finally, negative production levels do not make sense, so we require that $x_1 \geq 0$ and $x_2 \geq 0$.

The above described problem can be formulated as follows:

$$\begin{aligned} \max\ 0.65x_1 + 0.45x_2 & \\ \text{s.t. } [1.5, 2.0]x_1 + [2.0, 3.0]x_2 &\leq [400000, 450000], \\ [2.0, 3.0]x_1 + [1.0, 1.5]x_2 &\leq [300000, 320000], \\ x_1 &\leq 90000, \\ x_1 \text{ and } x_2 &\geq 0. \end{aligned}$$

```
Optimistic obj. value fn of maximization interval LP:
MAXIMIZE
0.65*X_1 + 0.45*X_2 + 0.0
SUBJECT TO
_C1: 1.5 X_1 + 2 X_2 <= 450000

_C2: 2 X_1 + X_2 <= 320000

_C3: X_1 <= 90000

VARIABLES
X_1 Continuous
X_2 Continuous

Status Optimal
Optimistic obj. value fn of this maximization interval LP is: 125000.0

X_1 = 76000.0
X_2 = 168000.0
```

```
Pessimistic obj. value fn of maximization interval LP:
MAXIMIZE
0.65*X_1 + 0.45*X_2 + 0.0
SUBJECT TO
_C1: 2 X_1 + 3 X_2 <= 400000

_C2: 3 X_1 + 1.5 X_2 <= 300000

_C3: X_1 <= 90000

VARIABLES
X_1 Continuous
X_2 Continuous

Status Optimal
Pessimistic obj. value fn of this maximization interval LP is: 77500.0

X_1 = 50000.0
X_2 = 100000.0
```

```
Minimax regret of this interval linear programming problem:
MINIMIZE
1*R_1 + 0
SUBJECT TO
_C1: 2 X_1 + 3 X_2 <= 400000

_C2: 3 X_1 + 1.5 X_2 <= 300000

_C3: X_1 <= 90000

_C4: - R_1 - 0.65 X_1 - 0.45 X_2 <= -125000

VARIABLES
R_1 Continuous
X_1 Continuous
X_2 Continuous

Optimal
Minimax regret is: 47500.0
R_1 = 47500.0
X_1 = 50000.0
X_2 = 100000.0
```

Figure 4.1: Solutions of Healthy Pet Food problem

In each production, if the company can make each package of Meaties contains 1.5 pounds of cereal and 2 ponds of meat, each package of Yummies contains 2 pounds cereal and 1 pounds of meat and it can buy up to 450,000 pounds of cereal and 320,000 pounds of meat per month, then the company will earn a monthly maximum profit of $125,000. That is, the company should make 76,000 packages of Meaties and 168,000 packages of Yummies each month. On the other hand, if the company can make each package of Meaties contains 2 pounds of cereal and 3 ponds of meat, each package of Yummies contains 3 pounds cereal and 1.5 pounds of meat and it can buy up to 400,000 pounds of cereal and 300,000 pounds of meat per month, then it will earn a monthly profit of $77,500. That is, the company should make 76,000 packages of Meaties and 168,000 packages of Yummies each month where the difference of monthly profits between these two situations (optimistic and pessimistic situations) is $47,500.

4.2 Miscellaneous application

Diet problem

The diet problem can be easily stated as follows:

Minimize the cost of food eaten during one day

Subject to the requirements that the diet satisfy a person's nutritional requirements and that not too few of any one food be eaten.

Consider the problem of diet optimization. There are six different foods: bread, milk, cheese, potato, fish, and yogurt. The cost and nutrition values per unit are displayed in Table 4.3. The objective is to find a minimum-cost diet that contains at least 280-300 calories, not less than 7.5-9 grams of protein, not less than 8-10 grams of carbohydrates, and not less than 5-8 grams of fat. In addition, the diet should contain at least 1 unit of fish and no more than 1 unit of milk.

	Bread	Milk	Cheese	Potato	Fish	Yogurt
Variables	x_1	x_2	x_3	x_4	x_5	x_6
Protein, g	4-5	8-8.5	7-8.5	1.3-1.6	8-9	9.2-9.4
Fat, g	1-2	5-7	9-9.5	0.1-0.2	7-8	1-2
Carbohydrates, g	15-17	11.7-12	0.4–0.6	22.6-22.8	0	17-17.5
Calories, kcal	90-92.5	120-125	106-109	97-100	130-134	180 -183

Table 4.3: Cost and nutrition values

The above data can be formulated as interval linear programming as follows:

$$
\begin{aligned}
\text{min } 2x_1 + 3.5x_2 + 8x_3 + 1.5x_4 + 11x_5 + x_6 & \\
\text{s.t. } [4.0, 5.0]x_1 + [8.0, 8.5]x_2 + [7.0, 8.5]x_3 + [1.3, 1.6]x_4 & \\
+[8.0, 9.0]x_5 + [9.2, 9.4]x_6 & \geq [7.5, 9.0], \\
[1.0, 2.0]x_1 + [5.0, 7.0]x_2 + [9.0, 9.5]x_3 + [0.1, 0.2]x_4 & \\
+[7.0, 8.0]x_5 + [1.0, 2.0]x_6 & \geq [5.0, 8.0], \\
[15, 17]x_1 + [11.7, 12.0]x_2 + [0.4, 0.6]x_3 + [22.6, 22.8]x_4 & \\
+[17.0, 17.5]x_6 & \geq [8.0, 10.0], \\
[90, 92.5]x_1 + [120, 125]x_2 + [106, 109]x_3 + [97, 100]x_4 & \\
+[130, 134]x_5 + [180, 183]x_6 & \geq [280, 300], \\
x_5 & \geq 1, \\
x_2 & \leq 1, \\
x_1, x_2, x_3, x_4, x_5 \text{ and } x_6 & \geq 0.
\end{aligned}
$$

```
Optimistic obj. value fn of minimization interval LP:
MINIMIZE
2.0*X_1 + 3.5*X_2 + 8.0*X_3 + 1.5*X_4 + 11.0*X_5 + 1.0*X_6 + 0.0
SUBJECT TO
_C1: 5 X_1 + 8.5 X_2 + 8.5 X_3 + 1.6 X_4 + 9 X_5 + 9.4 X_6 >= 7.5

_C2: 2 X_1 + 7 X_2 + 9.5 X_3 + 0.2 X_4 + 8 X_5 + 2 X_6 >= 5

_C3: 17 X_1 + 12 X_2 + 0.6 X_3 + 22.8 X_4 + 17.5 X_6 >= 8

_C4: 92.5 X_1 + 125 X_2 + 109 X_3 + 100 X_4 + 134 X_5 + 183 X_6 >= 280

_C5: X_5 >= 1

_C6: - X_2 >= -1

VARIABLES
X_1 Continuous
X_2 Continuous
X_3 Continuous
X_4 Continuous
X_5 Continuous
X_6 Continuous

Status:  Optimal
Optimistic obj. value fn of minimization interval LP is: 11.79781421

X_1 = 0.0
X_2 = 0.0
X_3 = 0.0
X_4 = 0.0
X_5 = 1.0
X_6 = 0.79781421
```

```
Pessimistic obj. value fn of minimization interval LP:
MINIMIZE
2.0*X_1 + 3.5*X_2 + 8.0*X_3 + 1.5*X_4 + 11.0*X_5 + 1.0*X_6 + 0.0
SUBJECT TO
_C1: 4 X_1 + 8 X_2 + 7 X_3 + 1.3 X_4 + 8 X_5 + 9.2 X_6 >= 9

_C2: X_1 + 5 X_2 + 9 X_3 + 0.1 X_4 + 7 X_5 + X_6 >= 8

_C3: 15 X_1 + 11.7 X_2 + 0.4 X_3 + 22.6 X_4 + 17 X_6 >= 10

_C4: 90 X_1 + 120 X_2 + 106 X_3 + 97 X_4 + 130 X_5 + 180 X_6 >= 300

_C5: X_5 >= 1

_C6: - X_2 >= -1

VARIABLES
X_1 Continuous
X_2 Continuous
X_3 Continuous
X_4 Continuous
X_5 Continuous
X_6 Continuous

Status:  Optimal
Pessimistic obj. value fn of minimization interval LP is: 11.9807692355
X_1 = 0.0
X_2 = 0.012820513
X_3 = 0.0
X_4 = 0.0
X_5 = 1.0
X_6 = 0.93589744


Minimax regret of this interval linear programming problem:
MINIMIZE
1*R_1 + 0
SUBJECT TO
_C1: 4 X_1 + 8 X_2 + 7 X_3 + 1.3 X_4 + 8 X_5 + 9.2 X_6 >= 9

_C2: X_1 + 5 X_2 + 9 X_3 + 0.1 X_4 + 7 X_5 + X_6 >= 8

_C3: 15 X_1 + 11.7 X_2 + 0.4 X_3 + 22.6 X_4 + 17 X_6 >= 10

_C4: 90 X_1 + 120 X_2 + 106 X_3 + 97 X_4 + 130 X_5 + 180 X_6 >= 300

_C5: X_5 >= 1

_C6: - X_2 >= -1

_C7: - R_1 + 2 X_1 + 3.5 X_2 + 8 X_3 + 1.5 X_4 + 11 X_5 + X_6 <= 11.79781421

VARIABLES
R_1 Continuous
X_1 Continuous
X_2 Continuous
X_3 Continuous
X_4 Continuous
X_5 Continuous
X_6 Continuous

Optimal
Minimax regret is: 0.18295502
R_1 = 0.18295502
X_1 = 0.0
X_2 = 0.012820513
X_3 = 0.0
X_4 = 0.0
X_5 = 1.0
X_6 = 0.93589744
```

Figure 4.2: Solutions to the diet problem

There are many ways to eat these six different foods for satisfying a person's conditions but to minimize cost of food eaten during one day if each food is prime (the nutrition values of each food reach its pinnacle value) then the minimum cost will be $11.797814 which a person have to eat 1 unit of fish and 0.797814 unit of of yogurt in a day. Unfortunately, if each food is not that good (the nutrition values of each food just only reach its lowest value) then the minimum cost will be $11.980769 which is greater than the minimum cost of the prime situation is $0.182955 and in this situation, a person have to eat 0.012821 unit of milk, 1 unit of fish and 0.935897 unit of yogurt in a day.

The applications of interval linear programming do not end here because when the uncertainties appear, there are many linear programming problems will become as the interval linear programming problems (see a bunch of linear programming applications in [9]). The interval linear programming is widely used in real-world which the using is up to the purpose of each user so that there are many more applications of interval linear programming that will be used to manipulate and deal with the uncertainties.

Chapter 5
Summary and Conclusion

5.1 Summary

In this project, we employ the optimistic, the pessimistic and the minimax regret approaches to help in finding and describing the solutions of the interval linear programming (IvLP) problem, by starting with the solution of the optimistic IvLP problem. The process of finding any solution is stop when the optimistic IvLP problem is infeasible. On the other hand, if it is feasible, we will continue to find the other solutions. Then, the pessimistic IvLP problem will be solved. If the pessimistic IvLP problem is feasible, we will continue finding the minimax regret and conclude the results. Nonetheless, if it is infeasible, we will reduce the size of each interval entry until the reduced IvLP is feasible. "Reduced IvLP" is what we call the IvLP whose its interval sizes are reduced. After that, we will continue finding the minimax regret of the reduced IvLP problem and conclude the results. The computer programs facilitate the users for solving the IvLP problems.

5.2 Conclusion

In this project, an interval linear programming is employed for solving linear programming problems with imprecise data in the pattern of intervals. Three approaches are used in this project to help users in describing and choosing their own best solutions. The first two approaches are optimistic and pessimistic approaches. The optimistic approach estimates each decision alternative in terms of the optimal value among all the objective values of the IvLP problem. If the problem is to maximize then the maximum objective value of all objective values is the optimistic objective value. On the other hand, the pessimistic approach is the antithesis of the previous approach. As a consequence, the minimum objective value among all objective values of the IvLP problem is the pessimistic objective value, if the problem is to maximize. The last approach is a minimax regret approach. As mentioned before, this approach is to minimize the worst-case (maximum) regret which its purpose is to operate as closely as possible to the real optimal value. In this project, the minimax regret (min R) is obtained as the different value from the optimistic objective value and the pessimistic objective value.

To reiterate the point of this project as follows, the optimistic and the minimax regret approaches are suitable for people who are risk lover and risk aversion, respectively. The other approach is suitable for considering as the worst situation that can happen among all the circumstance of each problem. Finally, when the uncertainties appear, we do not know what situation of problems will occur, in other words, we do not know the real value of each parameter in the linear programming so generating the linear programming is worthless. As a result, we use the interval linear programming as a tool to deal with the uncertainties, each solution of its in this project (optimistic, pessimistic and minimax regret solution) is just an approximated value and just a representation of the many solutions of the linear programming with inexact data.

5.3 Suggestions

Reducing size of the interval

There are several ways to reduce size of the interval. It is not neccesary that in reducing the size of the interval, each interval will be simultaneously increased value of the lower bound and decreased value of the upper bound. For example, each user can decide to reduce only at the lower bound or just only interval matrix $\boldsymbol{A}$ will be reduced, etc.

Language programs

Using Python language is not essential and neccesary. Users can build on the algorithms by following their convinient computer language.

Utilization from the project's aim

Since project owner is not good at computer language, the algorithms are not concise as much as it can. The series of algorithms in this project may not be suitable for all types of real-world problems. As a consequence, the algorithms still require the development by users in order to make it more effective and more suitable for some problems.

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Appendix

Appendix A: Code for Python

Figure 5.1: QR code of Python's code

URL: https://github.com/paanq/Kanokwan-Senoir_project/blob/dddaa1cd9e629eab7d11d408beddcd6892065765/Code%20for%20Python%20-%20IvLP.ipynb

Appendix B: Project Proposal

The Project Proposal of Course 2301399 Project Proposal

Academic Year 2020

Project Tittle (Thai)	ผลเฉลยของกำหนดการเชิงเส้นแบบช่วง
Project Tittle (English)	Solutions of Interval Linear Programming
Project Advisor	Associate Professor Phantipa Thipwiwatpotjana, Ph.D.
By	Miss Kanokwan Suchantabut ID 6033501223 Mathematics, Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University

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Background and Rationale

In the real life, uncertainty is the only certain thing. As we can see in our daily life, when handling real-world optimization problems by means of mathematical modeling, it is often necessary to treat inexact or uncertain input data due to various measurement errors or estimations. The parameters are not often known exactly and most of them have to be estimated. Unfortunately, the linear programming system requires constant parameters.

There are several different tools to handle uncertainty and inexactness in mathematical modeling, such as stochastic programming and fuzzy set theory, each bearing their own advantages and disadvantages. An interval linear programming is also one of the tools for solving real-world optimization problems under interval-valued uncertainty. It is a linear programming where its coefficients and parameters are in the pattern of intervals. Instead of approximating or estimating crisp input data, the coefficients of an interval program may perturb independently within the given lower and upper bounds.

This project provides many types of solutions of an interval linear programming by using knowledge in Operations Research II and three approaches which are optimistic, pessimistic and minimax regret approaches for finding the solutions. The most appropriate solution relies on the purpose of each decision maker.

Objectives

Obtain an algorithm to solve interval linear programming problems that have interval uncertain data (in it) by using optimistic, pessimistic and minimax regret approaches to describe the solutions of the problems.

Scopes

1. Explain the general idea of optimistic, pessimistic and minimax regret approaches.
2. Apply optimistic, pessimistic and minimax regret approaches in a linear programming problem with interval data.
3. Provide an application of linear programming problem with interval data.

Project Activities

1. Study linear programming problems.
2. Study the interval linear programming system.
3. Study optimistic and pessimistic approaches and apply them to an interval linear programming problem.
4. Study minimax regret approach and apply it to an interval linear programming problem.
5. Study strong solvability/feasibility of system of interval linear inequality.

6. Study relationship between regret and strong solution of an interval linear programming.
7. Use the knowledge about optimistic, pessimistic and regret appreaches to explain the solutions of an interval linear programming.
8. Build an algorithm for solving the interval linear programming problem.
9. Recheck and modify the algorithm.
10. Conclude the results and write a report.

Duration

Procedue	Month								
	May.	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	Jan.
1.Study linear programming problems.	■								
2.Study the interval linear programming system.	■	■	■	■	■	■	■		
3.Study optimistic and pessimistic approaches and apply them to an interval linear programming problem.			■	■	■				
4.Study minimax regret approach and apply it to an interval linear programming problem.				■	■	■			
5.Study strong solvability/ feasibility of system of interval linear inequality.				■	■	■	■		
6.Study relationship between regret and strong solution of interval linear programming.					■	■	■		
7.Use the knowledge about optimistic, pessimistic and regret appreaches to explain the solutions of an interval linear programming.						■	■		
8.Build an algorithm for solving the interval linear programming problem.						■	■	■	
9.Recheck and modify the algorithm.							■	■	
10.Conclude the results and write a report.								■	■

Benefits

1. The benefits of the project owner.

   I Apply knowledge from Operations Research II course to solve linear programming problems and built on this knowledge to solve interval linear programming problems.

   II Use Python version 3.7.9 to solve the problem.

2. The benefits of project users.

   I Increase convenience and efficiency in each decision making, especially someone who is such a risk aversion.

   II Increase alternative ways for desicion making which users capable of choosing solution depend on their satisfaction.

   III Can use the interval linear programming method helped in finding many types solutions and can apply this method to many interval linear programming applications.

Equipments

1. Hardware

   I Notebook computer Intel(R) Core(TM) i7(i7 - 7700HQ, 2.80 GHz, 6MB)

   II Printer

   III Flash drive

2. Software

   I Python version 3.7.9

   II TeXworks software version 0.6.5

Budgets

1. Wireless Mouse	900	Baht
2. Tray DVD Drive for HHD	280	Baht
3. Flash drive	140	Baht
4. SSD 240 GB	920	Baht
5. Notebook Service Fee	500	Baht
6. Battery Notebook	1,260	Baht
7. Bluetooth Keyboard	1,000	Baht
Total	5,000	Baht

Appendix

Linear programming

Linear programming problem (LP) can be formulated as follows:

$$\text{min } z := c^t x \tag{5.1}$$

$$\text{s.t. } Ax <\cdot> \ b, \tag{5.2}$$

$$x \geq 0 \tag{5.3}$$

where c and x are n dimentional vectors, b is an m dimentional vector, A is an $m \times n$ matrix, and $<\cdot>$ can be $\geq, \leq, =$.

Linear programming with N uncertainties

The linear programming problem with N uncertainties is formulated as :

$$\begin{aligned} \text{min } z &:= \boldsymbol{c}^t x \\ \text{s.t. } \boldsymbol{A}x &\geq \boldsymbol{b}, \\ x &\geq 0 \end{aligned}$$

where x is the vector of decision variables. The realizations of $[\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}]$ could be $[A^1, b^1, c^1], [A^2, b^2, c^2], ..., [A^N, b^N, c^N]$.

Pessimistic and optimistic approaches

A linear programming problem with N discrete uncertain data can be extented into N deterministic standard linear programming problem as follows:

Minimization LP problem

$$
\begin{aligned}
z(\xi^k) := \text{min } & c^t x \\
\text{s.t. } & A^k x \geq b^k, \\
& x \geq 0
\end{aligned}
$$

where $\xi^k = (A^k, b^k)$ for $k = 1, 2, ..., N$ is the k^{th} realization of uncertain data.

A pessimistic objective function value of these minimization LP problems is the maximum value of all N optimal solutions, i.e., $\exists k^* \in \{1, 2, ..., N\}$ s.t.

$$z(\xi^{k^*}) = \text{max } \{z(\xi^k),\ k = 1, 2, ..., N\}.$$

An optimistic objective function value of these minimization LP problems is the minimum value of all N optimal solutions, i.e., $\exists k_* \in \{1, 2, ..., N\}$ s.t.

$$z(\xi^{k_*}) = \text{min } \{z(\xi^k),\ k = 1, 2, ..., N\}.$$

Maxmization LP problem

$$
\begin{aligned}
z(\xi^k) := \text{max } & c^t x \\
\text{s.t. } & A^k x \geq b^k, \\
& x \geq 0
\end{aligned}
$$

where $\xi^k = (A^k, b^k)$ for $k = 1, 2, ..., N$ is the k^{th} realization of uncertain data.

A pessimistic objective function value of these maximization LP problems is the minimum value of all N optimal solutions, i.e., $\exists k_* \in \{1, 2, ..., N\}$ s.t.

$$z(\xi^{k_*}) = \text{min } \{z(\xi^k),\ k = 1, 2, ..., N\}.$$

An optimistic objective function value of these maximization LP problems is the maximum value of all N optimal solutions, i.e., $\exists k^* \in \{1, 2, ..., N\}$ s.t.

$$z(\xi^{k^*}) = \text{max } \{z(\xi^k),\ k = 1, 2, ..., N\}.$$

Example 1

Let $[A^1, b^1]$ and $[A^2, b^2]$ be the realizations of $[\boldsymbol{A}, \boldsymbol{b}]$. Each realization has the same objective value function i.e., $10x + 18y$.

$$A^1 = \begin{pmatrix} 4 & 10 \\ -2 & 1 \\ 7 & -2 \end{pmatrix} \qquad A^2 = \begin{pmatrix} -2 & 10 \\ 3 & -10 \\ 21 & 5 \end{pmatrix}$$

$$b^1 = \begin{pmatrix} 150 \\ 8 \\ 85 \end{pmatrix} \qquad b^2 = \begin{pmatrix} 120 \\ 20 \\ 325 \end{pmatrix}$$

LP1:

$$\begin{aligned} \text{max } z(\xi^1) &:= 10x + 18y \\ \text{s.t. } 4x + 10y &\leq 150, \\ -2x + y &\leq 8, \\ 7x - 2y &\leq 85, \\ x \text{ and } y &\geq 0. \end{aligned}$$

The optimal value is 330 where $x = 15$ and $y = 10$.

LP2:

$$\begin{aligned} \text{max } z(\xi^2) &:= 10x + 18y \\ \text{s.t. } -2x + 10y &\leq 120, \\ 3x - 10y &\leq 20, \\ 21x + 5y &\leq 325, \\ x \text{ and } y &\geq 0. \end{aligned}$$

The optimal value is 379.8198 where $x = 12.0456$ and $y = 14.4091$.

A pessimistic objective value function of this uncertainty linear programming problem is the minimum value of all optimal solution, i.e.,

$$\min\{z(\xi^k), k = 1, 2\} = \min\{330, 379.8198\} = 330 \text{ where } (x, y)^t = (15, 10)^t.$$

An optimistic objective value function of this uncertainty linear programming problem is the maximum value of all optimal solution, i.e.,

$$\max\{z(\xi^k), k = 1, 2\} = \max\{330, 379.8198\} = 379.8198$$

where $(x, y)^t = (12.0456, 14.4091)^t$.

Minimax regret model

The regret function $r_k(x)$ shows the amount a candidate solution x[†] deviates form the true objective value $z(\xi^k)$, i.e.,

$$r_k(x) = c^t x - z(\xi^k), \; k = 1, 2, ..., N.$$

As a consequence, the maximum (worst) regret of a candidate solution x is

$$R(x) = \max\{r_k(x), \; k = 1, 2, ..., N\}.$$

As a result, a general *minimax regret model* is

$$\begin{aligned} \min\ & R \\ \text{s.t. } R &\geq r_k(x), \; k = 1, 2, ..., N, \\ A^k x &\geq b^k, \; k = 1, 2, ..., N, \\ x &\geq 0. \end{aligned}$$

Since the value of regret function $r_k(x)$ can reiterate for each $k = 1, 2, ..., N$, then the candidate solution x that gives the maximum regret can be more than one. Hence, the optimal solution of the minimax regret model does not unique.

[†] A candidate solution x is a member in a set of possible solutions to a given problem. A candidate solution does not have to be a likely or reasonable solution to the problem – it is simply in the set that satisfies all constraints.

Example 2

From Example 1, $minimax\ regret\ model$ is

$$r(x,y) = z(\xi^k) - (10x + 18y) \text{ for } k = 1,2.$$

Hence, the maximum regret of a candadate solution (x, y) is

$$R(x,y) = \text{max}\{z(\xi^1) - (10x+18y), z(\xi^2) - (10x+18y)\}.$$

We formulate the minimax regret model for the problem as:

$$\begin{aligned} \text{min } R & & \\ \text{s.t. } R &\geq z(\xi^k) - (10x+18y),\ \ k = 1,2, \\ -2x + 10y &\leq 120, \\ 3x - 10y &\leq 20, \\ 21x + 5y &\leq 325, \\ 4x + 10y &\leq 160, \\ -2x + y &\leq 8, \\ 7x - 2y &\leq 85, \\ x \text{ and } y &\geq 0. \end{aligned}$$

Since R is a max$\{z(\xi^k) - (10x+18y), k = 1,2\}$, the minimum possible R is $379.8198 - (10x+18y)$. Then the new concise minimax regret model is

$$\begin{aligned} \text{min } R & & \\ \text{s.t. } R &\geq 379.8198 - (10x+18y), \\ -2x + 10y &\leq 120, \\ 3x - 10y &\leq 20, \\ 21x + 5y &\leq 325, \\ 4x + 10y &\leq 160, \\ -2x + y &\leq 8, \\ 7x - 2y &\leq 85, \\ x \text{ and } y &\geq 0. \end{aligned}$$

The optimal solution is $(x, y)^t = (12.894737, 10.842105)^t$.

The minimax regret is obtained as $R = 55.714537$.

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