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# การศึกษาริงซึ่งไอดีลทางขวาเฉพาะเป็นไอดีลทางขวาที่ไม่แปรเปลี่ยนอย่างเต็มทั้งหมด

นายปรเมศ แสงจันทร์

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# STUDY ON RINGS WHOSE PRIME RIGHT IDEALS ARE TOTALLY FULLY INVARIANT

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ปรเมศ แสงจันทร์: การศึกษาริงซึ่งไอดีลทางขวาเฉพาะเป็นไอดีลทางขวาที่ไม่แปรเปลี่ยนอย่างเต็มทั้งหมด. (STUDY ON RINGS WHOSE PRIME RIGHT IDEALS ARE TOTALLY FULLY INVARIANT) อ.ที่ปรึกษาโครงงานหลัก : รศ.ดร.เอื้อมพร ฟักสุวรรณ, อ.ที่ปรึกษาโครงงานร่วม : รศ.ดร.ศจี เพียรสกุล, 27 หน้า.

ในโครงงานนี้เราใช้แนวคิดของไอดีลทางขวา (ซ้าย) ที่ไม่แปรเปลี่ยนอย่างเต็มทั้งหมดในการนิยามริงที่ไอดีลทางขวา (ซ้าย) เฉพาะเป็นไอดีลทางขวา (ซ้าย) ที่ไม่แปรเปลี่ยนอย่างเต็มทั้งหมด เรียกริงนี้ว่าริงคู่เสมือนอย่างแรงทางขวา (ซ้าย) เฉพาะ เราศึกษาสมบัติบางประการของริงคู่เสมือนอย่างแรงทางขวา (ซ้าย) เฉพาะ ยิ่งไปกว่านั้นเราสนใจการศึกษาไอดีล $J^*(R)$ ของริงคู่เสมือนอย่างแรงทางขวาเฉพาะ $R$ ซึ่งคืออินเตอร์เซกชันของไอดีลทางขวาเฉพาะทั้งหมด เราพบว่าถ้าริง $R$ เป็นริงกรรมพันธุ์และริงคู่เสมือนอย่างแรงทางขวาเฉพาะแล้ว $R \,/\, J^*(R)$ เป็นริงคู่เสมือนอย่างแรงทางขวาเฉพาะ ในส่วนสุดท้ายเราศึกษาสมบัติของตัวหารศูนย์ในริงใดๆ ในริงคู่เสมือนอย่างแรงทางขวาเฉพาะและในริงคู่เสมือนอย่างแรงทางซ้ายเฉพาะ

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PORAMATE SANGCHAN: STUDY ON RINGS WHOSE PRIME RIGHT IDEALS ARE TOTALLY FULLY INVARIANT. ADVISOR : ASSOC. PROF. OUAMPORN PHUKSUWAN, PH.D., CO-ADVISOR : ASSOC. PROF. SAJEE PIANSKOOL, PH.D., 27 PP.

In this project, we use the notion of totally fully invariant right (left) ideal to define the ring whose prime right (left) ideals are totally fully invariant. This ring is called a prime right (left) strongly quasi-duo ring. We investigate some properties of a prime right strongly quasi-duo ring. Moreover, we are interested in studying the ideal $J^*(R)$ of a prime right strongly quasi-duo ring $R$ which is the intersection of all prime right ideals of $R$. We find that if $R$ is a hereditary and a prime right strongly quasi-duo ring, then $R/J^*(R)$ is a prime right strongly quasi-duo ring. Finally, we study zero divisors in an arbitrary ring, in a prime right strongly quasi-duo ring and in a prime left strongly quasi-duo ring.

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Poramate Sangchan
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# Contents

# Chapter 1

# Introduction

S. Safaeeyan [5] studied some properties of a right ideal $I$ of a ring $R$. That right ideal is called *totally fully invariant* and defined by for each $a \in I$ and $b \in R$, $ann_r(a) \subseteq ann_r(b)$ implies that $b \in I$, where $ann_r(x)$ is the set of all right annihilators of $x \in R$; that is, $ann_r(x) = \{y \in R : xy = 0\}$. A special property of a totally fully invariant right ideal is that every totally fully invariant right ideal of $R$ is a two-sided ideal. There have been many studies about some special rings such as right (left) duo rings, and right (left) quasi-duo rings. A right (left) duo ring is a ring that any right (left) ideal of $R$ is a two-sided ideal. In a noncommutative ring, the ring whose all maximal right (left) ideals are two-sided ideals is called a right (left) quasi-duo ring. By the definition of a right (left) duo ring and a right (left) quasi-duo ring, any right (left) duo ring is a right (left) quasi-duo ring. Safaeeyan defined another ring, called a *right strongly quasi-duo ring*, as the ring whose maximal right ideals are totally fully invariant right ideals. Since every totally fully invariant right ideal is a two-sided ideal, it follows that every right strongly quasi-duo ring is a right quasi-duo ring. Safaeeyan got many properties of right strongly quasi-duo rings. He discovered that if $R$ is a right strongly quasi-duo ring, then the set of all units in $R$ is $\{a \in R : ann_r(a) = \{0\}\}$. By using this result, he noticed that $R$ is a division ring if and only if $R$ is a right strongly quasi-duo domain. He added the condition on a hereditary ring to a right strongy quasi-duo ring and realized that if $R$ is a hereditary ring and a right strongly quasi-duo ring, then $R/J(R)$ is a right strongly quasi-duo ring, where $J(R)$ is the intersection of all maximal right ideals of $R$. Moreover, he identified that if $R$ is a right strongly quasi-duo ring, then nonequal maximal right ideals of $R$ are not isomorphic.

In our work, we study on rings whose all prime right ideals are totally fully invariant right ideals, namely a prime right strongly quasi-duo ring. We know from [3] that all prime right strongly quasi-duo rings are right strongly quasi-duo rings. In this project, we investigate some properties of prime right strongly quasi-duo rings. We prove some results on rings whose prime right ideals are totally fully invariant right ideals which are analogous to the results of Safaeeyan. In final part, we study about zero divisors in a ring $R$. We denote $LZD(R), RZD(R), TZD(R)$, and $STZD(R)$ to be the set of all left zero divisors, the set of all right zero divisors, the set of all two-sided zero divisors, and the set of all strongly two-sided zero divisors, respectively. We obtain some relations of these sets and also with the set of all units and the set of all right (left) annihilators in an arbitary ring $R$. Moreover, in a prime right (left) strongly quasi-duo ring, we get more precise results such as, in a prime right strongly quasi-duo ring $R$, $TZD(R) = RZD(R)$.

# Chapter 2

# Background

In this report, a "ring" means an associative ring with identity. In this chapter, we first give the definitions of a right (left) ideal, a two-sided ideal, a maximal right ideal, a prime right ideal and some background knowledge in algebra course quoted from [1]. We also provide the proofs of some theorems here.

**Definition 2.1.** Let $R$ be a ring. A subgroup $I$ of $R$ is called a *right (left) ideal* if for each $a \in I$ and $r \in R$, $ar \in I\,(ra \in I)$.

**Definition 2.2.** Let $R$ be a ring. A subgroup $I$ of $R$ is called a *two-sided ideal* if it is both a left ideal and a right ideal.

**Definition 2.3.** Let $R$ be a ring. A proper right (left) ideal $M$ of $R$ is called a *maximal right (left) ideal* if for every right (left) ideal $J$ of $R$, $M \subseteq J \subseteq R$ implies that $M = J$ or $J = R$.

**Definition 2.4.** Let $R$ be a ring. A proper right (left) ideal $P$ of $R$ is called a *prime right (left) ideal* if for every right (left) ideals $A$ and $B$, $AB \subseteq P$ impiles that $A \subseteq P$ or $B \subseteq P$.

**Theorem 2.5.** [3] *Let $R$ be a ring. Then every maximal right (left) ideal is a prime right (left) ideal.*

**Theorem 2.6.** [1] *If $R$ is a finite commutative ring, then maximal ideals and prime ideals coincide.*

For each two-sided ideal $J$ of a ring $R$, the set of all $n \times n$ matrices with entries from $J$ is denoted by $M_n(J)$.

**Theorem 2.7.** [1] *Let $R$ be a ring. Then every two-sided ideal of $M_n(R)$ is of the form $M_n(J)$ where $J$ is a two-sided ideal of $R$.*

**Theorem 2.8.** *Let $R$ be a ring, $I$ be a two-sided ideal of $R$ and $J$ be a subgroup of $R$ such that $I \subseteq J$. Then $J$ is a right ideal of $R$ if and only if $J/I$ is a right ideal of $R/I$.*

*Proof.* ($\rightarrow$) Assume that $J$ is a right ideal of $R$. Since $I$ is a two-sided ideal of $R$, $I$ is a group. Since $J$ is a subgroup of $R$ such that $I \subseteq J$, $I$ is a subgroup of $J$. Since $I$ is a two-sided ideal of $R$ and $J \subseteq R$, it follows that $ab \in I$ and $ba \in I$ for each $a \in J$ and for each $b \in I$. Hence $I$ is a two-sided ideal of $J$. Then $J/I$ is a ring (note here that $J/I$ is an associative ring but it may not have an identity). Since $J$ is a subring of $R$, $J/I$ is a subring of $R/I$. To show that $J/I$ is a right ideal of $R/I$, let $x + I \in R/I$ where $x \in R$ and $y + I \in J/I$ where $y \in J$. Thus $(y + I)(x + I) = yx + I$. Since $J$ is a right ideal of $R$, $yx \in J$. Then $yx + I \in J/I$ . Hence $(y + I)(x + I) \in J/I$.
($\leftarrow$) Assume that $J/I$ is a right ideal of $R/I$. By the assumption, $J$ is a subgroup of $R$. It remains to show that $J$ is a right ideal of $R$. We will show that for each $x \in R$ and for each $y \in J$, $yx \in J$. Let $x \in R$ and $y \in J$. Then $x + I \in R/I$ and $y + I \in J/I$. Thus $(y + I)(x + I) = yx + I$. Since $J/I$ is a right ideal of $R/I$, $yx + I \in J/I$. Then there exists $a \in J$ such that $yx+I = a+I$. Thus $yx-a \in I \subseteq J$, so $yx = yx-a+a \in J$. $\square$

Let $R$ be a ring and $I$ a two-sided ideal of $R$. Define $\pi : R \rightarrow R/I$ by

$$\pi(a) = a + I \text{ for all } a \in R.$$

Then $\pi$ is called the *canonical projection* from $R$ to $R/I$. Moreover, $\pi$ is a surjective homomorphism of $R$. Note that if $L$ is a right ideal of $R$ such that $I \subseteq L$, then $\pi(L) = L/I$ where $\pi(L)$ is the image of $L$ under $\pi$. To prove this, let $L$ be a right ideal of $R$ such that $I \subseteq L$.
" $\subseteq$ " Let $\pi(x) \in \pi(L)$ where $x \in L$. Then $\pi(x) = x + I \in L/I$.
" $\supseteq$ " Let $x + I \in L/I$ where $x \in L$. Then $x + I = \pi(x) \in \pi(L)$.

**Proposition 2.9.** *Let $I$ be a two-sided ideal of $R$. Then there is a one to one correspondence between a right ideal of $R$ containing $I$ and a right ideal of $R/I$ given by $J \mapsto \pi(J)$ where $J$ is a right ideal of $R$ such that $I \subseteq J$.*

This proposition describes right ideals of $R/I$ using right ideals of $R$.

*Proof.* We will show that there is a one to one correspondence between
$\mathcal{A} := \{J : J \text{ is a right ideal of } R \text{ such that } I \subseteq J\}$ and
$\mathcal{B} := \{K : K \text{ is a right ideal of } R/I\}$.
Define $\phi : \mathcal{A} \rightarrow \mathcal{B}$ by

$$\phi(J) = \pi(J) = J/I \text{ for all } J \in \mathcal{A}.$$

(1) To show that $\phi$ is injective, let $G, H \in \mathcal{A}$ and $\phi(G) = \phi(H)$. Thus $G/I = H/I$. Next we will show that $G \subseteq H$. Let $g \in G$. Then $g + I \in G/I = H/I$. Thus there exists $r \in H$ such that $g + I = r + I$. Hence $g - r \in I \subseteq H$. Then $g = g - r + r \in H$. Similarly, we obtain $H \subseteq G$. Then $G = H$. Hence $\phi$ is injective.

(2) To show that $\phi$ is surjective, let $L \in \mathcal{B}$. Since $\pi$ is a ring homomorphism and $L$ is a right ideal of $R/I$, $\pi^{-1}(L)$ is a right ideal of $R$. Since for each $a \in I$ we get $\pi(a) = a + I = 0 + I \in L$, it follows that $a \in \pi^{-1}(L)$ for all $a \in I$. Thus $I \subseteq \pi^{-1}(L)$. Hence $\pi^{-1}(L) \in \mathcal{A}$. Since $\pi$ is surjective, $\phi(\pi^{-1}(L)) = \pi(\pi^{-1}(L)) = L$. Hence $\phi$ is surjective. $\square$

**Theorem 2.10.** *Let $I$ be a two-sided ideal of $R$ and $P$ be a right ideal of $R$ such that $I \subseteq P$. If $P/I$ is a prime right ideal of $R/I$, then $P$ is a prime right ideal of $R$.*

*Proof.* Assume that $P/I$ is a prime right ideal of $R/I$. Let $A$ and $B$ be right ideals of $R$ such that $AB \subseteq P$. Then $A + I$ is a right ideal of $R$ since $A$ and $I$ are right ideals of $R$. It is obvious that $I \subseteq A + I$. Similarly, we have $B + I$ is a right ideal of $R$ and $I \subseteq B + I$. By Theorem 2.8 $(A + I)/I$ and $(B + I)/I$ are right ideals of $R/I$.

We will show $((A+I)/I)((B+I)/I) \subseteq P/I$. Note that $(A+I)/I = \{a+I : a \in A\}$ and $(B + I)/I = \{b + I : b \in A\}$. Let $(a + I)(b + I) \in ((A + I)/I)((B + I)/I)$ where $a \in A$ and $b \in B$. Then $(a + I)(b + I) = ab + I$. Since $ab \in AB \subseteq P$, $(a + I)(b + I) = ab + I \in P/I$. Thus $((A + I)/I)((B + I)/I) \subseteq P/I$. Since $P/I$ is a prime right ideal of $R/I$, $(A + I)/I \subseteq P/I$ or $(B + I)/I \subseteq P/I$.

Case$(A + I)/I \subseteq P/I$.

Let $a \in A$. Then $a + I \in (A + I)/I \subseteq P/I$. Thus $a + I = u + I$ for some $u \in P$. Thus $a - u \in I \subseteq P$, so $a = a - u + u \in P$. Hence $A \subseteq P$.

Case$(B + I)/I \subseteq P/I$.

It is similar to the case $(A + I)/I \subseteq P/I$, so we obtain $B \subseteq P$. $\square$

Next, we introduce the definitions of right $R$-modules and module $R$-homomorphisms where $R$ is a ring. The concept of module $R$-homomorphisms is used to define a projective module and then to define a hereditary ring, which is the additional condition to give some results in our work as will be appeared later.

**Definition 2.11.** Let $R$ be a ring. *A right $R$-module* or a *right module over $R$* is a set $M$ together with

(1) a binary operation $+$ on $M$ under which $M$ is an abelian group, and

(2) an action of $R$ on $M$ (that is, a map $M \times R \to M$), denoted the image of $(m, r)$ by $mr$, for all $r \in R$ and for all $m \in M$, which satisfies

(a) $m(r + s) = mr + ms$, for all $r, s \in R, m \in M$,

(b) $m(rs) = (mr)s$, for all $r, s \in R, m \in M$, and

(c) $(m + n)r = mr + nr$, for all $r \in R, m, n \in M$.

If a ring $R$ has an identity 1 and $M$ satisfies the additional axiom below, then $M$ is called a *unital module.*

(d) $m1 = m$, for all $m \in M$.

Note that the descriptor right in the above definition indicates that the ring elements appear on the right. Moreover, left $R$-modules can be defined analogously

An example of a right $R$-module is a right ideal of $R$. This is a consequence from distribution and association of $R$ under operations addition and multiplication from $R$.

We use a similar agument of an example of a right $R$-module. Thus an example of a left $R$-module is a left ideal of $R$.

**Definition 2.12.** Let $R$ be a ring and let $M$ and $N$ be right $R$-modules. A map $\phi : M \to N$ is an *R-module homomorphism* if it respects the $R-$module structures of $M$ and $N$, i.e.,
(a) $\phi(x + y) = \phi(x) + \phi(y)$, for all $x, y \in M$, and
(b) $\phi(xr) = \phi(x)r$, for all $r \in R, x \in M$.

**Definition 2.13.** Let $R$ be a ring, $M$ and $N$ be right $R$-modules. Define $Hom_R(M, N)$ to be the set of all $R$-module homomorphisms from $M$ into $N$.

**Definition 2.14.** [4] Let $P$ be an $R$-module. Then $P$ is a *projective module* if and only if for any surjectivetive $R$-module homomorphism $\alpha : B \to C$ and any $R$-module homomorphism $\beta : P \to C$, there is an $R$-module homomorphism $\gamma : P \to B$ such that $\beta = \alpha\gamma$.

**Definition 2.15.** [2] A ring $R$ is called *hereditary* if all right ideals of $R$ are projective as $R$-modules and all left ideals are projective as left $R$-modules.

Next, we define certain sets which play important role in this project. To define a totally fully invariant right (left) ideal, we must know what $ann_r(x)$ and $ann_l(x)$ are for all $x \in R$. Moreover, we study the set of all units in $R$, denoted by $U(R)$; that is

$$U(R) = \{a \in R : \exists b \in R,\ ab = 1 = ba\}.$$

**Definition 2.16.** Let $R$ be a ring. For each $x \in R$, let

$$ann_r(x) = \{y \in R : xy = 0\} \text{ and } ann_l(x) = \{y \in R : yx = 0\}.$$

We will show that for each $a \in R$, $ann_r(a)$ is a right ideal of $R$. Let $a \in R$. It is obvious that $0 \in ann_r(a)$. Hence $ann_r(a) \neq \emptyset$. Since for each $x, y \in ann_r(a)$, $a(x - y) = ax - ay = 0 - 0 = 0$, $ann_r(a)$ is a subgroup of $R$. To show that $ann_r(a)$ is a right ideal of $R$, let $g \in ann_r(a)$ and $k \in R$. Then $a(gk) = (ag)k = (0)k = 0$. Thus

$ann_r(a)$ is a right ideal of $R$. It can be proved similarly that $ann_l(a)$ is a left ideal of $R$.

Next, we give the definitions and some results of right strongly quasi-duo rings that are studied by S. Safaeeyan in [5].

**Definition 2.17.** A right ideal $I$ of $R$ is called a *totally fully invariant right ideal* if for each $a \in I$ and $b \in R$, $ann_r(a) \subseteq ann_r(b)$ implies that $b \in I$.

A left ideal $I$ of $R$ is called a *totally fully invariant left ideal* if for each $a \in I$ and $b \in R$, $ann_l(a) \subseteq ann_l(b)$ implies that $b \in I$.

**Example 2.18.** (1) Trivial ideals of a ring are totally fully invariant right ideals.
(2) Every ideal of $\mathbb{Z}_n$ $(n \geq 2)$ is both a totally fully invariant right ideal and a totally fully invariant left ideal.

**Theorem 2.19.** [5] *Every totally fully invariant right ideal of $R$ is a two-sided ideal.*

**Theorem 2.20.** [5] *A right ideal $I$ of a ring $R$ is totally fully invariant if and only if for each right ideal $J$ of $R$ contained in $I$ and each $f \in Hom_R(J, R)$, $f(J) \subseteq I$.*

Theorem 2.20 gives another way to show that $I$ is a totally fully invariant right ideal. The definition of a right strongly quasi-duo ring relies on the concept of totally fully invariant as follows.

**Definition 2.21.** [5] A ring $R$ is called a *right strongly quasi-duo ring* if every maximal right ideal of $R$ is a totally fully invariant right ideal.

We know from Example 2.18 that every ideal of $\mathbb{Z}_n$ $(n \geq 2)$ is a totally fully invariant right ideal, so every maximal right ideal of $\mathbb{Z}_n$ $(n \geq 2)$ is a totally fully invariant right ideal. Thus $\mathbb{Z}_n$ $(n \geq 2)$ is an example of right strongly-quasi duo rings.

**Proposition 2.22.** [5] *Let $R$ be a right strongly quasi-duo ring. Then*

$$U(R) = \{a \in R : ann_r(a) = \{0\}\}.$$

**Corollary 2.23.** [5] *$R$ is a division ring if and only if $R$ is a right strongly quasi-duo domain.*

Recall that $J(R)$ is the Jacobson radical of $R$, which is the intersection of all maximal right ideals of $R$.

**Theorem 2.24.** [5] *Let $R$ be a hereditary and right strongly quasi-duo ring. Then $R/J(R)$ is a right strongly quasi-duo ring.*

**Theorem 2.25.** [5] *Let $R$ be a right strongly quasi-duo ring and $I$ be a proper right ideal of $R$. Then $Hom_R(R, I)$ has no monomorphism element.*

**Corollary 2.26.** [5] *If $R$ is a right strongly quasi-duo ring, then $R$ is not isomorphic as $R$-module to any proper right ideal of itself.*

**Proposition 2.27.** [5] *If $R$ is a right strongly quasi-duo ring, then nonequal maximal right ideals of $R$ are not isomorphic as $R$-module.*

In our work, we change the condition that "every maximal right ideal is totally fully invariant" of a right strongly quasi-duo ring to the new condition that "every prime right ideal is totally fully invariant".

**Definition 2.28.** A ring $R$ is called a *prime right (left) strongly quasi-duo ring* if every prime right (left) ideal of $R$ is a totally fully invariant right (left) ideal.

By using Theorem 2.6 and the fact that all ideals of $\mathbb{Z}_n$ $(n \geq 2)$ are totally fully invariant right ideals, $\mathbb{Z}_n$ $(n \geq 2)$ is an example of prime right strongly quasi-duo rings. Moreover, an obvious example of prime right strongly quasi-duo rings is a division ring because, by Example 2.18(1), trivial ideals are totally fully invariant right ideals and all ideals of a division ring are only trivial ideals. In particular, all fields are right strongly quasi-duo rings.

If $R$ is a commutative ring and $x \in R$, then the set of all right annihilators of $x$ equals the set of all left annihilators of $x$. Hence an example of a prime left strongly quasi-duo ring is $\mathbb{Z}_n$ $(n \geq 2)$.

The next proposition shows the relation between a right strongly quasi-duo ring and a prime right strongly quasi-duo ring by using Theorem 2.5.

**Proposition 2.29.** *If $R$ is a prime right strongly quasi-duo ring, then $R$ is a right strongly quasi-duo ring.*

*Proof.* It is obtained directly from Theorem 2.5. □

One can say that prime right strongly quasi-duo rings and right strongly quasi-duo rings are identical when $R$ is a finite commutative ring by Theorem 2.6 and Proposition 2.29.

**Proposition 2.30.** *Let $R$ be a finite commutative ring. Then $R$ is a prime right strongly quasi-duo ring if and only if $R$ is a right strongly quasi-duo ring.*

*Proof.* $(\rightarrow)$ This is true by Proposition 2.29.

$(\leftarrow)$ Assume that $R$ is a right strongly quasi-duo ring. Hence every maximal right ideal is totally fully invariant. Since $R$ is a finite commutative ring, it follows from Theorem 2.6 that maximal ideals and prime ideals coincide . Thus every prime right ideal is totally fully invariant. Hence $R$ is a prime right strongly quasi-duo ring.

# Chapter 3

# Main results

## 3.1 Some Properties of Prime Right Strongly Quasi-Duo Rings

In this section, we give some results of prime right strongly quasi-duo rings. By Proposition 2.29, prime right strongly quasi-duo rings are right strongly quasi-duo rings, so we can derive some results of prime right strongly quasi-duo rings directly from the results of right strongly quasi-duo rings as in Theorems 3.1-3.3.

**Theorem 3.1.** *Let $R$ be a prime right strongly quasi-duo ring. Then*

$$U(R) = \{a \in R : ann_r(a) = \{0\}\}.$$

**Theorem 3.2.** *Let $R$ be a prime right strongly quasi-duo ring and $I$ be a proper right ideal of $R$. Then $Hom_R(R, I)$ has no monomorphism element.*

**Theorem 3.3.** *If $R$ is a prime right strongly quasi-duo ring, then $R$ is not isomorphic as $R$-module to any proper right ideal of itself.*

However, we also get other results of prime right strongly quasi-duo rings that are analogous to the ones of right strongly quasi-duo rings.

**Theorem 3.4.** *$R$ is a division ring if and only if $R$ is a prime right strongly quasi-duo domain.*

*Proof.* Since all ideals of a division ring are trivial ideals and by Example 2.18(1), the first direction is done. Conversely, the statement holds by Theorem 2.29 and Corollary 2.23 □

**Theorem 3.5.** *Let $R$ be a prime right strongly quasi-duo ring. If $P_1$ and $P_2$ are prime right ideals of $R$ which are isomorphic as $R$-module, then $P_1 = P_2$.*

*Proof.* Assume that $P_1$ and $P_2$ are two prime right ideals of $R$ which are isomorphic. Then $P_1$ and $P_2$ are right $R$-modules and there exists $f \in Hom_R(P_1, P_2)$ such that $f$ is a bijection. To show that $P_1 \subseteq P_2$, let $a \in P_1$. We will show $ann_r(f(a)) \subseteq ann_r(a)$. Let $y \in ann_r(f(a))$. Then $f(a)y = 0$. Thus $f(ay) = 0 = f(0)$. Since $f$ is one to one, $ay = 0$. Then $y \in ann_r(a)$. Hence $ann_r(f(a)) \subseteq ann_r(a)$. Since $f(a) \in P_2$ and $P_2$ is a totally fully invariant right ideal, $a \in P_2$. Similarly, we also get $P_2 \subseteq P_1$. Hence $P_1 = P_2$. □

Theorem 3.5 states that, in prime right strongly quasi-duo rings, nonequal prime right ideals of $R$ are not isomorphic as $R$-module.

Let $R$ be a ring. Define $J^*(R)$ to be the intersection of all prime right ideals of $R$.

**Theorem 3.6.** *If $R$ is a prime right strongly quasi-duo ring, then $J^*(R)$ is a totally fully invariant right ideal of $R$.*

*Proof.* Assume that $R$ is a prime right strongly quasi-duo ring. Let $a \in J^*(R)$ and $b \in R$. Suppose $ann_r(a) \subseteq ann_r(b)$. Let $P$ be a prime right ideal of $R$. Since $a \in J^*(R)$, $a \in P$. Since $R$ is a prime right strongly quasi-duo ring, $P$ is a totally fully invariant right ideal. Since $a \in P$ and $ann_r(a) \subseteq ann_r(b)$, $b \in P$. Hence $b \in J^*(R)$. Then $J^*(R)$ is a totally fully invariant right ideal of $R$. □

**Theorem 3.7.** *Let $R$ be a hereditary and prime right strongly quasi-duo ring. Then $R/J^*(R)$ is a prime right strongly quasi-duo ring.*

*Proof.* Since $R$ is a prime right strongly quasi-duo ring, by Theorem 3.6 $J^*(R)$ is a totally fully invariant right ideal of $R$. Hence by Theorem 2.19, $J^*(R)$ is a two-sided ideal of $R$. Thus $R/J^*(R)$ is a ring. By Proposition 2.9, a right ideal of $R/J^*(R)$ is of the form $L/J^*(R)$ where $L$ is a right ideal of $R$ such that $J^*(R) \subseteq L$.

Let $M/J^*(R)$ be a prime right ideal of $R/J^*(R)$ where $M$ is a right ideal of $R$ such that $J^*(R) \subseteq M$. Since by Theorem 2.10 and $M/J^*(R)$ is a prime right ideal of $R/J^*(R)$, $M$ is a prime right ideal of $R$. To show that $M/J^*(R)$ is a totally fully invariant right ideal by using Theorem 2.20, let $N/J^*(R)$ be a right ideal of $R/J^*(R)$ where $N$ is a right ideal of $R$ such that $J^*(R) \subseteq N$ and $N/J^*(R) \subseteq M/J^*(R)$ and let $f \in Hom_R(N/J^*(R), R/J^*(R))$. Moreover, let $\pi_1$ be the canonical map from $N$ into $N/J^*(R)$. Since for all $a, b \in N$, $\pi_1(a+b) = (a+b)+J^*(R) = (a+J^*(R))+(b+J^*(R)) = \pi_1(a)+\pi_1(b)$ and for all $r \in R$, $\pi_1(ar) = ar + J^*(R) = (a + J^*(R))r = \pi_1(a)r$, it follows that $\pi_1$ is an $R$-module homomorphism. Let $\pi_2$ be the canonical map from $R$ into $R/J^*(R)$. By the same argument, $\pi_2$ is an $R$-module homomorphism. It is obvious that $\pi_1$ and $\pi_2$ are epimorphisms.

Next, we will show that $f \circ \pi_1$ is an $R$-module homomorphism from $N$ to $R/J^*(R)$.

Let $x, y \in N$ and $r \in R$. Then

$$\begin{aligned}(f \circ \pi_1)(x+y) &= f(x+y+J^*(R)) \\ &= f(x+J^*(R)) + f(y+J^*(R)) \\ &= f(\pi_1(x)) + f(\pi_1(y)) \qquad \text{and} \\ (f \circ \pi_1)(xr) &= f(xr+J^*(R)) \\ &= f((x+J^*(R))r) \\ &= ((f \circ \pi_1)(x))r.\end{aligned}$$

Hence $f \circ \pi_1$ is an $R$-module homomorphism. Since $N$ is a right ideal of $R$ and $R$ is hereditary, $N$ is projective as a right $R$-module. Since $f \circ \pi_1$ is an $R$-module homomorphism from $N$ to $R/J^*(R)$, $\pi_2$ is a surjective $R$-module homomorphism from $R$ to $R/J^*(R)$ and $N$ is a projective $R$-module, it follows that there exists an $R$-module homomorphism $g : N \to R$ such that $\pi_2 \circ g = f \circ \pi_1$. Since $R$ is a prime right strongly quasi-duo ring and $M$ is a prime right ideal of $R$, $M$ is a totally fully invariant right ideal. Since $N/J^*(R) \subseteq M/J^*(R)$, $N \subseteq M$. Since $M$ is a totally fully invariant right ideal of $R$, $N$ is a right ideal of $R$ contained in $M$ and $g$ is an $R$-homomorphism from $N$ to $R$, by Theorem 2.20, $g(N) \subseteq M$. Thus

$$f(N/J^*(R)) = f(\pi_1(N)) = \pi_2(g(N)) \subseteq \pi_2(M) = M/J^*(R).$$

By Theorem 2.20, $M/J^*(R)$ is a totally fully invariant right ideal of $R/J^*(R)$ . $\square$

## 3.2 Zero Divisors of Prime Right Strongly Quasi-Duo Rings

**Definition 3.8.** Let $R$ be a ring. For each $x \in R$, (1) $x$ is a *left zero divisor* if there exists $b \in R \setminus \{0\}$ such that $xb = 0$,

(2) $x$ is a *right zero divisor* if there exists $b \in R \setminus \{0\}$ such that $bx = 0$,

(3) $x$ is a *two-sided zero divisor* if $x$ is both a left zero divisor and a right zero divisor, i.e., there exist $b, c \in R \setminus \{0\}$ such that $xb = 0$ and $cx = 0$,

(4) $x$ is a *strongly two-sided zero divisor* if and only if there exists $b \in R \setminus \{0\}$ such that $xb = 0 = bx$.

Moreover, we give the following notations.

$LZD(R)$ is the set of all left zero divisors of $R$,

$RZD(R)$ is the set of all right zero divisors of $R$,

$TZD(R)$ is the set of all two-sided zero divisors of $R$, and

$STZD(R)$ is the set of all strongly two-sided zero divisors of $R$.

From the above definitions, it is clear that

$$STZD(R) \subseteq TZD(R) \subseteq LZD(R) \text{ and } STZD(R) \subseteq TZD(R) \subseteq RZD(R).$$

Moreover, for each $a \in R \setminus \{0\}$, we have

$$ann_r(a) \subseteq RZD(R) \text{ and } ann_l(a) \subseteq LZD(R).$$

We consider the structures of $LZD(R)$,$RZD(R)$,$TZD(R)$ and $STZD(R)$. We find that they may not be subgroups of $R$. For example, in the ring $R = \mathbb{Z}_6$, we have

(1) $LZD(\mathbb{Z}_6)$ is not closed under addition.
Since $\bar{3} \cdot \bar{2} = \bar{0} = \bar{2} \cdot \bar{3}$, $\bar{2}, \bar{3} \in LZD(\mathbb{Z}_6)$. Since $\bar{2} + \bar{3} = \bar{5}$ and for each $g \in \mathbb{Z}_6 \setminus \{\bar{0}\}$, $\bar{5} \cdot g \neq \bar{0}$, $\bar{2} + \bar{3} \notin LZD(\mathbb{Z}_6)$.

(2) $RZD(\mathbb{Z}_6)$ is not closed under addition.
By similar argument as in (1), we conclude that $RZD(\mathbb{Z}_6)$ is not closed under addition.

(3) $TZD(\mathbb{Z}_6)$ is not closed under addition.
This is a consequence of the fact that $TZD(\mathbb{Z}_6) \subseteq RZD(\mathbb{Z}_6) \cap LZD(\mathbb{Z}_6)$ and by (1), (2) above.

(4) $STZD(\mathbb{Z}_6)$ is not closed under addition.
Since $\bar{3} \cdot \bar{2} = \bar{0} = \bar{2} \cdot \bar{3}$, $\bar{2}, \bar{3} \in STZD(\mathbb{Z}_6)$. Since $\bar{2} + \bar{3} = \bar{5}$ and for each $g \in \mathbb{Z}_6 \setminus \{\bar{0}\}$, $\bar{5} \cdot g \neq \bar{0}$, $\bar{5} \notin STZD(R)$.

Although $LZD(R), RZD(R), TZD(R)$ are not groups under addition, they are closed under multiplication. To prove this, firstly let $x, y \in RZD(R)$. Hence there exists $a \in R \setminus \{0\}$ such that $ax = 0$ Hence $axy = 0$. Thus $xy \in RZD(R)$. Similarly, $LZD(R)$ is closed under multiplication. Let $a, b \in TZD(R)$. Hence $a, b$ are both left divisors and right zero divisors. Thus $a, b \in RZD(R)$ and $a, b \in LZD(R)$. Since $RZD(R)$ and $LZD(R)$ are closed under multiplication, $ab \in RZD(R)$ and $ab \in LZD(R)$. Hence $ab$ is a right zero divisor and $ab$ is a left zero divisor. Then $ab \in TZD(R)$.

**Definition 3.9.** Let $R$ be a ring. Let

$$\hat{l} = \{x \in R : ann_l(x) = \{0\}\} \text{ and } \hat{r} = \{x \in R : ann_r(x) = \{0\}\}.$$

**Theorem 3.10.** *Let $R$ be a ring. Then*

(1) $LZD(R) = R \setminus \hat{r} = \{x \in R : ann_r(x) \neq \{0\}\}$,
(2) $RZD(R) = R \setminus \hat{l} = \{x \in R : ann_l(x) \neq \{0\}\}$,
(3) $TZD(R) = LZD(R) \cap RZD(R)$,
(4) $STZD(R) = \{x \in R : ann_r(x) \cap ann_l(x) \neq \{0\}\}$.

*Proof.* (1) "$\subseteq$" Let $y \in LZD(R)$. Then there exists $a \in R \setminus \{0\}$ such that $ya = 0$. Hence $a \in ann_r(y)$. Since $a \neq 0$, $ann_r(y) \neq \{0\}$. Thus $y \notin \hat{r}$, so $y \in R \setminus \hat{r}$.
"$\supseteq$" Let $y \in R \setminus \hat{r}$. Then $ann_r(y) \neq \{0\}$. Thus there exists $b \neq 0$ such that $yb = 0$. This implies $y \in LZD(R)$.
(2) The result follows similarly to the proof of (1).

(3) It is obvious from the remark that $TZD(R)$ is a subset of both $LZD(R)$ and $RZD(R)$.
(4) " $\subseteq$ " Let $y \in STZD(R)$. Then there exists $a \in R \setminus \{0\}$ such that $ay = 0 = ya$. Hence $a \neq 0$, $a \in ann_r(y)$ and $a \in ann_l(y)$. Thus $ann_r(y) \cap ann_l(y) \neq \{0\}$.
"$\supseteq$" Let $y \in \{x \in R : ann_r(x) \cap ann_l(x) \neq \{0\}\}$. Then $ann_r(y) \cap ann_l(y) \neq \{0\}$. Hence there exists $a \in R \setminus \{0\}$ such that $a \in ann_r(y) \cap ann_l(y)$. Thus $ya = 0 = ay$. Hence $y \in STZD(R)$. □

**Theorem 3.11.** *Let $R$ be a ring. Then $U(R) \subseteq \hat{l}$ and $U(R) \subseteq \hat{r}$.*

*Proof.* Let $x \in U(R)$. Then there exists $y \in R$ such that $xy = 1 = yx$. We will show that $x \in \hat{r}$. Thus we must show that $ann_r(x) = \{0\}$. Since $\{0\} \subseteq ann_r(x)$, it suffices to show only that $ann_r(x) \subseteq \{0\}$. Let $k \in ann_r(x)$. Hence $xk = 0$. Then $k = (1)k = (yx)k = y(xk) = y(0) = 0$. Hence $ann_r(x) = \{0\}$. Thus $x \in \hat{r}$. Hence $U(R) \subseteq \hat{r}$. Similarly, we get $U(R) \subseteq \hat{l}$. □

Note that it follows from Theorem 3.11 that $U(R) \subseteq \hat{r} \cap \hat{l}$.

**Corollary 3.12.** *Let $R$ be a ring. Then*
(1) $LZD(R) \setminus RZD(R) = \hat{l} \setminus \hat{r}$,
(2) $RZD(R) \setminus LZD(R) = \hat{r} \setminus \hat{l}$.

*Proof.* They are consequences of Theorems 3.10(1) and 3.10(2). □

In this part, we discuss about zero divisors in a prime right strongly quasi-duo ring and a prime left strongly quasi-duo ring. We find that if $R$ is a prime right strongly quasi-duo ring or a prime left strongly quasi-duo ring, there are more additional properties of zero divisors that are not true in general.

Recall the Zorn's lemma that if $A$ is a nonempty partially ordered set in which every chain (a linearly ordered subset of $A$) has an upper bound, then $A$ has a maximal element.

**Theorem 3.13.** *Let $R$ be a ring. Then*
(1) *if $R$ is a prime right strongly quasi-duo ring, then $U(R) = \hat{r}$,*
(2) *if $R$ is a prime left strongly quasi-duo ring, then $U(R) = \hat{l}$.*

*Proof.* (1) Assume that $R$ is a prime right strongly quasi-duo ring. Then it is true by Theorem 3.1.
(2) Assume that $R$ is a prime left strongly quasi-duo ring. By Theorem 3.11, $U(R) \subseteq \hat{l}$. It remains to show that $\hat{l} \subseteq U(R)$. Let $a \in \hat{l}$. Then $ann_l(a) = \{0\}$. Suppose to contary that $Ra \neq R$. Then $Ra \subset R$. We know that $Ra$ is a left ideal of $R$. Next, we will show that there exists a maximal left ideal $K$ such that $Ra \subseteq K$ by using the Zorn's Lemma.

Let $\mathcal{J} = \{J : J \neq R$ and $J$ is a left ideal of $R$ containing $Ra\}$. Since $Ra \in \mathcal{J}$,

$\mathcal{J} \neq \emptyset$. Let $C$ be a chain in $\mathcal{J}$. Since $0 \in J$ for each $J \in \mathcal{J}$, $0 \in \cup C$, so $\cup C \neq \emptyset$. Let $x, y \in \cup C$. Then there exists $J_1 \in C$ such that $x \in J_1$ and there exists $J_2 \in C$ such that $y \in J_2$. Since $C$ is a chain and $J_1, J_2 \in C$, without loss of generality, assume that $J_1 \subseteq J_2$. Then $x, y \in J_2$. Since $J_2$ is a subgroup of $R$ and $J_2 \in C$, $x - y \in J_2 \subseteq \cup C$. Thus $\cup C$ is a subgroup of $R$.

Let $v \in \cup C$ and $s \in R$. Then there exists $J_3 \in C$ such that $v \in J_3$. Since $J_3$ is a left ideal of $R$ and $J_3 \in C$, $sv \in J_3 \subseteq \cup C$. Thus $\cup C$ is a left ideal of $R$.

It is obvious that for each $J_4 \in \mathcal{J}$, $J_4$ contains $Ra$, so $\cup C$ contains $Ra$. Suppose that $R = \cup C$. Hence $1 \in J_5$ for some $J_5 \in C$. Thus $R = J_5$, contradicts to the properties of $J_5 \in \mathcal{J}$. Hence $R \neq \cup C$. Thus $\cup C \in \mathcal{J}$.

It is clear that $\cup C$ is an upper bound of $C$(order by subset). By Zorn's lemma, $\mathcal{J}$ has a maximal element. Then there exists a maximal element $K \in \mathcal{J}$. We will show that $K$ is a maximal left ideal of $R$. Let $L$ be a left ideal of $R$ such that $K \subseteq L \subseteq R$. Hence $Ra \subseteq K \subseteq L \subseteq R$. Suppose $L \neq R$. Thus $L \in \mathcal{J}$. Since $K \subseteq L \in \mathcal{J}$ and $K$ is a maximal element in $\mathcal{J}$, $K \not\subsetneq L$. Thus $K = L$. Hence $K$ is a maximal left ideal of $R$ such that $Ra \subseteq K$. Since $1 \in R$, $a = (1)a \in K$.

We want to show that $R \subseteq K$, let $b \in R$. Since $ann_l(a) = \{0\}$, $ann_l(a) = \{0\} \subseteq ann_l(b)$. Since $K$ is a maximal left ideal of $R$, it follows from Theorem 2.5 that $K$ is a prime left ideal of $R$.

Since $R$ is a prime left strongly quasi-duo ring and $ann_l(a) = \{0\} \subseteq ann_l(b)$ where $a \in K$, it follows that $b \in K$. Hence $K = R$ which contradicts the maximality of $K \in \mathcal{J}$. Thus $Ra = R$. Then there exists $c \in R$ such that $ca = 1$.

Next, we will show $ann_l(c) = \{0\}$. Let $y \in ann_l(c)$. Thus $yc = 0$. Hence $y = y(1) = y(ca) = (yc)a = (0)a = 0$. Then $ann_l(c) = \{0\}$. Similarly as above, we get $Rc = R$. Then there exists $d \in R$ such that $dc = 1$. Since $ca = 1$ and $dc = 1$, $d = d(1) = d(ca) = (dc)a = (1)a = a$. Thus $ac = dc = 1 = ca$. Hence $a \in U(R)$. □

**Corollary 3.14.** *Let $R$ be a ring. Then*

(1) *if $R$ is a prime right strongly quasi-duo ring, then $\hat{r} \subseteq \hat{l}$ and*

(2) *if $R$ be a prime left strongly quasi-duo ring, then $\hat{l} \subseteq \hat{r}$.*

*Proof.* (1) Assume that $R$ is a prime right strongly quasi-duo ring. By Theorem 3.13(1) and Theorem 3.11, we have $U(R) = \hat{r}$ and $U(R) \subseteq \hat{l}$. Then $\hat{r} = U(R) \subseteq \hat{l}$.
(2) Similarly to (1), the result is obtained. □

**Corollary 3.15.** *Let $R$ be a prime right strongly quasi-duo ring. Then*

(1) *$LZD(R) = R \setminus U(R)$,*

(2) *$RZD(R) \subseteq LZD(R)$,*

(3) *$TZD(R) = RZD(R)$.*

*Proof.* (1) It is obtained from Theorem 3.10(1) and Theorem 3.13(1).
(2) It is obtained from Corollary 3.14(1), Theorem 3.10(1) and Theorem 3.10(2).
(3) It is obtained from (2) and Theorem 3.10(3). □

**Corollary 3.16.** *Let $R$ be a prime left strongly quasi-duo ring. Then*

(1) $RZD(R) = R \setminus U(R)$,
(2) $LZD(R) \subseteq RZD(R)$,
(3) $TZD(R) = LZD(R)$.

*Proof.* (1) It is obtained from Theorem 3.10(2) and Theorem 3.13(2).
(2) It is obtained from Corollary 3.14(2), Theorem 3.10(1) and Theorem 3.10(2).
(3) It is obtained from (2) and Theorem 3.10(3). □

**Corollary 3.17.** *Let $R$ be a prime left strongly quasi-duo ring and a prime right strongly quasi-duo ring. Then $LZD(R) = TZD(R) = RZD(R)$.*

*Proof.* It is a consequence of Corollary 3.15(3) and Corollary 3.16(3). □

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# APPENDIX

## Appendix

**The Project Proposal of Course 2301399 Project Proposal**
**Academic Year 2020**

Project Tittle (Thai) การศึกษาริงซึ่งไอดีลทางขวาเฉพาะเป็นไอดีลทางขวาที่ไม่แปรเปลี่ยนอย่างเต็มทั้งหมด

Project Tittle (English) Study on rings whose prime right ideals are totally fully invariant

Project Advisor Associate Professor Dr. Ouamporn Phuksuwan

Project Co-advisor Associate Professor Dr. Sajee Pianskool

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**1.Background and Rationale**

Throughout this project, all rings are associative with identity. Let $R$ be such a ring. A subgroup $I$ of $R$ is called a right (or left) ideal if $ar \in I$ $(ra \in I)$ for any $a \in I$ and $r \in R$ and I is called a two-sided ideal if it is both a left ideal and a right ideal. A proper right ideal $M$ of $R$ is said to be maximal if for every right ideal $J$ of $R$,$M \subseteq J \subseteq R$ that $M = J$ or $J = R$. A right ideal $I$ of $R$ is said to be totally fully invariant if for each $a \in I$ and $b \in R$ , $ann(a) \subseteq ann(b)$ implies that $b \in I$, where $ann(c) = \{x \in R|cx = 0\}$ for any $c \in R$ A ring $R$ is said to be right strongly quasi-duo if any maximal right ideal of $R$ is totally fully invariant. An example of right strongly quasi-duo rings $\mathbb{Z}_n$ for each $n \geq 2$ S. Safaeeyan [1] studied totally fully invariant ideals, and found that a right ideal which is totally fully invariant is two-sided. Moreover, he studied strongly quasi-duo rings $R$ and proved that the set of all unit elements in $R$ is the set $\{a \in R|ann(a) = \{0\}\}$.

A proper right ideal $p$ of $R$ is called a prime right ideal if for every right ideals $A$ and $B$, $AB \subseteq P$ impiles that $A \subseteq P$ or $B \subseteq P$ . In this project, we study rings whose prime right ideals are totally fully invariant and investigate some of their properties.

**2.Objectives**

1. Study rings whose prime right ideals are totally fully invariant.
2. Investigate some properties of rings whose prime right ideals are totally fully invariant.

**3.Project Activities**

1. Literature reviews on right strongly quasi-duo rings.
2. Study rings whose prime right ideals are totally fully invariant.
3. Investigate some properties of rings whose prime right ideals are totally fully invariant.
4. Write a report.

**Activities Table**

| **Project activities** | **August 2020-April 2021** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Aug | Sep | Oct | Nov | Dec | Jan | Feb | Mar | Apr |
| 1. Literature reviews on right strongly quasi-duo rings. | | | | | | | | | |
| 2. Study rings whose prime right ideals are totally fully invariant. | | | | | | | | | |
| 3. Investigate some properties of rings whose prime right ideals are totally fully invariant. | | | | | | | | | |
| 4. Write a report. | | | | | | | | | |

**4.Benefits**

Obtain some properties of rings whose prime right ideals are totally fully invariant.

**5.Budget**

1. Books 5000 Bahts

**Total 5000 Bahts**

**6.Reference**

[1] S. Safaeeyan, Strongly Quasi-Duo Rings, Journal of Mathematical Extension, Vol. 5, No. 2(1), (2011), pp 67-74.

# Biography

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