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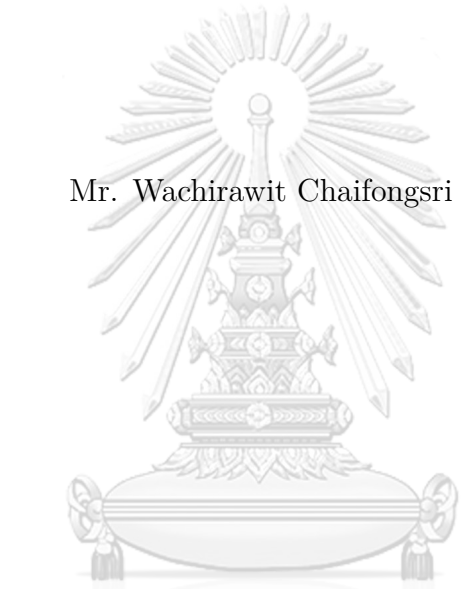
HAMILTONIAN CYCLES OF EQUATIONAL GRAPHS OVER FINITE FIELDS

Mr. Wachirawit Chaifongsri



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กำหนดให้ $\mathbb{F}_q$ เป็นฟิลด์จำกัดที่มีสมาชิก q ตัว สำหรับแต่ละสมการ $E(X,Y)=0$ เหนือ $\mathbb{F}_q$ กราฟเชิงสมการที่สอดคล้องกับสมการนี้เป็นไดกราฟที่มีเซตของจุดยอดเป็น $\mathbb{F}_q$ และ สำหรับ x และ y ใน $\mathbb{F}_q$ มีเส้นเชื่อมจาก x ไปยัง y ก็ต่อเมื่อ $E(x,y)=0$ ในงานวิจัยครั้งนี้ เราสมมติให้ $q-1\geq k$ และศึกษากราฟเชิงสมการ $\mathcal{G}^{(k)}(\lambda,f)$ ที่สอดคล้องกับสมการ

$$(Y^k-f(X))(\lambda Y^k-f(X))\dots(\lambda^{k-1}Y^k-f(X))=0$$

เมื่อ X และ Y เป็นตัวแปร $f(t)$ เป็นพหุนามใน $\mathbb{F}_q[t]$ และ λ เป็นสมาชิกใน $\mathbb{F}_q^\times$ ที่มีอันดับอย่างน้อย k เราศึกษาความเชื่อมโยงอย่างเข้มและการมีอยู่ของวัฏจักรแฮมิลตันของกราฟ $\mathcal{G}^{(k)}(\lambda,f)$ นอกจากนี้เราสามารถจำแนกกราฟเชิงสมการ $\mathcal{G}^{(3)}(\lambda,f)$ เชิงสมสัณฐานเมื่อ $f(t)$ เป็นพหุนามใน $\mathbb{F}_q[t]$ และหาเงื่อนไขบางอย่างของการมีอยู่ของส่วนประกอบที่มีจำนวนจุดยอดไม่มาก

ภาควิชา . . คณิตศาสตร์และวิทยาการคอมพิวเตอร์ . . ลายมือชื่อนิสิต . . . . . .
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WACHIRAWIT CHAIFONGSRI: HAMILTONIAN CYCLES OF EQUATIONAL GRAPHS OVER FINITE FIELD
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Let $\mathbb{F}_q$ be the finite field of q elements. For any equation $E(X,Y)=0$ over $\mathbb{F}_q$, the equational graph of this equation is a digraph whose vertex set is $\mathbb{F}_q$ and for $x,y\in\mathbb{F}_q$ there is the edge from x to y if $E(x,y)=0$. In this work, we assume that $q-1\geq k$ and work on the equational graph $\mathcal{G}^{(k)}(\lambda,f)$ associated with the equation

$$(Y^k-f(X))(\lambda Y^k-f(X))\dots(\lambda^{k-1}Y^k-f(X))=0$$

with variables X and Y, where $f(t)\in\mathbb{F}_q[t]$ and λ is an element in $\mathbb{F}_q^\times$ of order at least k. We study strongly connectivity and the existence of Hamiltonian cycle of the graph $\mathcal{G}^{(k)}(\lambda,f)$. Moreover, we classify the equational graph $\mathcal{G}^{(3)}(\lambda,f)$ up to isomorphism where $f(t)$ is a permutation polynomial in $\mathbb{F}_q[t]$ and find some conditions for the existence of components with small vertices.

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Field of Study	Mathematics	Advisor's Signature	.....
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Wachirawit Chaifongsri

Contents

CHAPTER I
PRELIMINARIES

1.1 Directed graphs and equational graphs

A *directed* graph (digraph) G consists of a vertex set $V(G)$ and an *arc* set $E(G)$ where a directed edge is an ordered pair of vertices in $V(G)$, a direction of an edge is indicated with an arrow, as in Figure 1.1 below. This might happen that an edge connects a vertex to itself and we obtain a *loop*.

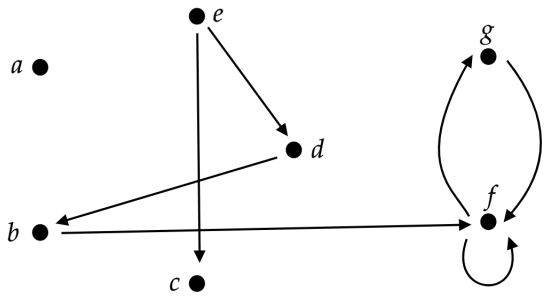


Figure 1.1: A directed graph G

For a directed graph, a vertex x is a *predecessor* of y if there is an edge from x to y. Otherwise, x is a *successor* of y if there is an edge from y to x. The *in-degree* of a vertex is the number of its predecessors and the *out-degree* of a vertex is the number of its successors. Two directed graphs G_1 and G_2 are *isomorphic* if there is a bijection τ from $V(G_1)$ to $V(G_2)$ which preserves adjacency conditions, that is, there is an edge from u to v in G_1 if and only if there is an edge from $\tau(u)$ in to $\tau(v)$ in G_2. A (directed) *path* of length r from x to y in a directed graph is a sequence of $r+1$ distinct vertices $x = a_1, a_2, \ldots, a_{r+1} = y$ such that for every $s \in \{1, 2, \ldots, r\}$ there is an edge from a_s to a_{s+1}. A *cycle* in a directed graph is a

directed path (with at least one edge) whose first and last vertices are the same. If a path in a directed graph visits each vertex exactly once, we call this *Hamiltonian path*. A *Hamiltonian cycle* is a Hamiltonian path that is a cycle. For example, the graph G in Figure 1.2 has a Hamiltonian cycle, its Hamiltonian path is as follows: $a_1 \to a_2 \to a_3 \to a_4 \to a_1$.

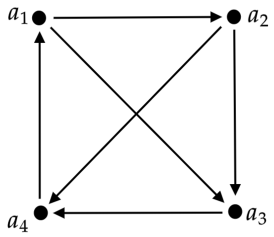


Figure 1.2: The directed graph G with a Hamiltonian cycle

Note that a directed graph containing a Hamiltonian path may not have a Hamiltonian cycle. For instance, the directed graph H in Figure 1.3 has no out-degree of the vertex b_0, so a Hamiltonian cycle does not exist. But its Hamiltonian path is as follows: $b_1 \to b_2 \to b_3 \to b_4 \to b_5 \to b_6 \to b_0$.

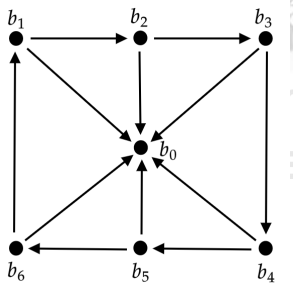


Figure 1.3: The directed graph H without Hamiltonian cycles

A directed graph is *bipartite* if its vertex set can be partitioned into two sets G_1 and G_2, where every edge in the graph goes from a vertex in G_1 to a vertex in G_2 and there is no adjacent edge between two vertices in the G_1 or G_2. For example, the graph $\mathcal{G}$ whose vertex set can be partitioned into two sets $\mathcal{G}_1 = \{a, b\}$ and $\mathcal{G}_2 = \{c, d, e\}$ (see Figure 1.4).

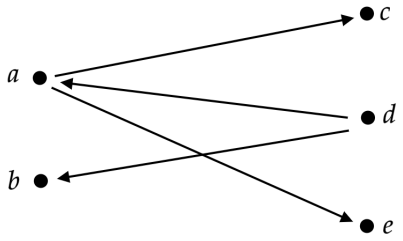


Figure 1.4: The bipartite graph $\mathcal{G}$

A *weak path* in any digraph is a sequence $a_0, a_1, \ldots, a_r$ of distinct vertices for which there is an undirected edge between a_{i-1} and a_i for each $i = 1, 2, \ldots, r$. A digraph is said to be *weakly connected* if any two vertices can be joined by a weak path. We say that a directed graph is *strongly connected* if it is possible to reach any vertex starting from any vertex in a graph. A *weakly connected component* of a directed graph is a maximal weakly connected subgraph. It is well known that a directed graph can be partitioned into a disjoint union of weakly connected components. For the notation, we abbreviate a *component* for a weakly connected component. A strongly connected directed graph is always weakly connected, but a weakly connected directed graph may not be strongly connected (see Figure 1.5).

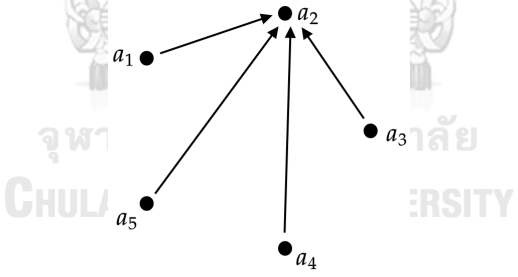


Figure 1.5: A non-strongly connected directed graph

Let $\mathbb{F}_q$ be the finite field of q elements. For a function f on $\mathbb{F}_q$, we define the *functional graph* of f as a digraph of q vertices labeled by the elements of $\mathbb{F}_q$, where there is an edge from u to v if and only if $f(u) = v$. By working on polynomials over $\mathbb{F}_q$, Knoyagin et al. [4] found algorithms to estimate the number of non-isomorphism functional graphs and provided an upper bound on the number of functional graphs. Later, Mans et al. [5] provided algorithms on quadratic polynomials over finite

fields to approximate the number of connected functional graphs, the number of graphs having a maximal cycle, the number of cycles of fixed size and the number of components of fixed size.

Mans et al. [5] suggested that almost all of the functional graphs generated by the polynomial $f(t) = t^2 + a \in \mathbb{F}_q[t]$ are weakly connected.

More generally, for an equation over $\mathbb{F}_q$:

$$E(X, Y) = 0$$

with variables X and Y, we may define a digraph by letting elements in $\mathbb{F}_q$ as vertices and drawing an edge from u to v in $\mathbb{F}_q$ if and only if $E(u, v) = 0$. We call this graph an *equational graph* associated with the above equation.

Mans et al. [6] studied the equational graph associated with the equation

$$E(X, Y) = (Y^2 - f(X))(\lambda Y^2 - f(X)) = 0$$

with variables X and Y, $f(t)$ is a polynomial over $\mathbb{F}_q$, where q is odd and λ is a non-square element in $\mathbb{F}_q$. It is denoted by $\mathcal{G}^{(2)}(\lambda, f)$ (see Figure 1.6).

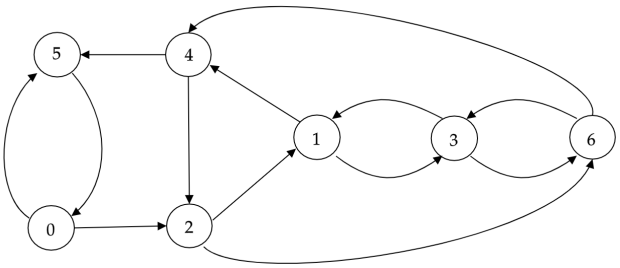


Figure 1.6: The equational graph $\mathcal{G}^{(2)}(3, 2t + 4)$ over $\mathbb{Z}_7$

A polynomial $f(t)$ in $\mathbb{F}_q[t]$ is a *permutation polynomial* if the map $x \mapsto f(x)$ gives a bijection from $\mathbb{F}_q$ onto itself. Mans et al. obtained properties on vertices when $f(t)$ is a permutation polynomial as follows.

Proposition 1.1.1. *[6] Let $f(t)$ be a permutation polynomial over $\mathbb{F}_q$ where q is odd. For the graph $\mathcal{G}^{(2)}(\lambda, f)$,*

1. *if $f(0) \neq 0$, then the vertex 0 has in-degree 1 and out-degree 2, the vertex $f^{-1}(0)$ has in-degree 2 and out-degree 1, and the vertex x ($x \neq 0$ and $x \neq f^{-1}(0)$) has in-degree 2 and out-degree 2, and*

2. *if* $f(0) = 0$*, then the vertex* 0 *has in-degree* 1 *and out-degree* 1*, and the vertex* x $(x \neq 0)$ *has in-degree* 2 *and out-degree* 2.

It follows from the above proposition that every vertex lies in a cycle and also every edge lies in a cycle. It also implies that every Mans' graph is not a bipartite graph. Note that there is a strongly connected digraph whose some component does not have a Hamiltonian cycle (see Corollary 3.8.2 of [2]). Mans et al. showed that

Theorem 1.1.2. *[6] If* $f(t)$ *is a permutation polynomial in* $\mathbb{F}_q[t]$ *where* q *is odd, then every component of the graph* $\mathcal{G}^{(2)}(\lambda, f)$ *has a Hamiltonian cycle.*

Some examples for the non-permutation polynomials were also provided in [6]. Mans et al. also analyzed the graphs and obtained some algorithms for searching Hamiltonian cycles when $f(t)$ is of degree $1, 2$ or 3. In addition, they [6] classified the equational graphs $\mathcal{G}^{(2)}(\lambda, f)$ up to isomorphism where $f(t)$ is a permutation polynomial of degree one (a linear polynomial). They showed that $\mathcal{G}^{(2)}(\lambda, at + b)$ is isomorphic to $\mathcal{G}^{(2)}(\lambda, t + a^{-1}b)$ for any $a \neq 0$ and $b \in \mathbb{F}_q$. Thus, they studied components with $2, 3$ or 4 vertices of $\mathcal{G}^{(2)}(\lambda, t + b)$ for every $b \in \mathbb{F}_q$. Their results suggested that almost of graphs $\mathcal{G}^{(2)}(\lambda, f)$ are connected where $f(t)$ is a linear polynomial in $\mathbb{F}_q[t]$.

1.2 The kth power mapping on finite fields

Let $\mathbb{F}_q$ be the finite fields of q elements. We write $\mathbb{F}_q^\times$ for its multiplicative group of nonzero elements. Let $k \geq 2$ be a positive integer. Consider the kth power mapping φ_k on $\mathbb{F}_q^\times$ given by $\varphi_k(a) = a^k$ for all $a \in \mathbb{F}_q^\times$. The kernel of φ_k is $\{a \in \mathbb{F}_q^\times : a^k = 1\}$ and the image of φ_k is $\{a^k : a \in \mathbb{F}_q^\times\}$, the set of the kth power elements. The following result gives the structure of the multiplicative group $\mathbb{F}_q^\times$.

Theorem 1.2.1. *[1] The multiplicative group* $\mathbb{F}_q^\times$ *is a cyclic of order* $q - 1$*. Its generators are called primitive elements of* $\mathbb{F}_q$*.*

Next, we discuss some basic tools in group theory. Let ϕ be the *Euler* ϕ*-function*, that is, a function counting positive integers up to a given natural number n that are relatively prime to n.

Theorem 1.2.2. *[3] Let G be a cyclic group of order n generated by a.*

1. *For each positive divisor d of n, the group G has exactly one subgroup of order d, namely $\langle a^{n/d} \rangle$.*

2. *If d is a positive divisor of n, then the number of elements of order d in a cyclic group G of order n is $\phi(d)$.*

3. $\sum_{d|n} \phi(d) = n$.

We proceed to compute the size of $\ker \varphi_k$. By Theorem 1.2.2, if $d \mid q-1$, then the number of elements of order d in $\mathbb{F}_q^\times$ is $\phi(d)$ and there are no elements of order d in $\mathbb{F}_q^\times$ otherwise. Since $\ker \varphi_k$ consists of elements in $\mathbb{F}_q^\times$ of order a divisor of k, by Theorem 1.2.2 (3), we have

$$|\ker \varphi_k| = \sum_{d|k} |\{a \in \mathbb{F}_q^\times : \circ(a) = d\}| = \sum_{d|\gcd(k,q-1)} \phi(d) = \gcd(k, q-1).$$

It follows that

$$|\operatorname{im} \varphi_k| = |\mathbb{F}_q^\times / \ker \varphi_k| = \frac{q-1}{\gcd(k, q-1)}.$$

In particular, we have φ_k is injective if and only if $\gcd(k, q-1) = 1$, and in this case, every element in $\mathbb{F}_q^\times$ is the kth power. On the other hand, if $k \mid q-1$, then $|\ker \varphi_k| = k$ and the polynomial $t^k - 1$ splits into monic linear factors in $\mathbb{F}_q[t]$ and $\mathbb{F}_q^\times / \operatorname{im} \varphi_k$ is a cyclic group of order k. We can apply these observations to study the generalized equational graphs introduced in Section 1.1 of Mans et al. [6].

1.3 Our graphs

Let $k \geq 2$ be a positive integer. Assume that $k \leq q-1$. Then there is an element λ in $\mathbb{F}_q^\times$ of order at least k. A directed graph whose vertex set is $\mathbb{F}_q$ and for $x, y \in \mathbb{F}_q$ there is a directed edge from x to y if

$$(y^k - f(x))(\lambda y^k - f(x)) \dots (\lambda^{k-1} y^k - f(x)) = 0.$$

is called *the digraph associated with the polynomial* $f(t) \in \mathbb{F}_q[t]$. It is denoted by $\mathcal{G}^{(k)}(\lambda, f)$. If q is odd and $k = 2$, it is a Mans' graph. Mans et al. suggested the above equation in their comments in Section 7 of [6].

In this thesis, we use the kth power mapping to study the graph $\mathcal{G}^{(k)}(\lambda, f)$ where $f(t)$ is a permutation polynomial in $\mathbb{F}_q[t]$. The work is organized as follows. In Chapter 2, we assume that $\lambda \operatorname{im} \varphi_k$ generates the quotient group $\mathbb{F}_q^\times / \operatorname{im} \varphi_k$ and prove that every vertex has a positive in-degree and out-degree in Section 2.1. This implies the strong connectivity of the graph. Moreover, if $k \mid (q-1)$, then a Hamiltonian cycle exists. This proof is presented in Section 2.2 together with some examples. In Section 3.1, we classify the equational graphs $\mathcal{G}^{(3)}(\lambda, f)$ up to isomorphism where $f(t)$ is a permutation polynomial in $\mathbb{F}_q[t]$. Finally, we find some conditions for the existence of components with three or four vertices in Section 3.2.

CHAPTER II
CONNECTIVITY AND HAMILTONIAN CYCLES

2.1 In-degree, out-degree and strongly connectivity

Assume that $\lambda \operatorname{im} \varphi_k$ generates the quotient group $\mathbb{F}_q^\times / \operatorname{im} \varphi_k$. This can easily hold if λ is a primitive element of $\mathbb{F}_q$ or the size of $\mathbb{F}_q^\times / \operatorname{im} \varphi_k$ is a small prime. First, we study the in-degrees and out-degrees of every vertex of $\mathcal{G}^{(k)}(\lambda, f)$ where $f(t) \in \mathbb{F}_q[t]$ is a permutation polynomial. It follows that every component of the graph $\mathcal{G}^{(k)}(\lambda, f)$ is strongly connected and the graph is not bipartite. We begin with an algebraic lemma.

Lemma 2.1.1. *Assume that* $\gcd(k, q-1) = d$. *Let* $f(t)$ *be a polynomial in* $\mathbb{F}_q[t]$ *and let* $x \in \mathbb{F}_q$ *with* $f(x) \neq 0$. *Then there exists* $l \in \{0, 1, \ldots, d-1\}$ *such that* $\lambda^{-l} f(x) \in \operatorname{im} \varphi_k$. *Moreover, for* $j \in \{0, 1, \ldots, k-1\}$, *there exists* $y \in \mathbb{F}_q^\times$ *such that* $\lambda^j y^k = f(x)$ *if and only if* $j = l + md$ *for some* $m \in \{0, 1, \ldots, k/d - 1\}$. *In particular, the number of* y*'s in* $\mathbb{F}_q$ *such that*

$$(y^k - f(x))(\lambda y^k - f(x)) \ldots (\lambda^{k-1} y^k - f(x)) = 0$$

is k.

Proof. Let $j \in \{0, 1, \ldots, k-1\}$. Suppose that there exists $y \in \mathbb{F}_q^\times$ such that $\lambda^j y^k = f(x)$. Since $\lambda^{-l} f(x) \in \operatorname{im} \varphi_k$, $f(x) = \lambda^l g^k$ for some $g \in \mathbb{F}_q^\times$. Then $\lambda^j y^k = \lambda^l g^k$, so $\lambda^{j-l} = (gy^{-1})^k \in \operatorname{im} \varphi_k$. Since the order of $\lambda \operatorname{im} \varphi_k$ is $d = |\mathbb{F}_q^\times / \operatorname{im} \varphi_k|$, we have $d \mid (j - l)$.

Conversely, we suppose that $j = l + md$ for some $m \in \{0, 1, \ldots, k/d - 1\}$. Since the order of $\lambda \operatorname{im} \varphi_k$ is d, we have

$$\lambda^{-j} f(x) \operatorname{im} \varphi_k = \lambda^{-l} f(x) \operatorname{im} \varphi_k = \operatorname{im} \varphi_k.$$

Then $\lambda^{-j} f(x) \in \operatorname{im} \varphi_k$, so $\lambda^{-j} f(x) = y^k$ for some $y \in \mathbb{F}_q^\times$.

Finally, for each $j \equiv l \mod d$, the number of y's such that $\lambda^j y^k = f(x)$ is d because $|\ker \varphi_k| = d$. Since the number of j's in $\{0, 1, \ldots, k-1\}$ such that $j \equiv l \mod d$ is k/d, it follows that our desired number is k. □

Proposition 2.1.2. *Let $f(t)$ be a permutation polynomial in $\mathbb{F}_q[t]$. For the graph $\mathcal{G}^{(k)}(\lambda, f)$,*

1. *if $f(0) = 0$, then the vertex 0 has in-degree 1 and out-degree 1, and the vertex x ($x \neq 0$) has in-degree k and out-degree k, and*

2. *if $f(0) \neq 0$, then the vertex 0 has in-degree 1 and out-degree k, the vertex $f^{-1}(0)$ has in-degree k and out-degree 1, and the vertex x ($x \neq 0$ and $x \neq f^{-1}(0)$) has in-degree k and out-degree k.*

Proof. 1. Assume that $f(0) = 0$. We first count the in-degree and out-degree of the vertex 0. Let u be a successor of the vertex 0. Then,

$$\lambda^{k(k-1)/2} u^{k^2} = (u^k - f(0))(\lambda u^k - f(0)) \ldots (\lambda^{k-1} u^k - f(0)) = 0,$$

and so $u = 0$. Hence, the vertex zero has in-degree 1. Let v be a predecessor of the vertex 0. Then, $f(v) = 0 = f(0)$. Since f is a one-to-one, $v = 0$. The vertex 0 has out-degree 1. Next, we let x be a nonzero vertex in $\mathbb{F}_q$. Since f is one-to-one and $f(0) = 0$, we have $f(x) \neq 0$. By Lemma 2.1.1, there are k successors of x. Now, we let s be a predecessor associated with x. Then

$$(x^k - f(s))(\lambda x^k - f(s)) \ldots (\lambda^{k-1} x^k - f(s)) = 0.$$

Therefore, $s = f^{-1}(\lambda^j x^k)$ where $j \in \{0, 1, \ldots, k-1\}$ gives k solutions.

2. Assume that $f(0) \neq 0$. By Lemma 2.1.1, there are k successors of the vertex 0. Let v be a predecessor of the vertex 0. Then $f(v) = 0$ and so $v = f^{-1}(0) \neq 0$ because f is one-to-one. Therefore, there is only one predecessor of the vertex 0.

Now, we let c be a successor of $f^{-1}(0)$. Then, $\lambda^{k(k-1)/2}c^{k^2} = 0$, so $c = 0$. Next, we let h be a predecessor of $f^{-1}(0)$. Then,

$$((f^{-1}(0))^k - f(h))(\lambda(f^{-1}(0))^k - f(h)) \dots (\lambda^{k-1}(f^{-1}(0))^k - f(h)) = 0,$$

so $h = f^{-1}(\lambda^{-j}(f^{-1}(0))^k)$ where $j \in \{0, 1, \dots, k-1\}$. Since $f(0) \neq 0, f^{-1}(0) \neq 0$. Hence, the vertex $f^{-1}(0)$ has in-degree 1 and out-degree k.

Lastly, let x be a vertex such that $x \neq 0$ and $x \neq f^{-1}(0)$. Then $f(x) \neq 0$. By Lemma 2.1.1, there are k successors of x. Let s be a predecessor associated with x. Then

$$(x^k - f(s))(\lambda x^k - f(s)) \dots (\lambda^{k-1}x^k - f(s)) = 0.$$

Thus, $s = f^{-1}(\lambda^j x^k)$ where $j \in \{0, 1, \dots, k-1\}$. Since $x \neq 0$, the vertex s varies k solutions. □

Remark. From the above proposition, if $k \geq 3$ and $f(t) \in \mathbb{F}_q[t]$, then there is no components of $\mathcal{G}^{(3)}(\lambda, f)$ with two vertices. Mans et al. gave a condition in which $\mathcal{G}^{(2)}(\lambda, f)$ has a component with two vertices (see Proposition 4.5 of [6]).

Lemma 2.1.3. *Let $f(t)$ be a permutation polynomial in $\mathbb{F}_q[t]$. Then every vertex of $\mathcal{G}^{(k)}(\lambda, f)$ lies in a cycle, and every edge of $\mathcal{G}^{(k)}(\lambda, f)$ lies in a cycle. Moreover, every component of the graph is strongly connected.*

Proof. Let $\mathcal{C}$ be a connected component of $\mathcal{G}^{(k)}(\lambda, f)$. By Proposition 2.1.2, every vertex must be adjacent to some vertex in $\mathbb{F}_q$. Thus, we will show that, if there is an edge from x to y, then there is a path from the vertex y to the vertex x and thus we obtain our desired cycle. Proposition 2.1.2 also says that every vertex in $\mathbb{F}_q$ has positive in-degree as well as out-degree. Let G_1 be a subgraph of $\mathcal{G}^{(k)}(\lambda, f)$ that we start from x, then we draw the predecessors of x, and the predecessors of the predecessors of x, and so on. On the other hand, we let G_2 be a subgraph of $\mathcal{G}^{(k)}(\lambda, f)$ that we start from y, then we draw the successors of y, and the successors of the successors of y, and so on.

Next, we show that there is a common vertex in the subgraphs G_1 and G_2. Suppose that G_1 and G_2 have no common vertex. Then, the edge from x to y is not in G_1, and also not in G_2.

Case 1. G_2 does not contain the vertex 0. So $y \neq 0$ and $y \neq f^{-1}(0)$. By Proposition 2.1.2, in G_2, the vertex y has out-degree k and in-degree at most $k-1$. Nonetheless, any other vertex in G_2 has out-degree k and in-degree at most k. Therefore, the sum of out-degrees in G_2 is not equal to the sum of in-degrees in G_2, a contradiction.
Case 2. G_2 contains the vertex 0. If $f(0) = 0$, then $\mathcal{C}$ contains only the vertex 0 which is impossible. So $f(0) \neq 0$. If $y = 0$, then $x = f^{-1}(0)$ is not in G_2, and hence any other vertex in G_2 has in-degree at most k and out-degree k. It follows that

$$\begin{aligned} k + k(|V(G_2)| - 1) &= \text{the sum of out-degrees in the subgraph } G_2 \\ &= \text{the sum of in-degrees in the subgraph } G_2 \\ &\leq 0 + k(|V(G_2)| - 1) \end{aligned}$$

which is absurd. For the case $y = f^{-1}(0)$, we have a successor of y must be a vertex 0, $|V(G_2)| \geq 2$, and in G_2, the vertex $f^{-1}(0)$ has in-degree at most $k-1$ and out-degree 1. By Proposition 2.1.2, we obtain the sum of in-degrees in the subgraph G_2 is at most $(k-1) + 1 + k(|V(G_2)| - 2)$ and the sum of out-degrees in the subgraph G_2 equals to $1 + k(|V(G_2)| - 1)$. Since, in G_2, the sum of in-degrees is equal to the sum of out-degrees, we have

$$1 + k(|V(G_2)| - 1) \leq k + k(|V(G_2)| - 2),$$

so $1 \leq 0$ which is a contradiction. Thus, we may assume that $y \neq 0$ and $y \neq f^{-1}(0)$, so any other vertex z in the subgraph G_2 which is not 0 and $f^{-1}(0)$ has out-degree k and in-degree at most k. Hence,

$$\begin{aligned} k + k + 1 + k(|V(G_2)| - 3) &= \text{the sum of out-degrees in the subgraph } G_2 \\ &= \text{the sum of in-degrees in the subgraph } G_2 \\ &\leq (k-1) + 1 + k + k(|V(G_2)| - 3) \end{aligned}$$

which is also a contradiction, and we can conclude that G_1 and G_2 have a common vertex. □

Proposition 2.1.4. *Assume that q is odd. If $f(t)$ is a permutation polynomial in $\mathbb{F}_q[t]$, then the graph $\mathcal{G}^{(k)}(\lambda, f)$ is not a bipartite graph.*

Proof. Assume that the graph $\mathcal{G}^{(k)}(\lambda, f)$ is a bipartite graph. Then the vertex set is divided into two disjoint subsets, say G_1 and G_2, where there is no adjacent edge between two vertices in the same vertex subset G_1 or G_2. Thus, there is no loops in $\mathcal{G}^{(k)}(\lambda, f)$ and so $f(0) \neq 0$. Without loss of generality, we suppose that $f^{-1}(0) \in G_1$ and $0 \in G_2$. Let $|G_1| = m$ and $|G_2| = n$. By Proposition 2.1.2, the sum of out-degrees of the vertices in G_1 equals $k(m-1)+1$, and the sum of in-degree of the vertices in G_2 is equal to $k(n-1)+1$. Since $\mathcal{G}^{(k)}(\lambda, f)$ is a bipartite graph, the sum of out-degrees of the vertices in G_1 and the sum of in-degrees of the vertices in G_1 are equal. Hence, $m = n$ and $q = m+n$ is an even integer which is a contradiction. □

2.2 Existence of Hamiltonian cycles

Next, we obtain some conditions for the existence of a Hamiltonian cycle in the graph $\mathcal{G}^{(k)}(\lambda, f)$.

Theorem 2.2.1. *If $k \mid (q-1)$ and $f(t)$ is a permutation polynomial in $\mathbb{F}_q[t]$, then every component of the graph $\mathcal{G}^{(k)}(\lambda, f)$ has a Hamiltonian cycle.*

Proof. Suppose on the contrary that there exists a component $\mathcal{C}$ in $\mathcal{G}^{(k)}(\lambda, f)$ which does not have a Hamiltonian cycle. By Lemma 2.1.3, we can choose a maximal cycle, say M, in $\mathcal{C}$. So the cycle M cannot be enlarged and does not go through all vertices of $\mathcal{C}$. Also, there is a vertex in M whose successor is outside M. Let x_0 be a vertex in M such that y_0 is a successor of x_0 outside the cycle M. If $y_0 = 0$, then x_0 has out-degree at least 2 which is impossible. Thus, $y_0 \neq 0$, and so the in-degree of y_0 is k.

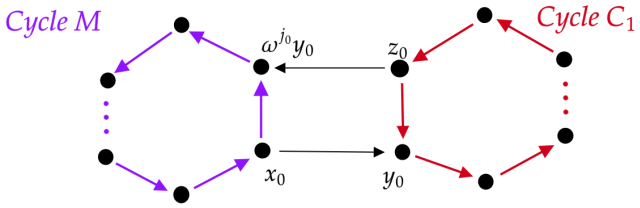
Next, we show that y_0 has a predecessor outside M. Suppose that all predecessors of y_0 are inside the cycle M. Let u_0 be a predecessor of y_0. Then

$$(y_0^k - f(u_0))(\lambda y_0^k - f(u_0)) \dots (\lambda^{k-1} y_0^k - f(u_0)) = 0.$$

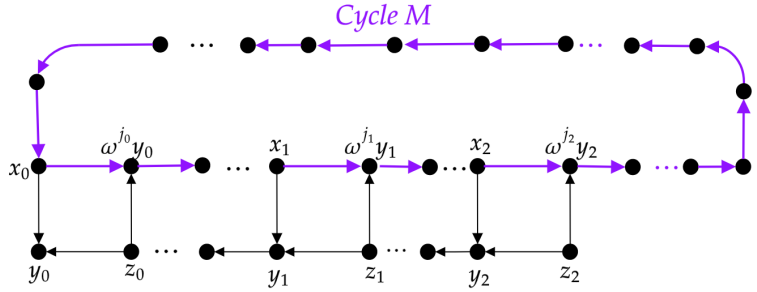
Since f is one-to-one, $u_0 = f^{-1}(\lambda^l y_0^k)$ for some $l \in \{0, 1, \dots, k-1\}$. Since $k \mid (q-1)$, there is a primitive kth root of unity ω in $\mathbb{F}_q$. Then, for each $i \in \{0, 1, \dots, k-1\}$,

$$((\omega^i y_0)^k - f(u_0))(\lambda(\omega^i y_0)^k - f(u_0)) \dots (\lambda^{k-1}(\omega^i y_0)^k - f(u_0)) = 0.$$

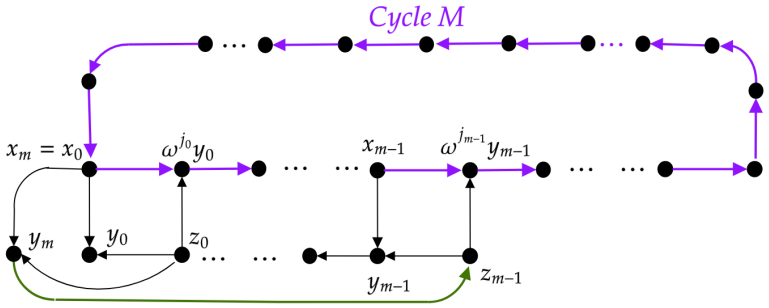
Thus, k vertices $f^{-1}(y_0^k), f^{-1}(\lambda y_0^k), \ldots, f^{-1}(\lambda^{k-1} y_0^k)$ have common successors $y_0, \omega y_0$, $\omega^2 y_0, \ldots, \omega^{k-1} y_0$ and $x_0 = f^{-1}(\lambda^l y_0^k)$ for some $l \in \{0, 1, \ldots, k-1\}$. Since the vertex y_0 is outside the cycle M, the successor of x_0 inside the cycle M is $\omega^{j_0} y_0$ for some $j_0 \in \{1, 2, \ldots, k-1\}$. Since $y_0, \omega y_0, \omega^2 y_0, \ldots, \omega^{k-1} y_0$ are common successors of $f^{-1}(y_0^k), f^{-1}(\lambda y_0^k), \ldots, f^{-1}(\lambda^{k-1} y_0^k)$, there is a predecessor of y_0 whose out-degree is at least $k+1$, a contradiction. Hence, there is a predecessor of y_0 outside the cycle M, say z_0. By Lemma 2.1.3, the edge from z_0 to y_0 lies in a cycle, say C_1. If the cycle C_1 does not intersect M, then we can enlarge the cycle M by cutting edges by edges from x_0 to $\omega^{j_0} y_0$ and from z_0 to y_0 (see the figure below), which contradicts the maximality of M, so C_1 must intersect M.



Let x_1 be a vertex in C_1 and also in M such that, along with the cycle C_1, the path from x_1 to y_0 does not intersect with M, except the vertex x_1. Similarly, we can find a successor $y_1 \neq 0$ of x_1 outside the cycle M, and there exists a predecessor of y_1 outside M, say z_1. By Lemma 2.1.3, the edge from z_1 to y_1 lies in a cycle, say C_2. As before, the cycle C_2 must intersect M, and there exist a vertex say x_2, in C_2 and also in M such that along the cycle C_2, the path from x_2 to y_1 does not intersect M except the vertex x_2. Then we draw vertices $y_2, \omega^{j_2} y_2, z_2$ and the edges among them as before. Repeating this previous process, we obtain a sequence of vertices in M, $x_0, x_1, x_2, \ldots$ in a similar manner (see the next figure).



Since M is a finite cycle, $x_n = x_m$ for some $n < m$. Without loss of generality, we assume that $n = 0$. Since $\omega^{j_m} y_m$ is a successor of x_m in the cycle M, we have $y_m = \omega^l y_0$ for some $l \in \{0, 1, \ldots, k-1\}$. Also, there is a path from y_m to z_{m-1} and there is an edge from z_0 to $y_m = \omega^l y_0$, shown in the next figure.



Thus $C : x_0, \omega^{j_0} y_0, \ldots, x_{m-1}, y_{m-1}, \ldots, z_0, y_m = \omega^l y_0, z_{m-1}, \omega^{j_{m-1}} y_{m-1}$ is a cycle in $\mathcal{C}$ extending M which contradicts the maximality of M. Therefore, every component has a Hamiltonian cycle. □

Example 2.2.2. Consider the finite field $\mathbb{Z}_3[i]$, where $i^2 = -1$ in $\mathbb{Z}_3$, $\lambda = 1 + i$ and $f(t) = t$. Indeed, $f(t)$ is a permutation polynomial over $\mathbb{Z}_3[i]$ and $\gcd(4, 9-1) = 4$. From Figure 2.1, the graph $\mathcal{G}^{(4)}(1+i, t)$ over $\mathbb{Z}_3[i]$ has two components and its Hamiltonian paths are as follows: $0 \to 0$ and $1 - i \to 1 \to -1 \to -1 - i \to 1 + i \to -i \to i \to -1 + i \to 1 - i$.

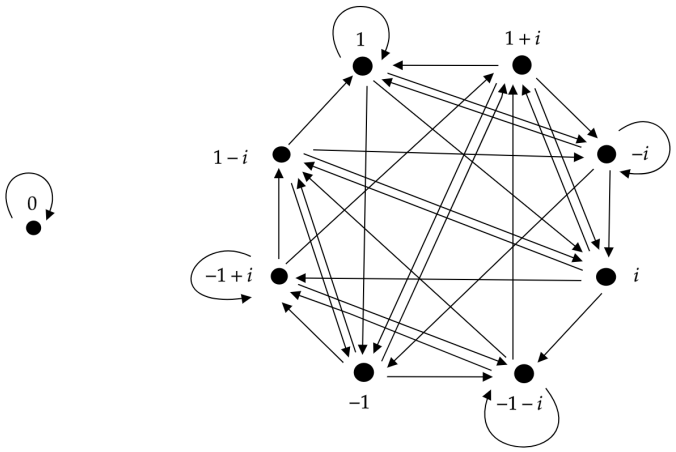


Figure 2.1: The graph $\mathcal{G}^{(4)}(1+i, t)$ over $\mathbb{Z}_3[i]$

Example 2.2.3. If $f(t) \in \mathbb{F}_q[t]$ is not a permutation polynomial, then there is a graph $\mathcal{G}^{(k)}(\lambda, f)$ whose component does not have a Hamiltonian cycle. For example, in $\mathbb{Z}_7$, $f(t) = t^3 + 1 \in \mathbb{Z}_7[t]$ is not a permutation polynomial and we consider $\lambda = 3$. From Figure 2.2, the graph $\mathcal{G}^{(3)}(3, t^3 + 1)$ has only one component and there is no predecessor of vertices $3, 5$ and 6. Hence, the Hamiltonian cycle does not exist. On the other hand, $f(t) = t^3 \in \mathbb{Z}_7[t]$ is not a permutation polynomial but every component of the graph $\mathcal{G}^{(3)}(2, t^3)$ over $\mathbb{Z}_7$ has a Hamiltonian cycle (see Figure 2.3).

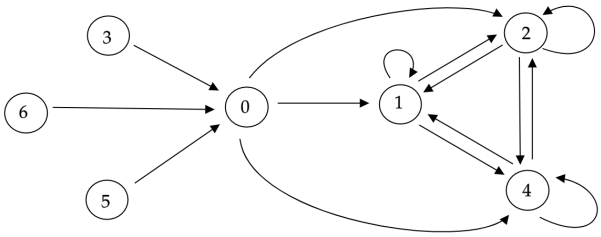


Figure 2.2: The graph $\mathcal{G}^{(3)}(3, t^3 + 1)$ over $\mathbb{Z}_7$

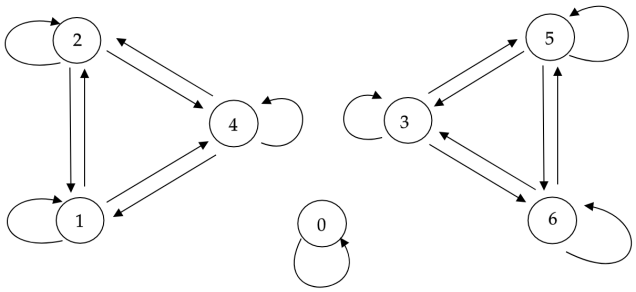


Figure 2.3: The graph $\mathcal{G}^{(3)}(3, t^3)$ over $\mathbb{Z}_7$

CHAPTER III
FURTHER RESULTS

3.1 Isomorphism Classses

Throughout this section, we assume that q is odd and fix a non-square element c in $\mathbb{F}_q$. We work on isomorphism classes of the graph $\mathcal{G}^{(3)}(\lambda, at+b)$ where $a, b \in \mathbb{F}_q$ with $a \neq 0$. We shall distinguish $a \neq 0$ in two cases, namely, square and non-square, as follows.

Proposition 3.1.1. *Let $\alpha \neq 0$ in $\mathbb{F}_q$ and $b \in \mathbb{F}_q$, the graph $\mathcal{G}^{(3)}(\lambda, \alpha^2 t + b)$ is isomorphic to the graph $\mathcal{G}^{(3)}(\lambda, t + \alpha^{-3}b)$.*

Proof. Let τ be the bijection map on $\mathbb{F}_q^\times$ defined by $\tau(x) = \alpha^{-1}x$ for all $x \in \mathbb{F}_q$. To show that τ preserves the adjacency conditions, we let u and v be in $\mathbb{F}_q$ and compute

$$\begin{aligned}
&\left(v^3 - (\alpha^2 u + b)\right)\left(\lambda v^3 - (\alpha^2 u + b)\right)\left(\lambda^2 v^3 - (\alpha^2 u + b)\right) = 0 \\
&\Leftrightarrow \alpha^{-3}\left(v^3 - (\alpha^2 u + b)\right)\alpha^{-3}\left(\lambda v^3 - (\alpha^2 u + b)\right)\alpha^{-3}\left(\lambda^2 v^3 - (\alpha^2 u + b)\right) = 0 \\
&\Leftrightarrow \left((\alpha^{-1}v)^3 - (\alpha^{-1}u + \alpha^{-3}b)\right)\left(\lambda(\alpha^{-1}v)^3 - (\alpha^{-1}u + \alpha^{-3}b)\right)\left(\lambda^2(\alpha^{-1}v)^3 - (\alpha^{-1}u + \alpha^{-3}b)\right) = 0 \\
&\Leftrightarrow \left((\tau(v))^3 - (\tau(u) + \alpha^{-3}b)\right)\left(\lambda(\tau(v))^3 - (\tau(u) + \alpha^{-3}b)\right)\left(\lambda^2(\tau(v))^3 - (\tau(u) + \alpha^{-3}b)\right) = 0.
\end{aligned}$$

Hence, τ is an isomorphism from the graph $\mathcal{G}^{(3)}(\lambda, \alpha^2 t + b)$ onto the graph $\mathcal{G}^{(3)}(\lambda, t + \alpha^{-3}b)$. □

Proposition 3.1.2. *Assume that a is a non-square element in $\mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$. Then ac is a square element and the graph $\mathcal{G}^{(3)}(\lambda, at + b)$ is isomorphic to the graph $\mathcal{G}^{(3)}(\lambda, c^{-1}t + \beta^{-3}b)$ where $\beta \in \mathbb{F}_q$ and $\beta^2 = ac$.*

Proof. Consider the square mapping φ_2. Since $\ker \varphi_2 = \{1, -1\}$, $|\mathbb{F}_q^\times / \operatorname{im} \varphi_2| = 2$, so $a \operatorname{im} \varphi_2 = c \operatorname{im} \varphi_2 = c^{-1} \operatorname{im} \varphi_2$. Then $ac \in \operatorname{im} \varphi_2$, so $ac = \beta^2$ for some $\beta \in \mathbb{F}_q^\times$. Let

τ be the bijection map on $\mathbb{F}_q$ defined by $\tau(x) = \beta^{-1}x$ for all $x \in \mathbb{F}_q$. To show that τ preserves the adjacency conditions, we let u and v be in $\mathbb{F}_q$. Since

$$\begin{aligned}
&\left(v^3 - (au+b)\right)\left(\lambda v^3 - (au+b)\right)\left(\lambda^2 v^3 - (au+b)\right) = 0 \\
&\Leftrightarrow \left(cv^3 - [(\beta^2 u) + cb]\right)\left(c\lambda(v)^3 - [(\beta^2 u) + bc]\right)\left(c\lambda^2(v)^3 - [(\beta^2 u) + bc]\right) = 0 \\
&\Leftrightarrow \beta^{-3}\left(cv^3 - [(\beta^2 u) + cb]\right)\beta^{-3}\left(c\lambda(v)^3 - [(\beta^2 u) + bc]\right)\beta^{-3}\left(c\lambda^2(v)^3 - [(\beta^2 u) + bc]\right) = 0 \\
&\Leftrightarrow \left(c(\beta^{-1}v)^3 - [(\beta^{-1}u) + \beta^{-3}cb]\right)\left(c\lambda(\beta^{-1}v)^3 - [(\beta^{-1}u) + \beta^{-3}bc]\right)\left(c\lambda^2(\beta^{-1})^3 - [(\beta^{-1}u) + \beta^{-3}bc]\right) = 0 \\
&\Leftrightarrow \left((\beta^{-1}v)^3 - [c^{-1}(\beta^{-1}u) + \beta^{-3}b]\right)\left(\lambda(\beta^{-1}v)^3 - [c^{-1}(\beta^{-1}u) + \beta^{-3}b]\right)\left(\lambda^2(\beta^{-1})^3 - [c^{-1}(\beta^{-1}u) + \beta^{-3}b]\right) = 0 \\
&\Leftrightarrow \left((\tau(v))^3 - [c^{-1}(\tau(u)) + b\beta^{-3}]\right)\left(\lambda(\tau(v))^3 - [c^{-1}(\tau(u)) + b\beta^{-3}]\right)\left(\lambda^2(\tau(v))^3 - [c^{-1}(\tau(u)) + b\beta^{-3}]\right) = 0,
\end{aligned}$$

τ is an isomorphism from the graph $\mathcal{G}^{(3)}(\lambda, at+b)$ onto the graph $\mathcal{G}^{(3)}(\lambda, t+\beta^{-3}b)$. □

By Propositions 3.1.1 and 3.1.2, we may focus the study on the graphs $\mathcal{G}^{(3)}(\lambda, t+b)$ and $\mathcal{G}^{(3)}(\lambda, ct+b)$ where $b \in \mathbb{F}_q$. In addition, we have

Proposition 3.1.3. *The graphs $\mathcal{G}^{(3)}(\lambda, t)$ and $\mathcal{G}^{(3)}(\lambda, ct)$ are not isomorphic.*

Proof. Consider the equation

$$(X^3 - X)(\lambda X^3 - X)(\lambda^2 X^3 - X) = 0.$$

If λ is a square element in $\mathbb{F}_q$, then there are seven solutions associated with the equation, and if λ is non-square, then there are five solutions of the equation. It follows that there are at least five fixed vertices in $\mathcal{G}^{(3)}(\lambda, t)$. Next, we consider another equation

$$(X^3 - cX)(\lambda X^3 - cX)(\lambda^2 X^3 - cX) = 0.$$

Since $X^2 = \lambda^{-2}c$ and $X^2 = c$ are insolvable, the equation has at most three solutions. Thus, there are at most three fixed vertices of the graph $\mathcal{G}^{(3)}(\lambda, ct)$. Hence, the number of fixed vertices of $\mathcal{G}^{(3)}(\lambda, t)$ and $\mathcal{G}^{(3)}(\lambda, ct)$ are different, so they are not isomorphic. □

Proposition 3.1.4. *Let $a \neq 0$ and $h \in \mathbb{F}_q$. Then the graph $\mathcal{G}^{(3)}(\lambda, at+h)$ is isomorphic to the graph $\mathcal{G}^{(3)}(\lambda, at-h)$.*

Proof. Let τ be the bijection on $\mathbb{F}_q$ defined by $\tau(x) = -x$ for all $x \in \mathbb{F}_q$. To show that τ preserves the adjacency conditions, we let u and v be in $\mathbb{F}_q$ and compute

$$\left(v^3 - (au+h)\right)\left(\lambda v^3 - (au+h)\right)\left(\lambda^2 v^3 - (au+h)\right) = 0$$

$$\Leftrightarrow \left(-v^3-(-au-h)\right)\left(-\lambda v^3-(-au-h)\right)\left(-\lambda^2 v^3-(-au-h)\right)=0$$
$$\Leftrightarrow \left((-v)^3-(a(-u)-h)\right)\left(\lambda(-v)^3-(a(-u)-h)\right)\left(\lambda^2(-v)^3-(a(-u)-h)\right)=0$$
$$\Leftrightarrow \left((\tau(v))^3-[a(\tau(u))-h]\right)\left(\lambda(\tau(v))^3-[a(\tau(u))-h]\right)\left(\lambda^2(\tau(v))^3-[a(\tau(u))-h]\right)=0.$$

Thus, we have the proposition. □

Proposition 3.1.5. *If $a_1, a_2 \in \mathbb{F}_q^\times$, then the graph $\mathcal{G}^{(3)}(\lambda, a_1 t)$ and $\mathcal{G}^{(3)}(\lambda, a_2 t+b)$ with $b \neq 0$ are not isomorphic.*

Proof. Since $b \neq 0$, $\mathcal{G}^{(3)}(\lambda, a_2 t+b)$ has no vertex with in-degree 1 and out-degree 1 by Proposition 2.1.2 (2). However, the vertex 0 in $\mathcal{G}^{(3)}(\lambda, a_1 t)$ has in-degree 1 and out-degree 1, so both graphs are not isomorphic. □

Proposition 3.1.6. *Assume that $3 \mid (q-1)$. Let $a \neq 0$ and $i, j \in \{0,1,2\}$. If λ is an element of order 3 in $\mathbb{F}_q^\times$, then the graph $\mathcal{G}^{(3)}(\lambda, at+\lambda^i)$ is isomorphic to $\mathcal{G}^{(3)}(\lambda, at+\lambda^j)$.*

Proof. Let τ be the bijection map on $\mathbb{F}_q$ defined by $\tau(x) = \lambda^{j-i}x$ for $x \in \mathbb{F}_q$. To show that τ preserves the adjacency conditions, we let u and v be in $\mathbb{F}_q$. Since

$$\left(v^3-(au+\lambda^i)\right)\left(\lambda v^3-(au+\lambda^i)\right)\left(\lambda^2 v^3-(au+\lambda^i)\right)=0$$
$$\Leftrightarrow \lambda^{j-i}\left(v^3-(au+\lambda^i)\right)\lambda^{j-i}\left(\lambda v^3-(au+\lambda^i)\right)\lambda^{j-i}\left(\lambda^2 v^3-(au+\lambda^i)\right)=0$$
$$\Leftrightarrow \left((\tau(v))^3-(a\tau(u)+\lambda^j)\right)\left(\lambda(\tau(v))^3-(a\tau(u)+\lambda^j)\right)\left(\lambda^2(\tau(v))^3-(a\tau(u)+\lambda^j)\right)=0,$$

we have the proposition. □

Proposition 3.1.7. *If $q \equiv 3 \mod 4$, then the graphs $\mathcal{G}^{(3)}(\lambda, -t+1)$ and $\mathcal{G}^{(3)}(\lambda, t+1)$ are not isomorphic.*

Proof. Suppose that both graphs are isomorphic with an isomorphism τ. By Proposition 2.1.2 (2), 0 is the only vertex in $\mathcal{G}^{(3)}(\lambda, -t+1)$ whose in-degree is 1 and out-degree is 3. Then only the vertex $\tau(0)$ in $\mathcal{G}^{(3)}(\lambda, t+1)$ has in-degree 1 and out-degree 3, so $\tau(0) = 0$. Moreover, 1 is only vertex in $\mathcal{G}^{(3)}(\lambda, -t+1)$ whose in-degree is 3 and out-degree is 1. Thus, only the vertex $\tau(1)$ in $\mathcal{G}^{(3)}(\lambda, t+1)$ has in-degree 1 and out-degree 3, so $\tau(1) = -1$. Since

$$\left(1^3-(-0+1)\right)\left(\lambda(1)^3-(-0+1)\right)\left(\lambda^2(1)^3-(-0+1)\right)=0,$$

there exists an edge from the vertex 0 to 1 in $\mathcal{G}^{(3)}(\lambda, -t+1)$. Since $q \equiv 3 \mod 4$, we know that -1 is non-square, so $\lambda^2 + 1 \neq 0$. On the other hand, we note that

$$\begin{aligned}\left((-1)^3 - (0+1)\right)\left(\lambda(-1)^3 - (0+1)\right)\left(\lambda^2(-1)^3 - (0+1)\right) \\ = (-2)(-\lambda-1)(-\lambda^2-1) \neq 0,\end{aligned}$$

there is no edges from the vertex 0 to -1 in $\mathcal{G}^{(3)}(\lambda, t+1)$. However, since τ preserves the adjacency condition, there is an edge from $\tau(0) = 0$ to $\tau(1) = -1$ in $\mathcal{G}^{(3)}(\lambda, t+1)$. Hence, we have a contradiction, so both graphs are not isomorphic. □

Example 3.1.8. In $\mathbb{Z}_7$ with $\lambda = 2$, by Propositions 3.1.1 to 3.1.5, we have at most eight isomorphism classes of the graph $\mathcal{G}^{(3)}(\lambda, at+b)$ where $a, b \in \mathbb{Z}_7$ with $a \neq 0$ as shown in the following table.

$\mathcal{G}^{(3)}(2, -t)$	$\mathcal{G}^{(3)}(2, t)$
$\mathcal{G}^{(3)}(2, -t+1)$	$\mathcal{G}^{(3)}(2, t+1)$
$\mathcal{G}^{(3)}(2, -t+2)$	$\mathcal{G}^{(3)}(2, t+2)$
$\mathcal{G}^{(3)}(2, -t+4)$	$\mathcal{G}^{(3)}(2, t+4)$

Since the order of λ is 3, by Proposition 3.1.6, the number of isomorphism classes is at most four, and it equals four by Proposition 3.1.7 as shown in the next table.

$\mathcal{G}^{(3)}(2, -t)$	$\mathcal{G}^{(3)}(2, t)$
$\mathcal{G}^{(3)}(2, -t+1)$	$\mathcal{G}^{(3)}(2, t+1)$

3.2 Components with small number of vertices

In this section, we first study the number of vertices in a component of $\mathcal{G}^{(k)}(\lambda, f)$ where $f(t)$ is a permutation polynomial in $\mathbb{F}_q[t]$ and $k \mid (q-1)$. This allows us to study the components of $\mathcal{G}^{(3)}(\lambda, f)$ of small vertices. We show some conditions for the existence of a component with three or four vertices of our desired graphs.

Lemma 3.2.1. *Assume that $k \mid (q-1)$ and $f(t)$ is a permutation polynomial in $\mathbb{F}_q[t]$. Let $\mathcal{C}$ be a component in $\mathcal{G}^{(3)}(\lambda, f)$ with the number of vertices $n_{\mathcal{C}}$. Then*

1. *if 0 is not a vertex of $\mathcal{C}$, then $n_{\mathcal{C}}$ is divisible by k, and*

2. if 0 *is a vertex of* $\mathcal{C}$*, then* $n_{\mathcal{C}} - 1$ *is divisible by* k*.*

Proof. Define

$$\mathcal{E} = \{(x, y) : x, y \in V(\mathcal{C}) \text{ and } y^k = f(x)\}.$$

For $y \in V(\mathcal{C})$, we write

$$\mathcal{A}_y = \{(x, y) : x \in V(\mathcal{C}) \text{ and } y^k = f(x)\}.$$

Then

$$\mathcal{E} = \bigcup_{y \in V(\mathcal{C})} \mathcal{A}_y = \bigcup_{y \in V(\mathcal{C})} \{(f^{-1}(y^k), y)\}$$

because f is one-to-one. Hence $|\mathcal{E}| = \sum_{y \in V(\mathcal{C})} 1 = n_{\mathcal{C}}$. For $x \in V(\mathcal{C})$, we write

$$\mathcal{B}_x = \{(x, y) : y \in V(\mathcal{C}) \text{ and } y^k = f(x)\},$$

and so $\mathcal{E} = \bigsqcup_{x \in V(\mathcal{C})} \mathcal{B}_x$, a disjoint union.

1. Let $x \in V(\mathcal{C})$. Assume that $\mathcal{B}_x \neq \emptyset$. Then there exists $y \in V(\mathcal{C})$ such that $y^k = f(x)$, so $f(x) = (y\omega^j)^k$ for all $j \in \{0, 1, \ldots, k-1\}$ and ω is an element of order k in $\mathbb{F}_q^\times$. Since $x \in V(\mathcal{C}), y\omega^j \in V(\mathcal{C})$ for all $j \in \{0, 1, \ldots, k-1\}$. Thus, $|\mathcal{B}_x| = k$ because $y \neq 0$. Hence, we have shown that $|\mathcal{B}_x|$ is either 0 or k, so $n_{\mathcal{C}} = \sum_{x \in V(\mathcal{C})} |\mathcal{B}_x|$ is divisible by k.

2. If $f(0) = 0$, then $|V(\mathcal{C})| = 1$ by Proposition 2.1.2 (1). Suppose that $f(0) \neq 0$. Then $|\mathcal{B}_{f^{-1}(0)}| = 1$. Let $z \in V(\mathcal{C}) \backslash \{f^{-1}(0)\}$. Assume that $\mathcal{B}_z \neq \emptyset$. Then there exists $y \in V(\mathcal{C})$ such that $y^k = f(z)$, so $f(z) = (y\omega^j)^k$ for all $j \in \{0, 1, \ldots, k-1\}$ and ω is an element of order k in $\mathbb{F}_q^\times$. Since $z \in V(\mathcal{C}), y\omega^j \in V(\mathcal{C})$ for all $j \in \{0, 1, \ldots, k-1\}$. Thus, $|\mathcal{B}_z| = k$ because $f(z) \neq 0$. Hence, we have shown that $|\mathcal{B}_z|$ is either 0 or k, so $n_{\mathcal{C}} - 1 = \sum_{z \in V(\mathcal{C}) \backslash \{f^{-1}(0)\}} |\mathcal{B}_z|$ is divisible by k. □

Example 3.2.2. Lemma 3.2.1 may not hold if $k \nmid (q-1)$. For example, the graph $\mathcal{G}^{(3)}(2, t+2)$ over $\mathbb{Z}_5$ has a component with five vertices whereas the vertex 0 is in this component (see Figure 3.1).

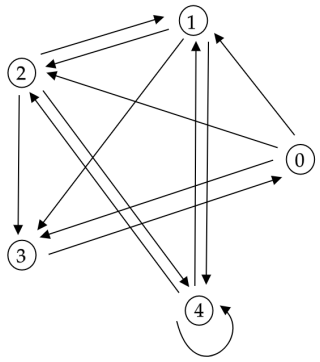
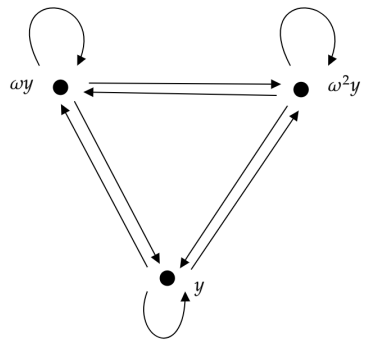


Figure 3.1: The graph $\mathcal{G}^{(3)}(2, t+2)$ over $\mathbb{Z}_5$

Proposition 3.2.3. *Assume that* $3 \mid (q-1)$. *If* $a \in \mathbb{F}_q$, *then the graph* $\mathcal{G}^{(3)}(\lambda, t+a)$ *has a component with three vertices if and only if* $a = 0$ *and the order of* λ *is* 3.

Proof. Let $\mathcal{C}$ be a component of the graph $\mathcal{G}^{(3)}(\lambda, t+a)$ with three vertices. By Lemma 3.2.1, $0 \notin V(\mathcal{C})$ and so $f^{-1}(0) \notin V(\mathcal{C})$. By Proposition 2.1.2, each vertex of $\mathcal{C}$ has in-degree 3 and out-degree 3. Since $3 \mid (q-1)$, we can let ω be a primitive 3rd root of unity in $\mathbb{F}_q^\times$. Let x be a vertex of $\mathcal{C}$ and y be a successor of x. By Lemma 2.1.1, $y^3 = \lambda^{-l}(x+a)$ for some $l \in \{0,1,2\}$, so $y, \omega y$ and $\omega^2 y$ are vertices of $\mathcal{C}$. We have only one possibility of $\mathcal{C}$ displayed below.



Also, we have three predecessors of y, namely $y^3 - a, \lambda y^3 - a$ and $\lambda^2 y^3 - a$. Hence, we obtain system of equations

$$y^3 - a = \omega^i y \tag{3.2.1}$$

$$\lambda y^3 - a = \omega^j y \tag{3.2.2}$$

$$\lambda^2 y^3 - a = \omega^k y \tag{3.2.3}$$

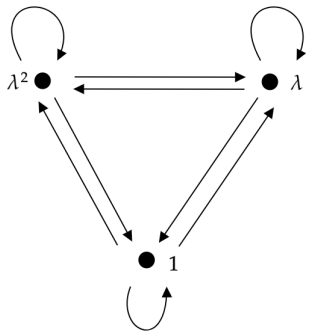
where $\{i, j, k\} = \{0, 1, 2\}$. By (3.2.1) and (3.2.2), we have

$$(\lambda - 1)y^3 = (\omega^j - \omega^i)y. \tag{3.2.4}$$

By (3.2.2) and (3.2.3), we have

$$\lambda(\lambda - 1)y^3 = (\omega^k - \omega^j)y. \tag{3.2.5}$$

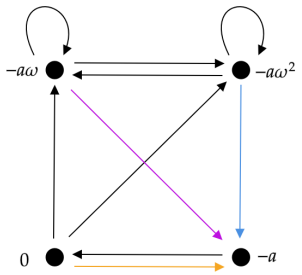
So, by (3.2.4) and (3.2.5), we obtain $\lambda = \dfrac{\omega^k - \omega^j}{\omega^j - \omega^i}$, and hence $\lambda^3 = 1$. By (3.2.1), (3.2.2) and (3.2.3), we have $a = \frac{1}{3}(1+\lambda+\lambda^2)y^3 = 0$. For the converse, the component



of $\mathcal{G}^{(3)}(\lambda, t)$ has three vertices. □

Proposition 3.2.4. *Assume that q is odd and $3 \mid (q-1)$. For each $a \in \mathbb{F}_q$, there is no component of the graph $\mathcal{G}^{(3)}(\lambda, t+a)$ with four vertices.*

Proof. Suppose on a contrary that the graph $\mathcal{G}^{(3)}(\lambda, t+a)$ has a component with four vertices, say $\mathcal{C}$. By Lemma 3.2.1, the vertex $0 \in V(C)$ and $a \neq 0$. Let y be a successor of the vertex 0. By Lemma 2.1.1, $y^3 = \lambda^{-l}a$ for some $l \in \{0, 1, 2\}$, so $y, \omega y$ and $\omega^2 y$ are vertices of $\mathcal{C}$. Since $-a$ is a nonzero predecessor of the vertex 0, we have $-a = y\omega^j$ for some $j \in \{0, 1, 2\}$ and so all successors of the vertex 0 are $-a, -a\omega$ and $-a\omega^2$ where ω is a primitive 3rd root of unity in $\mathbb{F}_q$. By Lemma 2.1.1 the vertex 0 has in-degree 1, and so the vertices $-a\omega$ as well as $-a\omega^2$ have three common successors $-a, -a\omega$ and $-a\omega^2$. We have only one possibility of $\mathcal{C}$ displayed below.



In addition, there are three predecessors of $-a$, namely $-a^3 - a, -\lambda a^3 - a$ and $-\lambda^2 a^3 - a$. Hence, we obtain

$$\lambda^i(-a)^3 = 0 + a \tag{3.2.6}$$

$$\lambda^j(-a)^3 = -a\omega + a \tag{3.2.7}$$

$$\lambda^k(-a)^3 = -a\omega^2 + a \tag{3.2.8}$$

where $\{i, j, k\} = \{0, 1, 2\}$. By the above equations, we have

$$\lambda^k - \lambda^j = (\omega - 1)(\lambda^j - \lambda^i). \tag{3.2.9}$$

We next permute i, j, k in six cases as follows.

Case 1. $i = 0, j = 1$ and $k = 2$. By (3.2.9), we have $\lambda = \omega - 1$. But, from (3.2.6) and (3.2.7), we have $\lambda = -\omega + 1$ which implies that $2 = 0$, a contradiction.

Case 2. $i = 0, j = 2$ and $k = 1$. By (3.2.9), we have $\lambda = \omega$. But, from (3.2.6) and (3.2.8), we have $\lambda = -\omega^2 + 1$ and thus $2 = 0$ which is a contradiction.

Case 3. $i = 1, j = 0$ and $k = 2$. By (3.2.9), we have $\lambda = -\omega$. But, from (3.2.6) and (3.2.7), we have $\lambda = \frac{1}{1-\omega}$ and so $2 = 0$ which is impossible.

Case 4. $i = 1, j = 2$ and $k = 0$. By (3.2.9), we have $\lambda = -\omega^2$. But, from (3.2.6) and (3.2.8), we have $\lambda = \frac{1}{1-\omega^2}$ and thus $2 = 0$ which is impossible.

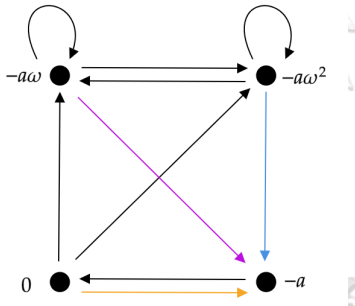
Case 5. $i = 2, j = 0$ and $k = 1$. By (3.2.9), we have $\lambda = \frac{\omega}{1-\omega}$. But from (3.2.7) and (3.2.8), we have $\lambda = \omega + 1$ and so $2 = 0$ which is absurd.

Case 6. $i = 2, j = 1$ and $k = 0$. By (3.2.9), we have $\lambda = \frac{1}{\omega-1}$. But from (3.2.7) and (3.2.8), we have $\lambda = \frac{1}{1+\omega}$ which also implies that $2 = 0$, a contradiction.

Hence, we obtain a contradiction and so there is no components with four vertices. □

Proposition 3.2.5. *Assume that q is even and $3 \mid (q-1)$. If $a \in \mathbb{F}_q$, then the graph $\mathcal{G}^{(3)}(\lambda, t+a)$ has a component with four vertices if and only if the order of λ is 3 and $a = \lambda^i$ for some $i \in \{0,1,2\}$.*

Proof. Since $\mathrm{char}\mathbb{F}_q = 2$, we have $\ker \varphi_2 = \{1\}$. Let $\mathcal{C}$ be a component of the graph $\mathcal{G}^{(3)}(\lambda, t+a)$ with four vertices. By Lemma 3.2.1, $0 \notin \mathcal{C}$ and $a \neq 0$. Let y be a successor of the vertex 0. By Lemma 2.1.1, $y^3 = \lambda^{-l}a$ for some $l \in \{0,1,2\}$, so $y, \omega y$ and $\omega^2 y$ are vertices of $\mathcal{C}$. Since $-a$ is a nonzero predecessor of the vertex 0, we have $-a = y\omega^j$ for some $j \in \{0,1,2\}$ and so all successors of the vertex 0 are $-a, -a\omega$ and $-a\omega^2$ where ω is a primitive 3rd root of unity in $\mathbb{F}_q$. By Lemma 2.1.1 the vertex 0 has in-degree 1, and so the vertices $-a\omega$ as well as $-a\omega^2$ have three common successors $-a, -a\omega$ and $-a\omega^2$. We have only one possibility of $\mathcal{C}$ displayed below.



In addition, there are three predecessors of $-a$, namely $-a^3 - a, -\lambda a^3 - a$ and $-\lambda^2 a^3 - a$. Hence, we obtain

$$\lambda^i(-a)^3 = 0 + a \tag{3.2.10}$$

$$\lambda^j(-a)^3 = -a\omega + a \tag{3.2.11}$$

$$\lambda^k(-a)^3 = -a\omega^2 + a \tag{3.2.12}$$

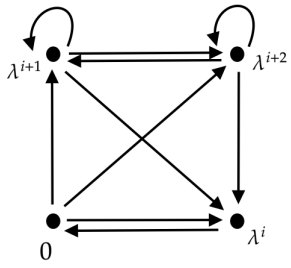
where $\{i,j,k\} = \{0,1,2\}$. By the above equations, we have

$$(1 + \lambda + \lambda^2)(-a^3) = 4a = 0$$

because $\mathrm{char}\mathbb{F}_q = 2$. So, we have $\lambda^3 = 1$. By equations (3.2.10), (3.2.11) and (3.2.12), we again have

$$\lambda^3(-a^9) = (a^3)(1-\omega)(1-\omega^2) = 3a^3 = a^3,$$

so $a^6 = 1$ and hence $a^3 = 1$ because $|\ker \varphi_2| = 1$. For the converse, the component



has four vertices. □

Example 3.2.6. Let $\mathbb{F}_4 = \{a+b\alpha : a, b \in \mathbb{Z}_2 \text{ and } \alpha^2+\alpha+1 = 0\} \cong \mathbb{Z}_2[t]/\langle t^2+t+1\rangle$. Let $f(t) = t + a$ be a linear permutation polynomial in $\mathbb{F}_4[t]$. Let $\mathcal{C}$ be a component in $\mathcal{G}^{(3)}(\lambda, f)$ not containing the vertex 0. By Lemma 3.2.1, the number of vertices in $\mathcal{C}$ is a multiple of 3. Let $\mathcal{D}$ be a component containing the vertex 0. Then the number of vertices in $\mathcal{D}$ is $3d+1$ for some $d \in \{0, 1\}$. If $d = 0$, then there is a graph containing a component with three vertices, for example, $\mathcal{G}^{(3)}(\alpha, t)$ (see Figure 3.2). If $d = 1$, then there is a graph containing a component with three vertices such as $\mathcal{G}^{(3)}(\alpha, t + \alpha)$ (see Figure 3.2). Our desired λ is provided by Proposition 3.2.5. Hence, any graph $\mathcal{G}^{(3)}(\lambda, t + a)$ is isomorphic to the graph in Figure 3.2 or Figure 3.3.

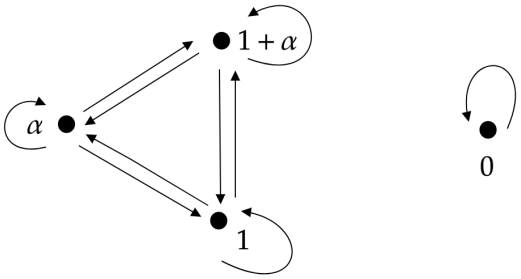


Figure 3.2: The graph $\mathcal{G}^{(3)}(\alpha, t)$ over $\mathbb{F}_4$

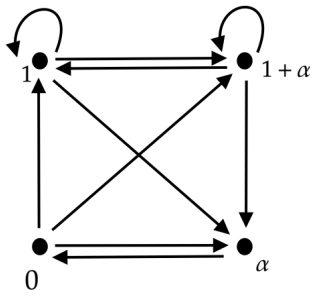


Figure 3.3: The graph $\mathcal{G}^{(3)}(\alpha, t+\alpha)$ over $\mathbb{F}_4$

Example 3.2.7. Let $f(t) = t + a$ be a permutation polynomial in $\mathbb{Z}_7[t]$. Let $\mathcal{C}$ be a component in $\mathcal{G}^{(3)}(\lambda, f)$ not containing the vertex 0. By Lemma 3.2.1, the number of vertices in $\mathcal{C}$ is a multiple of 3. Let $\mathcal{D}$ be a component containing the vertex 0. Then the number of vertices in $\mathcal{D}$ is $3d + 1$ for some $d \in \{0, 1, 2\}$. The graph $\mathcal{G}^{(3)}(\lambda, f)$ does not have a component with four vertices by Proposition 3.2.4, so d is 0 or 2. If $d = 0$, then there are two possibilities of our desired graphs. The first one is a graph containing a component with six vertices, for example $\mathcal{G}^{(3)}(3, t)$ (see Figure 3.4). Another one is a graph containing two components with three vertices such as $\mathcal{G}^{(3)}(4, t)$ (see Figure 3.5). This desired λ is given by Proposition 3.2.3 as well. If $d = 2$, then there is an equational graph $\mathcal{G}^{(3)}(\lambda, f)$ with 7 vertices, for instance $\mathcal{G}^{(3)}(3, t+1)$ (see Figure 3.6). Hence, by working on small components, we find that the graph $\mathcal{G}^{(3)}(\lambda, t + a)$ is isomorphic to the graph in Figure 3.4 or Figure 3.5 or Figure 3.6.

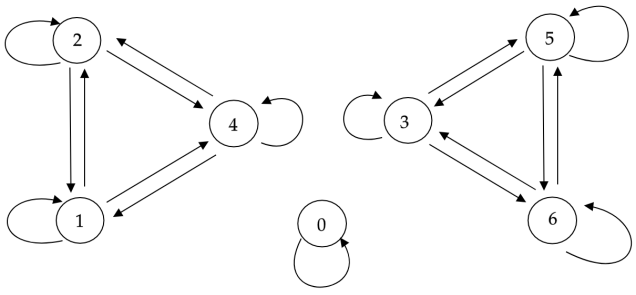


Figure 3.4: The graph $\mathcal{G}^{(3)}(4, t)$ over $\mathbb{Z}_7$

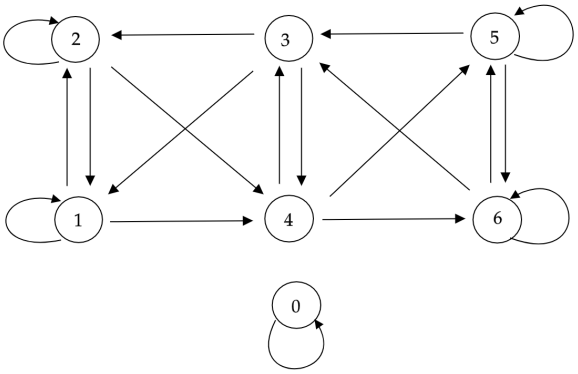


Figure 3.5: The graph $\mathcal{G}^{(3)}(3,t)$ over $\mathbb{Z}_7$

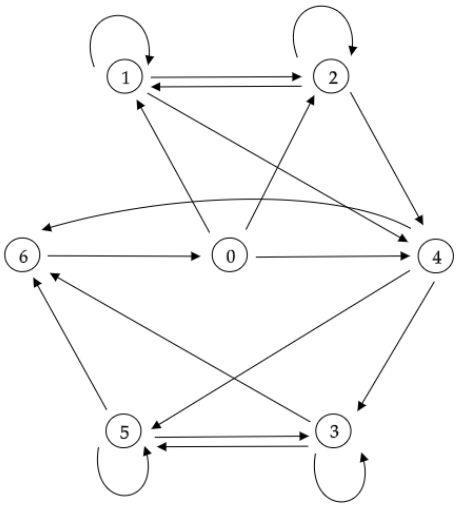


Figure 3.6: The graph $\mathcal{G}^{(3)}(3,t+1)$ over $\mathbb{Z}_7$

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