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In P-minimal structures with definable Skolem functions

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In P-minimal structures with definable Skolem functions

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หัวข้อโครงงาน โครงสร้างพีมินิมอลที่มีฟังก์ชันสโคเลมที่นิยามได้

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ให้ $\mathcal{L}$ เป็นภาษาที่นับได้และ $\mathfrak{M}$ เป็นโครงสร้างพีมินิมอลที่มีภาษา $\mathcal{L}$ เราจะแสดงว่าถ้า $\mathfrak{M}$ เป็นโครงสร้างพีมินิมอลที่มีฟังก์ชันสโคเลมที่นิยามได้และมีสมบัติการแลกเปลี่ยนซึ่งอิ่มตัวแบบอะเลปหนึ่ง และ $(G, \cdot)$ เป็นกรุปซึ่งนิยามได้โดย 0 และมี n มิติ แล้วจะมีทอพอโลยีที่ทำให้ $(G, \cdot)$ เป็นกรุปทอพอโลยี

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต พัชราภา หาญมุ่งธรรม

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Let $\mathcal{L}$ be a countable language and $\mathfrak{M}$ be a P-minimal $\mathcal{L}$-structure. We show that if $\mathfrak{M}$ admits definable Skolem function, has exchange property and is $\aleph_1$-saturated and $(G, \cdot)$ is a group definable and n-dimensional, then there is a topology that makes $(G, \cdot)$ a topological group.

Department	Mathematics and Computer Science	Student's Signature	พัชรภา หาญมุ่งธรรม
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Content

Chapter 1
Introduction

Throughout, let $\mathbb{N}$ be the set of all nonnegative integers, $\mathbb{Z}$ be the set of all integers and $\mathbb{Q}$ be the set of all rationals.

1.1 The field of p-adic numbers $\mathbb{Q}_p$

The absolute value $|\cdot|$ is a well-known norm on the field $\mathbb{Q}$ of rational numbers. Is there other norms on $\mathbb{Q}$? The answer to this question is yes. Let p be a prime number. The p-adic valuation $v : \mathbb{Z} \setminus \{0\} \to \mathbb{N}$ is defined by for each nonzero integer $x, v(x)$ is the largest n such that p^n divides x. We define the p-adic norm $|\cdot|_p$ on $\mathbb{Q}$ as follows: for each $x \in \mathbb{Q}$,

$$|x|_p = \begin{cases} 0, & \text{if } x = 0 \\ p^{-v(a)+v(b)}, & \text{if } x = \dfrac{a}{b} \text{ for some } a, b \in \mathbb{Z} \setminus \{0\} \end{cases}$$

The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$.

1.2 First-order Logic

We will introduce some definitions in first-order logic.

Definition 1.1. A language $\mathcal{L}$ is a disjoint union of

1. a set $\mathcal{R}$ of relation symbols; each $R \in \mathcal{R}$ has associated arity $\textbf{arity}(R) \in \mathbb{N}$;
2. a set $\mathcal{C}$ of constant symbols; each $c \in \mathcal{C}$ has associated arity 0;
3. a set $\mathcal{F}$ of function symbols; each $f \in \mathcal{F}$ has associated arity $\textbf{arity}\ (f) \in \mathbb{N} \setminus \{0\}$.

Definition 1.2. An $\mathcal{L}$-structure is a quadruple $\mathfrak{M} = (M, \mathcal{R}^{\mathfrak{M}}, \mathcal{C}^{\mathfrak{M}}, \mathcal{F}^{\mathfrak{M}})$ consisting of:

1. a nonempty set M, called the underlying set;
2. $\mathcal{R}^{\mathfrak{M}} = \{R^{\mathfrak{M}} : R \in \mathcal{R}\}$ where each $R^{\mathfrak{M}} \subseteq M^{arity(R)}$;
3. $\mathcal{C}^{\mathfrak{M}} = \{c^{\mathfrak{M}} : c \in \mathcal{C}\}$ where each $c^{\mathfrak{M}} \in M$;

4. $\mathcal{F}^{\mathfrak{M}} = \{f^{\mathfrak{M}} : f \in \mathcal{F}\}$ where each $f^{\mathfrak{M}} : M^{arity(f)} \to M$.

Definition 1.3. Let $\mathcal{L}$ be a language and $\mathfrak{M}$ be an $\mathcal{L}$-structure. Define the expansion of $\mathcal{L}, \mathcal{L}_M$, and the canonical expansion, $\mathfrak{M}_M$ of $\mathfrak{M}$ to $\mathcal{L}_M$ by $\mathcal{L}_M = \mathcal{L} \cup M$, and $a^{\mathfrak{M}_M} = a$ for every $a \in M$.

Definition 1.4. Let $\mathfrak{M}$ be an $\mathcal{L}$-structure. Without loss of generality, we may assume that $\mathfrak{M} = \mathfrak{M}_M$. Let σ be an $\mathcal{L}_M$-sentence. We recursively define σ to be true in $\mathfrak{M}$ (denoted by $\mathfrak{M} \vDash \sigma$) by requiring

1. $\mathfrak{M} \vDash \top$,
2. for $R \in \mathcal{R}$ and variable-free $\mathcal{L}_M$-terms $t_1, ..., t_{\text{ arity }(R)}$, $\mathfrak{M} \vDash Rt_1...t_{\text{ arity }(R)}$ if and only if $(t_1^{\mathfrak{M}}, ..., t^{\mathfrak{M}}_{\text{arity }(R)}) \in R^{\mathfrak{M}}$;
3. for variable-free $\mathcal{L}_M$-terms t_1, t_2, $\mathfrak{M} \vDash t_1 = t_2$ if and only if $t_1^{\mathfrak{M}} = t_2^{\mathfrak{M}}$;
4. if $\sigma = \neg\tau$ for some $\mathcal{L}_M$-sentence τ, then $\mathfrak{M} \vDash \sigma$ if and only if $\mathfrak{M} \nvDash \tau$;
5. if $\sigma = \tau_1 \wedge \tau_2$ for some $\mathcal{L}_M$-sentences τ_1, τ_2, then $\mathfrak{M} \vDash \sigma$ if and only if $\mathfrak{M} \vDash \tau_1$ and $\mathfrak{M} \vDash \tau_2$;
6. if $\sigma = \tau_1 \vee \tau_2$ for some $\mathcal{L}_M$-sentences τ_1, τ_2, then $\mathfrak{M} \vDash \sigma$ if and only if $\mathfrak{M} \vDash \tau_1$ or $\mathfrak{M} \vDash \tau_2$;
7. if $\sigma = \forall v_i \varphi(v_i)$ for some variable v_i and $\mathcal{L}_M$-formula $\varphi(v_i)$, then $\mathfrak{M} \vDash \sigma$ if and only if $\mathfrak{M} \vDash \varphi(a)$ for every $a \in M$;
8. if $\sigma = \exists v_i \varphi(v_i)$ for some variable v_i and $\mathcal{L}_M$-formula $\varphi(v_i)$, then $\mathfrak{M} \vDash \sigma$ if and only if $\mathfrak{M} \vDash \varphi(a)$ for some $a \in M$.

Definition 1.5. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures. We say that $\mathfrak{M}$ is a substructure of $\mathfrak{N}$, denoted by $\mathfrak{M} \subseteq \mathfrak{N}$, if

1. $M \subseteq N$;
2. $R^{\mathfrak{M}} = R^{\mathfrak{N}} \cap M^{\text{ arity(R) }}$ for every $R \in \mathcal{R}$;
3. $c^{\mathfrak{M}} = c^{\mathfrak{N}}$ for every $c \in \mathcal{C}$;
4. for every $f \in \mathcal{F}, f^{\mathfrak{M}}$ is the restriction of $f^{\mathfrak{N}}$ to $M^{\text{ arity(R) }}$.

Definition 1.6. Suppose $\mathfrak{M} \subseteq \mathfrak{N}$. We say that $\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$, (denoted by $\mathfrak{M} \preceq \mathfrak{N}$) if for all $n \in \mathbb{N}$, $\mathcal{L}$-formula $\varphi(x_1, ..., x_n)$, and $a_1, ..., a_n \in M$, $\mathfrak{M} \vDash \varphi(a_1, ..., a_n)$ if and only if $\mathfrak{N} \vDash \varphi(a_1, ..., a_n)$.

Now, we let $\mathfrak{M}$ be an $\mathcal{L}$-structure and A be a subset of an underlying set M of $\mathfrak{M}$.

Definition 1.7. Let $X \subseteq M^n$. We say that X is A-definable in $\mathfrak{M}$ if there exist an $\mathcal{L}$-formula $\varphi(x_1, ..., x_n, y_1, ..., y_k)$ (where $k \in \mathbb{N}$) and $a_1, ..., a_k \in A$ such that $X = \{(b_1, ..., b_n) \in M^n : \mathfrak{M} \vDash \varphi(b_1, ..., b_n, a_1, ..., a_k)\}$.

Let $f : X \to M^m$ be a function. We say that f is A-definable in $\mathfrak{M}$ if the group of f, $\{(x, y) \in M^{n+m} : f(x) = y\}$ is A-definable.

Definition 1.8. Let $\mathcal{L}_A$ be the language obtained by adding each element of A to $\mathcal{L}$ as a constant symbol. Note that $\mathfrak{M}$ is an $\mathcal{L}_A$-structure by trivial interpretation $a^{\mathfrak{M}} = a$ for each $a \in A$. Let $Th_A(\mathfrak{M}) = \{\sigma : \sigma$ is an $\mathcal{L}_A$-sentence and $\mathfrak{M} \vDash \sigma\}$. We say that p is an n-type over A if p is the set of all $\mathcal{L}_A$-formulas without free occurrences of variables besides $x_1, ..., x_n$.

Definition 1.9. Let p be an n-type over A. We say that $a \in M^n$ realizes p if $\mathfrak{M} \vDash \phi(a)$ for all $\phi \in p$.

We let $S_n^{\mathfrak{M}}(A)$ be the set of all complete and consistent n-types over A.

Definition 1.10. We say that $\mathfrak{M}$ is "$\aleph_1$-saturated" if for any countable $Z \subseteq M$, $p(x) \in S_n^{\mathfrak{M}}(Z)$, then there is $\bar{a} \in M^n$ such that $\bar{a}$ realized $p(x)$.

Theorem 1.11. *Let $\mathfrak{N}$ be an $\mathcal{L}$-structure and $S \subseteq N$. Then there is an $\mathcal{L}$-structure $\mathfrak{M}$ such that $\mathfrak{M} \preceq \mathfrak{N}, S \subseteq M$, and $|M| \leq |\mathcal{L}| + |S| + \aleph_0$.*

Definition 1.12. Let $(G, \cdot)$ be a group in $\mathfrak{M}$. We say that $(G, \cdot)$ is a group definable in $\mathfrak{M}$ if G is a definable subset of M^n, for some n, and $\cdot$ is also definable.

Definition 1.13. A "topological group" G is a group endowed with a topology such that the multiplication and the inversion are continuous.

Anand Pillay studies properties of definable groups in the field of p-adic numbers $\mathbb{Q}_p$ and o-minimal structures. In [5], he proved that if $(G, \cdot)$ is a group definable in the field of $p-$adic numbers, then there is a topology τ on G such that $\cdot : G \times G \to G$ is $\tau-$continuous and there exists $n > 0$ such that for all $g \in G$, there is an open set U in G such that $g \in U$ and U is isomorphic to $\mathbb{Q}_p^n$. A similar result in o-minimal context was shown in [4].

1.3 P-minimality

Let $\mathcal{L} = \{+, -, \cdot, 0, 1, \mathrm{Div}, (P_n)_{n \in \mathbb{N}}\}$ be a language where $0, 1$ are constant symbols, $+, \cdot$ are binary function symbols, $-$ is a unary function symbol, Div is a binary relation symbol and each P_n is a unary relation symbol. Let p be a prime number. Observe that the expansion of the field of p-adic numbers by $\mathrm{Div} = \{(a, b) \in \mathbb{Q}_p^2 : v(a) \leq v(b)\}$ and $P_n = \{x \in \mathbb{Q}_p : \exists y, y^n = x\}$(where $n \in \mathbb{N}$) is an $\mathcal{L}$-structure.

In [1], Deirdre Haskell and Dugald Macpherson generalize the concept of p-adic fields into P-minimal structures.

Throughout the rest of this section, let $\mathcal{L}'$ be a language extending $\mathcal{L}$ and $\mathfrak{M}$ be an $\mathcal{L}'$-structure such that (M, v) is a valued-field, $\mathrm{Div} = \{(a, b) \in M^2 : v(a) \leq v(b)\}$ and $P_n^{\mathfrak{M}} = \{x \in M : \exists y, y^n = x\}$.

Definition 1.14. We say that $\mathfrak{M}$ is P-minimal if for every $\mathfrak{M}'$ elementarily equivalent to $\mathfrak{M}$, every definable subset of $\mathfrak{M}'$ is defined by quantifier-free $\mathcal{L}$-formula.

Let $\mathfrak{M}$ be a P-minimal structure. Then the underlying set M can be equipped with the topology generated by $\{B_{v(a)}(c) : a, b \in M\}$ where $B_{v(a)}(c) = \{x \in M : \mathfrak{M} \models \mathrm{Div}(a, x-c)\}$ The following lemma was proved by Deidre Haskell and Dugald Macpherson in [1].

Lemma 1.15. *Let $\mathfrak{M}$ be a P-minimal structures. Then*

1. *Every infinite definable subset of M has non-empty interior in M.*
2. *Every non-empty definable subset of the value group vM which is bounded above has a greatest element.*

Definition 1.16. Let $(i_1, ..., i_n)$ be a sequence of zeroes and ones of length n. Let $P_k^* = P_k \setminus \{0\}$. An $(i_1, ..., i_n)$-cell is a definable subset of M^n defined by induction on n as follows:

1. A (0)-cell is a point of M and a (1)-cell is of the form $\{x \in M : \gamma_1 < v(x-c) < \gamma_2 \wedge \lambda(x-c) \in P_k^*\}$, where $\lambda_1, \lambda_2 \in v(M) \cup \{-\infty, \infty\}$, $c, \lambda \in M$ and $k \in \mathbb{N}$.
2. Suppose that $(i_1, ..., i_n)$-cells are already defined. Then an $(i_1, ..., i_n, 0)$-cells is the graph of a definable continuous function from an $(i_1, ..., i_n)$-cell to M, and an $(i_1, ..., i_n, 1)$-cell is a set of the form $\{(y, x) \in C \times M : v(a_1(y)) \square_1 v(x - c(y)) \square_2 v(a_2(y)) \wedge \lambda(x - c(y)) \in P_k^*\}$, where C is an $(i_1, ..., i_n)$-cell, a_1, a_2, c are definable continuous functions on C, λ is as in 1., and $\square_1$ and $\square_2$ are either $\leq, <$, or no condition.

Definition 1.17. Let $\mathfrak{M}$ be an $\mathcal{L}$-structure. We say that $\mathfrak{M}$ admits "definable Skolem function" if for every $\mathcal{L}$-formula $\varphi(x, y)$ such that for every $b \in M^k, \mathfrak{M} \models \exists x, \varphi(x, b)$, there is a definable function $f : M^k \to M^l$ such that $\mathfrak{M} \models \varphi(f(b), b)$, for every $b \in M^k$.

The following theorem was proved by Marie-Helene Mourgues in [3].

Theorem 1.18. *Let $\mathfrak{M}$ be a P-minimal structure with definable Skolem function. Then for each $n \in \mathbb{N}$, the following hold:*

1. *If A is a definable subset of M^n, then A can be partitioned into finitely many cells of M^n.*

2. *Given a definable function $f : A \to M$, where A is a definable subset of M^n, there exists a finite partition of A into cells such that the restriction of f to each cell is continuous.*

Definition 1.19. We say that $a \in \mathrm{acl}(A)$ if and only if there exists an $\mathcal{L}$-formula $\varphi(x, y)$ and $\bar{a} \in A^n$ such that $\mathfrak{M} \models \varphi(a, \bar{a})$ and $\{c \in M : \mathfrak{M} \models (c, \bar{a})\}$ is finite.

Definition 1.20. We say that $\mathfrak{M}$ has "Exchange Property" if for $A \subseteq M$, if $a \in \mathrm{acl}(A \cup \{b\})$ and $a \notin \mathrm{acl}(A)$, then $b \in \mathrm{acl}(A \cup \{a\})$.

1.4 Outline

In this project, we study properties of definable groups in P-minimal structures. In Chapter 2, we recall the notion of dimension. In Chapter 3, let $\mathcal{L}$ be a countable language and $\mathfrak{M}$ be a P-minimal $\mathcal{L}$-structure. Let $(G, \cdot)$ be a group such that $G \subseteq M^n$. For each open $V \subseteq G$, we define a topology τ_V on G by Q is τ_V-open if and only if $(a \cdot Q) \cap V$ is open for any $a \in G$. We show that if $\mathfrak{M}$ admits definable Skolem function, has Exchange Property and is $\aleph_1$-saturated, and G is an n-dimensional 0-definable subset of M^n, then there is a large definable subset V of G such that

1. G equipped with the topology τ_V is a topological group; and

2. V is a finite disjoint union of 0-definable τ_V-open sets.

Chapter 2
Dimension

In this chapter, we introduce a concept of dimension. We let $\mathfrak{M}$ be a P-minimal structure and A be a finite subset of an underlying set M of $\mathfrak{M}$.

Definition 2.1. Let $a \in M^n$. Then $\dim(a/A) =$ the least cardinality of a subtuple a' of a such that $a \subseteq \operatorname{acl}(A \cup a')$.

We let $S_n^{\mathfrak{M}}(A)$ be the set of all complete and consistent n-types over A.

Definition 2.2. Let $p(\bar{x}) \in S_n^{\mathfrak{M}}(A)$. Then $\dim p = \dim(\bar{a}/A)$ for some $\bar{a} \in M^n$ realizing p.

Definition 2.3. Let $X \subseteq M^n$ be A-definable. Then $\dim X = \max\{\dim(a/A) : a \in X\} = \max\{\dim p : p \in S_n^{\mathfrak{M}}(A) \text{ is realized in } X\}$.

Definition 2.4. Let Z be a definable subset of M^n. The topological dimension of Z, $\operatorname{topdim}(Z)$ is the greatest integer k for which there is a projection $\pi : M^n \to M^k$ such that $\pi(Z)$ has non-empty interior in M^k.

The following lemma was proved by Deirdre Haskell and Dugald Macpherson in [1].

Lemma 2.5. *If n, m are positive integers, X is a definable subset of M^n and $f : M^n \to M^m$ is a definable function, then* $\operatorname{topdim}(X) \geq \operatorname{topdim}(f(X))$.

The following is a consequence of Lemma 1.15 and Lemma 2.5.

Lemma 2.6. *If $\mathfrak{M}$ has Exchange Property, then for any positive integer n and definable $X \subseteq M^n$,* $\operatorname{topdim}(X) = \dim(X)$.

Lemma 2.7. *Let $X \subseteq M^n$ be A-definable and $k \leq n$. If $\mathfrak{M}$ has Exchange Property, then* $\dim X \geq k$ *if and only if some projection of X onto M^k has interior in M^k.*

Lemma 2.8. *Suppose that $\mathfrak{M}$ has Exchange Property. Let $\mathcal{V}(x_1 ... x_n, \bar{y})$ be a formula and for any $\bar{b}$ let $X_{\bar{b}}$ be the subset of M^n defined by $\mathcal{V}(\bar{x}, \bar{b})$. Then for any $k \leq n$ there is a formula $\psi_k(\bar{y})$ without parameters such that for any $\bar{b}$,* $\dim X_{\bar{b}} = k$ *if and only if $\psi_k(\bar{b})$.*

Proof. By Lemma 2.7. □

Definition 2.9. Let $Y \subseteq X \subseteq M^n$ be definable. We say that Y is large in X if $\dim(X \setminus Y) < \dim X$.

Definition 2.10. Let X be A-definable and $a \in X$. We will say that a is a generic point of X over A if $\dim(a/A) = \dim X$.

Lemma 2.11. *Let $Y \subseteq X$ be definable. Then Y is large in X if and only if for every A over which X and Y are defined, every generic point a of X over A is in Y.*

Proof. By Definition 2.9. □

Lemma 2.12. *Let $X \subseteq M^n$ be A-definable. Let $\phi(x_1, ..., x_n, \bar{y})$ be an $\mathcal{L}$-formula over $\emptyset$. Then $\{\bar{b} : \phi(\bar{x}, \bar{b})^M \cap X$ is large in $X\}$ is A-definable.*

Proof. By Lemma 2.11 □

Chapter 3
Definable groups

Throughout this chapter, let $\mathfrak{M}$ be an $\aleph_1$-saturated P-minimal structure that admits definable Skolem functions and has Exchange Property. Let $(G, \cdot)$ be a group that is 0-definable in $\mathfrak{M}$. Observe that the inversion $^{-1} : G \to G$ is also 0-definable in $\mathfrak{M}$.

Lemma 3.1. *Let $b \in G$ and let a be a generic of G over b. Then $b \cdot a$ is a generic of G over b.*

Proof. Let $b \in G$ and a be a generic of G over b. Then $\dim(a/b) = \dim G$. Since $\cdot$ and $^{-1}$ are 0-definable, $\dim(b \cdot a/b) = \dim G$. Thus, $b \cdot a$ is a generic of G over b. □

Lemma 3.2. *For any $b \in G$, there are generics b_1, b_2 of G such that $b = b_1 \cdot b_2$.*

Proof. Let $b \in G$. Let b_1 be a generic of G over b. Then $\dim(b_1/b) = \dim G$. Since the inversion is 0-definable, $\dim(b_1^{-1}/b) = \dim G$. Let $b_2 = b_1^{-1} \cdot b$. By Lemma 3.1, $b_2 = b_1^{-1} \cdot b$ is a generic of G over b such that $b_1 \cdot b_2 = b_1 \cdot b_1^{-1} \cdot b = b$. □

Definition 3.3. We say that a, b are mutually generic of G, if a is generic of G over b and also b is generic of G over a.

Lemma 3.4. *Let X be a large definable subset of G. Then finitely many translates of X cover G.*

Proof. Let X be $\{a_1, ..., a_n\}$-definable. First, we will construct $\mathfrak{M}_0$. By Theorem 1.11, there is an $\mathcal{L}$-structure $\mathfrak{M}_0$ such that $\mathfrak{M}_0 \preceq \mathfrak{M}$, $\{a_1, ..., a_n\} \subseteq M_0$ and $|M_0| \leq \aleph_0$. Next, we will show that for every $a \in G$ there exists $b \in G^{\mathfrak{M}_0}$ such that $a \in b \cdot X$. Let $a \in G$. Suppose to the contrary that for all $b \in G^{\mathfrak{M}_0}$, $a \notin b^{-1} \cdot X$ i.e. $b \notin X \cdot a^{-1}$. Then $G^{\mathfrak{M}_0} \subseteq G \setminus (X \cdot a^{-1})$. We have

$$\begin{aligned}
\dim G^{\mathfrak{M}_0} &= \max\{\dim p : p \in S_n^{\mathfrak{M}_0}(\emptyset) \text{ and } p \text{ is realized in } G^{\mathfrak{M}_0}\} \\
&\leq \max\{\dim p : p \in S_n^{\mathfrak{M}_0}(\emptyset) \text{ and } p \text{ is realized in } G \setminus (X \cdot a^{-1})\} \\
&= \max\{\dim p : p \in S_n^{\mathfrak{M}_0}(\emptyset) \text{ and } p \text{ is realized in } G \setminus (X \cdot a^{-1})\} \\
&\leq \max\{\dim p : p \in S_n^{\mathfrak{M}_0}(\{a\}) \text{ and } p \text{ is realized in } G \setminus (X \cdot a^{-1})\} \\
&= \dim(G \setminus X \cdot a^{-1}).
\end{aligned}$$

Since a^{-1} is bijection, by Lemma 2.5, $\dim(G \setminus X \cdot a^{-1}) = \dim(G \setminus X)$. Since X is large in G, we have $\dim(G \setminus X) < \dim G$. Then $\dim G^{\mathfrak{M}_0} < \dim(G \setminus X \cdot a^{-1}) =$

$\dim(G \setminus X) < \dim G$, which is absurd. Therefore, there exists $b \in G^{\mathfrak{M}_0}$ such that $a \in b \cdot X$. Suppose to the contrary that for all $r, b_1, ..., b_r \in G^{\mathfrak{M}_0}$, there exists $a \in G$ such that $a \notin b_i \cdot X$ for all $i \in \{1, ..., r\}$. Let $p(x) = \{x \notin b \cdot X : b \in G^{\mathfrak{M}_0}\}$. Let $\triangle$ be a finite subset of $p(x)$. Then $\triangle = \{x \notin b_1 \cdot X, ..., x \notin b_r \cdot X\}$ for some $b_1, ..., b_r \in G^{\mathfrak{M}_0}$. By hypothesis, there exists $a \in G$ such that for all $r, b_1, ..., b_r \in G^{\mathfrak{M}_0}$. Then $\triangle$ is consistent. By Compactness Theorem, $p(x)$ is consistent. That is $p(x) \in S_n^{\mathfrak{M}}(A)$. Since $G^{\mathfrak{M}_0} \subseteq M_0^k$ for some k, we have $|G^{\mathfrak{M}_0}| \leq |M_0|$. Since $\mathfrak{M}_0$ is countable, $G^{\mathfrak{M}_0}$ is countable. Since $\mathfrak{M}$ is $\aleph_1$-saturated, there is $\bar{a} \in M$ such that a realizes $p(x)$. Since $p(x)$ is consistent, there exists a such that for every $b \in G^{\mathfrak{M}_0}$ such that $a \notin b \cdot X$, which is absurd. Then there are $b_1, ..., b_r \in G^{\mathfrak{M}_0}$ such that every $a \in G$ is in $b_i \cdot X$ for some $i = 1, ..., r$. Hence, finitely many translates of X cover G. $\square$

Definition 3.5. Let V be an open subset of G. We define the topology τ_V on G by $\mathcal{O} \subseteq G$ is τ_V-open if and only if for all $g \in G$, $(g \cdot \mathcal{O}) \cap V$ is open.

Lemma 3.6. *Assume* $\dim G = n$. *Then there are a subset* V *of* G *and* Y *of* $G \times G$ *such that*

1. $V = U_1 \cup ... \cup U_r$ *where the* U_i*'s are open* 0*-definable subsets of* M^n*;*
2. V *is large in* G*;*
3. Y *is large in* $G \times G$*;*
4. *the inversion is a continuous map from* V *onto* V*;*
5. Y *is open in* $V \times V$ *and the multiplication is a continuous map from* Y *to* V*; and*
6. *for every* $a \in V$*, if* b *is a generic of* V *over* a*, then* $(b, a) \in Y$ *and* $(b^{-1}, b \cdot a) \in Y$*.*

Proof. By Theorem 1.18, we can write G as a finite disjoint union of 0-definable cells. Let $U_1, ..., U_r$ be the cells of dimension n. Let $V_0 = U_1 \cup ... \cup U_r$. Then V_0 is large in G. By Theorem 1.18, we can write U_i as a finite disjoint union of 0-definable cells on each of which either the inversion is not a map into U_j or the inversion is a continuous map into U_j. Noting that for each generic a of G over $\emptyset$, there are i, j with $a \in U_i$, $a^{-1} \in U_j$ and every 0-definable subset of G of dimension n contains a generic of G over $\emptyset$. Then for each i there are open subsets $U_i^1, ..., U_i^r$ of U_i such that $\bigcup_{j=1}^r U_i^j$ is large in U_i and the inversion is a continuous map from $U_i^j \to U_j$ for every j. Let $V_1 = \bigcup_{i,j} U_i^j$. Then V_1 is large and open in V_0 and the inversion is a continuous map $V_1 \to V_0$. Since U_i is the cell of dimension n for all $i \in \{1, ..., r\}$ and $V_0 = U_1 \cup ... \cup U_r$, we can find an open large 0-definable $Y_0 \subseteq V_0 \times V_0$ such that the multiplication is a continuous map from $Y_0 \to V_0$ such that Y_0 is large in $G \times G$. We define $V_1' = \{a \in V_1 :$ for every generic b of G over $a, (b,a) \in Y_0$ and $(b^{-1}, b \cdot a) \in Y_0\}$. By Lemma 2.12, V_1' is 0-definable. If a is a generic of G, then $a \in V_1$. Moreover for b generic of G over a, (b,a) and $(b^{-1}, b{\cdot}a)$ are generics of $G{\times}G$. Then $a \in V_1'$. By Lemma 2.6, V_1' is large in G. We can write V_1' as a finite disjoint union of 0-definable cell such that dimension n. We can find open large 0-definable $V_2 \subseteq V_1'$. Then V_2^{-1} is also open in V_0 and large in G. Let $V = V_2 \cap V_2^{-1}$. Then V is open in V_0 and large in G, also $V \times V$ is open in $V_0 \times V_0$ and large in $G \times G$. Let $Y = (V \times V) \cap \{(a,b) \in Y_0 : a \cdot b \in V\}$. Then Y is open in $V_0 \times V_0$ and for mutually generic a, b of G, $(a,b) \in Y_0$ and $a \cdot b$ is generic. Then $(a,b) \in Y$. By Lemma 2.11, Y is large in $G \times G$. Therefore,

1. $V = U_1 \cup ... \cup U_r$ where the $U_i's$ are open 0-definable subsets of M^n;
2. V is large in G;
3. Y is large in $G \times G$;
4. the inversion is a continuous map from V onto V;
5. Y is open in $V \times V$ and the multiplication is a continuous map from Y to V; and
6. for every $a \in V$, if b is a generic of V over a, then $(b,a) \in Y$ and $(b^{-1}, b{\cdot}a) \in Y$.

□

Lemma 3.7. *Let V be as in Lemma 3.6. Then*

1. *For any $a, b \in G$, the set $Z = \{x \in V : a \cdot x \cdot b \in V\}$ is open in V, and the map $x \to a \cdot x \cdot b$ is a homeomorphism $Z \to a \cdot Z \cdot b$.*
2. *For any $a, b \in G$, the set $Z = \{(x,y) \in V \times V : a \cdot x \cdot b \cdot y \in V\}$ is open in $V \times V$ and the map $(x,y) \mapsto a \cdot x \cdot b \cdot y$: $Z \to V$ is continuous.*

Proof. 1. Let $a, b \in G$ and $x_0 \in Z$. Write $b = b_1 \cdot b_2$ with $b_1, b_2 \in V$. Let $c \in G$ such that c is generic over $\{a, x_0, b_1, b_2\}$. Then $c \in V$ and $c{\cdot}a \in V$. Let $Z_0 = \{x \in V : (c{\cdot}a, x) \in Y, (c{\cdot}a{\cdot}x, b_1) \in Y, (c{\cdot}a{\cdot}x{\cdot}b_1, b_2) \in Y$ and $(c^{-1}, c{\cdot}a{\cdot}x{\cdot}b_1{\cdot}b_2) \in Y\}$. Then $Z_0 \subseteq Z$ is open in V and $x_0 \in Z_0$. Hence, Z is open in V. Next, we will show that the map $f : Z \to a \cdot Z \cdot b$ is a homeomorphism. Let W be open in

$a \cdot Z \cdot b$ and $k \in f^{-1}(W)$. Consider

$$\begin{aligned} f^{-1}(W) &= \{x \in Z : f(x) \in W\} \\ &= \{x \in Z : a \cdot x \cdot b \in W\} \\ &= \{x \in V : a \cdot x \cdot b \in V \cap W\} \\ &\subseteq \{x \in V : a \cdot x \cdot b \in W\}. \end{aligned}$$

Then $k \in V$ and $a \cdot k \cdot b \in W$. Let $d \in G$ such that d is generic over $\{a, k, b_1, b_2\}$. Let $K_0 = \{x \in V : (c \cdot a, x) \in Y, (c \cdot a \cdot x, b_1) \in Y, (c \cdot a \cdot x \cdot b_1, b_2) \in Y, (c^{-1}, c \cdot a \cdot x \cdot b_1 \cdot b_2) \in Y$ and $(a \cdot x \cdot b) \in W\}$. Since multiplication is continuous, K_0 is open in Z. Then $K_0 \subseteq f^{-1}(W)$ and $k \in K_0$. Therefore, $f^{-1}(W)$ is open in Z. Thus, the map $f : Z \to a \cdot Z \cdot b$ is a homeomorphism.

2. By a similar argument as in 1., we are done. □

Lemma 3.8. *Let V be as in Lemma 3.6. Let $\mathcal{O} \subseteq V$ and $a \in G$. Then $a \cdot \mathcal{O}$ is τ_V-open if and only if $\mathcal{O}$ is open in V.*

Proof. Assume that $a \cdot \mathcal{O}$ is τ_V-open. Then for all $g \in G$, $(g \cdot a \cdot \mathcal{O}) \cap V$ is open. Since G is a group, $a^{-1} \in G$. Hence, $(a^{-1} \cdot (a \cdot \mathcal{O})) \cap V = \mathcal{O}$ is open in V. Conversely, assume that $\mathcal{O}$ is open in V. Let $g \in G$ Then $(g \cdot (a \cdot \mathcal{O})) \cap V = ((g \cdot a) \cdot \mathcal{O}) \cap V$. Let $Z = \{x \in V : a^{-1} \cdot g^{-1} \cdot x \in V\}$, by Lemma 3.7, $g \cdot (a \cdot \mathcal{O}) \cap V$ is open in V. Thus, $a \cdot \mathcal{O}$ is τ_V-open. □

Theorem 3.9. *If $\mathfrak{M}$ admits definable Skolem function, has Exchange Property and is $\aleph_1$-saturated and G is an n-dimensional 0-definable subset of M^n, then there are a large definable subset V of G such that*

1. *G with the topology τ_V is a topological group;*
2. *V is a finite disjoint union of 0-definable sets $U_1, ..., U_r$ such that for each $i = 1, ..., r, U_i$ is τ_V-open in G.*

Proof. Let V be as in Lemma 3.6. Suppose $V = U_1 \cup \cdots \cup U_r$ where the U_i's are open 0-definable subsets of M^n. For each $i \in \{1, \ldots r\}$, since $U_i = e \cdot U_i$ and U_i is open, by Lemma 3.8, U_i is τ_V-open. To complete the proof of this theorem, it is enough to show that the inversion is a τ_V-continuous and the multiplication is τ_V-continuous on G. Let W be τ_V-open in G. We will show that the pre-images of W under the inversion and the multiplication are τ_V-open.

Claim 1. *For every $a \in G$, $(W \cap aV)^{-1}$ is τ_V-open.*

Proof of Claim 1. Let $g \in G$. We would like to show that $g \cdot (W \cap aV)^{-1} \cap V$ is open. Since W is τ_V-open in G, $a^{-1}W \cap V$ is open. Since $a^{-1}W \cap V = a^{-1}(W \cap aV)$, $a^{-1}(W \cap aV)$ is open. Since the inversion is continuous, $(W \cap aV)^{-1}a =$

$(a^{-1}(W \cap aV))^{-1}$ is open. By Lemma 3.7, $g \cdot ((W \cap aV)^{-1} \cdot a) \cdot a^{-1} \cap V$ is open. Then $g \cdot (W \cap aV)^{-1} \cap V$ is open. Hence, $(W \cap aV)^{-1}$ is τ_V-open. □

Claim 2. *For every $a \in G$, $\{(x, y) \in G \times G : x \cdot y \in W \cap aV\}$ is τ_V-open.*

Proof of Claim 2. We want to show that $\{(x, y) \in G \times G : x \cdot y \in W \cap aV\}$ is τ_V-open in $G \times G$. Let $g, h \in G$. We want to show that $\mathcal{O} = \{(g \cdot x, h \cdot y) \in V \times V : x \cdot y \in W \cap aV\}$ is open. Note that $\mathcal{O} = \{(g \cdot x, h \cdot y) \in G \times G : a^{-1} \cdot x \cdot y \in a^{-1}W \cap V\}$. By the same argument as in Claim 1, we have that $a^{-1}W \cap V$ is open. Since the map $(x, y) \mapsto a^{-1} \cdot g^{-1} \cdot x \cdot h^{-1} \cdot y$ is continuous, $\mathcal{O}$ is open. Hence $\{(x, y) \in G \times G : x \cdot y \in W \cap aV\}$ is τ_V-open. □

By Lemma 3.4, there exists $c_1, \ldots, c_k$ such that $W = (W \cap c_1 V) \cup ... \cup (W \cap c_k V)$. By Claim 1, we have that $(W \cap c_1 V)^{-1}, \ldots, (W \cap c_k V)^{-1}$ are τ_V-open. Hence, $W^{-1} = (W \cap c_1 V)^{-1} \cap \cdots \cap (W \cap c_k)^{-1}$ is τ_V-open. For each $i \in \{1, \ldots, k\}$, let $\mathcal{O}_i = \{(x, y) \in G \times G : x \cdot y \in W \cap c_i V\}$. By Claim 2, $\mathcal{O}_1, \ldots, \mathcal{O}_k$ are τ_V-open. Since

$$\begin{aligned} \cdot^{-1}[W] &= \cdot^{-1}[(W \cap c_1 V) \cup \cdots \cup (W \cap c_k V)] \\ &= \cdot^{-1}[W \cap c_1 V] \cup \cdots \cup \cdot^{-1}[W \cap c_k V] \\ &= \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k, \end{aligned}$$

the pre-image $\cdot^{-1}[W]$ is τ_V-open.

Therefore, the inversion and the multiplication are τ_V-continuous on G. □

References

[1] Deirdre Haskell and Dugald Macpherson. **A version of o-minimality for the p-adics.** The Journal of Symbolic Logic, 1997 : 1075-1092.

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APPENDIX A

The Project Proposal of Course 2301399 Project Proposal

Academic Year 2020

Project Tittle (Thai) โครงสร้างพีมินินอลที่มีฟังก์ชันสโคเลมที่นิยามได้

Project Tittle (English) In P-minimal structures with definable skolem functions.

Project Advisor Dr. Athipat Thamrongthanyalak

By 1. Patcharapa Hanmungtham ID 6033530423

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Background and Rationale

Let p be a prime number. The p-adic valuation v on $\mathbb{Z} \setminus \{0\}$ is a map from $\mathbb{Z} \setminus \{0\}$ to $\mathbb{N} \cup \{0\}$ that sends any nonzero integer x to the highest n such that p^n divides x. We define the p-adic norm $|\cdot|_p$ on $\mathbb{Q}$ as follows:

$$|x|_p = \begin{cases} 0, & \text{if } x = 0 \\ p^{-v(a)+v(b)}, & \text{if } x = \dfrac{a}{b} \text{ for some } a, b \in \mathbb{Z} \setminus \{0\} \end{cases}$$

The p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$. Let $L = \{+, -, \cdot, 0, 1, \mathrm{Div}, (P_n)_{n\in\mathbb{N}}\}$ be a language where $0, 1$ are constant symbols, $+, \cdot$ are binary function symbols, $-$ is a unary function symbol and Div is a binary relation symbol. Observe that the field of p-adic numbers with the interpretation of Div by $\mathrm{Div}(a, b)$ if and only if $v(a) \le v(b)$ and the interpretation of $(P_n)_{n\in\mathbb{N}}$ by $\{x \in M: \exists y, y^n = x\}$ is an L -structure. Let L' be a language extending L, and let $\mathcal{F}$ be an L'-structure. We say that $\mathcal{F}$ is P- minimal if, for every $\mathcal{F}'$ elementarily equivalent to $\mathcal{F}$, every definable subset of F' is quantifier-free definable by L-formula. In [1], Haskell and Macpherson generalize a concept of p-adic field into P- minimal structures. In [3], Pillay proved that if $(G,*)$ is a definable group in $\mathbb{Q}_p$, then there is a topology τ on G such that $*\colon G \times G \to G$ is τ −continuous and there exists $n > 0$ such that for all $g \in G$, there is an open set U in G such that $g \in U$ and U is homeomorphic to $\mathbb{Q}_p^n$. In this project, we are interested in properties of definable groups in P-minimal structures.

Objectives

To study properties of definable groups in P-minimal structures with definable Skolem functions.

Scope

In this project, we only study P-minimal structures with definable Skolem functions.

Project Activities

1. Literature review.
2. Studying properties of definable groups in P-minimal structures.
3. Writing the project report.

Duration of time

Project Activities	Month							
	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar
1. Literature review	■	■	■					
2. Studying properties of definable groups in P -minimal structures.			■	■	■	■		
3. Writing the project report.							■	■

Benefits

To know properties of definable groups in P-minimal structures.

Equipment

Software

1. Latex
2. Microsoft Word

Hardware

1. Notebook
2. Printer

Budget

1. Books 5,000 Baht

References

[1] Deirdre Haskell and Dugald Macpherson. A Version of o-Minimality for the p-Adics. The Journal of Symbolic Logic Volume 62 (December 1997): 1075-1092.

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