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Holographic Solutions from N=5 Gauged Supergravity

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Holographic Solutions from $N = 5$ Gauged Supergravity

by

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A senior project
presented to the Chulalongkorn University

Examining Committee Membership

The following served on the Examining Committee for this senior project.

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Abstract

In this senior project, we find holographic solutions in the forms of supersymmetric domain walls, supersymmetric Janus solutions, and supersymmetric AdS_4 black holes from $N = 5$ gauged supergravity with local SO(5) symmetry. There is only one $N = 5$ supersymmetric AdS_4 vacuum preserving the full $SO(5)$ symmetry dual to an $N = 5$ SCFT in three dimensions in the language of holographic duality. A number of supersymmetric domain wall solutions interpolating between this AdS_4 fixed point and singular geometries in the IR with $SO(4)$ and $SO(3)$ symmetries are presented. Holographically, these solutions describe RG flows from the $N = 5$ SCFT in the presence of mass deformations. Some of these solutions are precisely in agreement with the previously known mass deformations within the dual $N = 5$ SCFT. In addition, universal domain wall solutions subjected to specific unbroken supersymmetry are analyzed. Then, we provide a number of projectors and BPS equations required for the existence of Janus solutions in parallel to the case of domain wall solutions. As a result, we find supersymmetric Janus solutions describing two-dimensional conformal defects in the $N = 5$ SCFT with $N = (4, 1)$ and $N = (1, 1)$ supersymmetries on the defects in the form of AdS_3-sliced domain wall solutions. Finally, we perform an analysis as similar to the two previous cases for solutions of the form $AdS_2 \times \Sigma^2$, with $\Sigma^2 = S^2, H^2$ being a Riemann surface, corresponding to near horizon geometries of AdS_4 black holes. We study both magnetic and dyonic solutions and show that there exists a class of $AdS_2 \times H^2$ solutions with $SO(2)$ symmetry. In the language of holography, these solutions correspond to twisted compactification of $N = 5$ SCFT to superconformal quantum mechanics.

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Chapter 1

Introduction

Over the past twenty years, The AdS/CFT correspondence originally proposed in [1] see also [2] and [3], has provided the holographic duality between strongly coupled conformal field theory and the low-energy limit of the M-/superstring- theory living on the product of $AdS_d \times M_{11/10-d}$. This well-known duality has become a prominent tool describing various strongly couple systems ranging from (non-)conformal field theories, conformal defects, AdS-black holes, and condensed matter physics systems. Although the complete AdS/CFT duality is achieved only in the context of string/M-theory, a large number of remarkable results have been obtained from solutions of lower-dimensional gauged supergravities.

In many cases, the gauged supergravity theory under consideration is known to be consistently embedded in ten- or eleven-dimensional supergravities which are low energy effective theories of string/M-theory. The resulting holographic solutions can accordingly be uplifted to string/M-theory and can be interpreted as D- and M-brane configurations. Solutions of gauged supergravities with presently unknown higher-dimensional origin are also interesting in the sense that they can provide a bottom-up approach to the AdS/CFT duality and still give some insight to the dynamics of the dual field theories at strong coupling limits. These make studying solutions of gauged supergravities in various space-time dimensions and different numbers of supersymmetries worth considering.

Most of the previous studies concern with finding a particular class of solutions that preserve some amount of supersymmetry. These supersymmetric or BPS solutions play an important role in different aspects of the AdS/CFT correspondence. Gauged supergravities including possible massive deformations are known to exist in dimensions from two to ten. Among these theories, four-dimensional gauged supergravities are of particular interest since they give rise to holographic duals of three-dimensional superconformal field theories

(SCFTs) and possible deformations thereof. These SCFTs describe low energy dynamics of the world-volume theory on M2-branes which are fundamental objects in M-theory. The SCFTs in three dimensions take the form of Chern-Simons-Matter (CSM) theories since the usual gauge theories with Yang-Mills gauge kinetic terms are not conformal. Up to now, many of these SCFTs with different numbers of supersymmetries have been constructed, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

In this senior project, we are interested in supersymmetric solutions from $N = 5$ gauged supergravity with local $SO(5)$ constructed long ago in [25]. According to the AdS/CFT duality, these solutions could describe various aspects of strongly coupled $N = 5$ SCFT in three dimensions. There are ten scalars described by $SU(5,1)/U(5)$ coset. For the full construction of gauged supergravity with a given coset manifold G/H in various space-time dimensions and a number of supersymmetries, there is a complete prescription using a generalized electromagnetic duality and an embedding of gauge group inside the global group G, see an excellent review on this topic [26], which can be used to construct this theory and provide a generalization to other (non-)compact gauge group; however, this senior project will not cover this topic. In the case of $SU(5,1)/U(5)$, the scalar potential of this gauged supergravity has been analyzed in [27]. There is only one supersymmetric AdS_4 vacuum preserving the full $N = 5$ supersymmetry with unbroken $SO(5)$ symmetry. According to the AdS/CFT duality, this AdS_4 critical point is dual to an $N = 5$ SCFT in three dimensions. There is also another non-supersymmetric AdS_4 vacuum with unbroken $SO(3)$ gauge symmetry. This critical point is perturbatively stable as pointed out in [28] and has been extensively studied in the context of holographic superconductors in [29]. To the best of our knowledge, no supersymmetric solutions of $N = 5$ gauged supergravity have previously been considered. The present work will hopefully fill this gap in the existing literature.

We will look for various types of supersymmetric solutions of the aforementioned $N = 5$ gauged supergravity. We will firstly study supersymmetric domain walls interpolating between the supersymmetric AdS_4 vacuum and singular geometries. We will show explicitly the calculations involving the Killing spinor equations. Consequently, the solutions to these equations describe holographic RG flows from the dual $N = 5$ SCFT in the UV to non-conformal field theories in the IR obtained from mass deformations of the $N = 5$ SCFT. Similar solutions have extensively been studied in $N = 8, 4, 3, 2$ four-dimensional gauged supergravities, see for example [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43].

We will also find Janus solutions described by AdS_3-sliced domain walls interpolating between asymptotically AdS_4 spaces. These solutions are holographically dual to two-dimensional conformal defects within the $N = 5$ SCFT that break the superconformal symmetry in the three-dimensional bulk to a small superconformal symmetry on the two-dimensional surfaces resulting in conformal interfaces. Supersymmetric Janus solutions in other four-dimensional gauged supergravities have previously been studied in [41, 42, 44, 45, 46, 47].

We finally look for solutions interpolating between the supersymmetric AdS_4 and $AdS_2 \times \Sigma^2$ geometries with Σ^2 being a Riemann surface. These solutions describe supersymmetric black holes in an asymptotically AdS_4 space. Solutions of this type in other gauged supergravities can be found in [48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. In the dual field theory, the solutions are dual to RG flows from the $N = 5$ SCFT to another SCFT in one dimension or superconformal quantum mechanics. The latter is obtained from the former via twisted compactifications on Σ^2. This type of solution plays an important role in the microscopic computation of black hole entropy in asymptotically AdS_4 space, see for example [58, 59, 60].

The senior project is organized as follows. In section 2, we review the construction of four-dimensional $N = 5$ gauged supergravity with $SO(5)$ gauge group. In section 3, we first review a calculation used in deriving the supersymmetric domain wall solutions, such as the Killing spinor equations. We also mention a criterion conjectured to classify physical singularities obtained as solutions in the IR regime. Subsequently, we will look for supersymmetric domain wall solutions describing RG flows in the dual $N = 5$ SCFTs in three-dimensions. Similarly, we then study the case of supersymmetric Janus solutions in section 4 and finally consider possible supersymmetric AdS_4 black holes for both magnetic and dyonic solutions in section 5. Several calculations are based on the use of differential form and vielbein formalism; we review these rich topics and show some practical examples in appendix A. In Appendix B, we collect some useful manipulations involving gamma matrices. Finally, conclusions and comments on the results are given in section 6.

Chapter 2

A Review of $N = 5$ Gauged Supergravity

In this chapter, we review a basic structure describing $N = 5$ gauged supergravity theory which is the main theory we focus on.

We begin by providing the original form of $N = 5$ gauged supergravity with an $SO(5)$ gauge group first constructed in [25], with slight changes of the convention. $N = 5$ supersymmetry in four-dimensions does not allow for any matter multiplets due to a large number of supersymmetries, so the only allowed supermultiplet in this theory is the gravity multiplet with the following field content

$$\left(e_\mu^a, \psi_\mu^i, A_\mu^{ij}, \chi^{ijk}, \phi^i\right) \tag{2.1}$$

These fields correspond to the graviton e_μ^a, five gravitini ψ_μ^i, ten vectors $A_\mu^{ij} = -A_\mu^{ji}$, eleven 3-form spin-$\frac{1}{2}$ fields $\chi^{ijk} = \chi^{[ijk]}$ and a singlet spin-$\frac{1}{2}$ field χ together with five complex scalars $\phi^i = (\phi_i)^*$.

Space-time and tangent space indices are denoted by $\mu, \nu, ... = 0, 1, 2, 3$ and $a, b, ... = 0, 1, 2, 3$, respectively. $N = 5$ supergravity theory admits global $SU(5, 1)$ and local composite $U(5) \sim SU(5) \times U(1)$ symmetries. The latter is the R-symmetry for $N = 5$ supersymmetry. Indices $i, j, k, ... = 1, 2, ..., 5$ denote a fundamental representation of $SU(5)$. The ten scalars then parametrize $SU(5, 1)/U(5)$ coset manifold described by the coset representative

$$\Sigma^A{}_B = \left(\begin{array}{c|c} \delta^i{}_j - e_2\phi^i\phi_j & e_1\phi^i \\ \hline e_1\phi_j & e_1 \end{array}\right) \tag{2.2}$$

with $A, B = 1, 2, ..., 6$ being indices of $SU(5,1)$ fundamental representation. The quantities e_1 and e_2 are defined by

$$e_1 \equiv \frac{1}{\sqrt{1-|\phi|^2}} \quad \text{and} \quad e_2 \equiv \frac{1}{|\phi|^2}\left(1 - \frac{1}{\sqrt{1-|\phi|^2}}\right) \tag{2.3}$$

where $|\phi|^2 = \phi^i \phi_i$ with summation over i. It should be noted that this expression of coset representative can be given in terms of an exponential of non-compact generators on $SU(5,1)$ contracted with the scalar fields, i.e.

$$\Sigma^A{}_B = \exp\left(\begin{array}{c|c} 0_{5\times 5} & \tilde{\phi}^i \\ \hline \tilde{\phi}_j & 0_{1\times 1} \end{array}\right), \tag{2.4}$$

in which we can match two different scalars as $\phi^i = \tilde{\phi}^i \frac{\tanh|\tilde{\phi}|}{|\tilde{\phi}|}$.

In addition this parametrization restricts ϕ^i to be well-defined on $|\phi|^2 < 1$. Being an element of $SU(5,1)$, $\Sigma^A{}_B$ satisfies the following identity

$$\Sigma^{-1} = \eta \Sigma^\dagger \eta \tag{2.5}$$

in which $\eta = \text{diag}(1,1,1,1,1,-1)$ is the $SU(5,1)$ invariant tensor.

The ten vector fields A^{ij}_μ can be used to gauge the $SO(5) \subset SU(5) \subset SU(5,1)$ symmetry resulting in $N = 5$ gauged supergravity with $SO(5)$ gauge group. The corresponding bosonic Lagrangian is given by

$$\begin{aligned} e^{-1}\mathcal{L} &= \frac{1}{2}R - \frac{1}{2}P^i_\mu P^\mu_i - \frac{1}{8}\left[(2S^{ij,kl} - \delta^{ik}\delta^{jl})F^+_{\mu\nu ij}F^{+\mu\nu}_{kl}\right. \\ &\quad \left. +(2S_{ij,kl} - \delta_{ik}\delta_{jl})F^{-ij}_{\mu\nu}F^{-\mu\nu kl}\right] - V \end{aligned} \tag{2.6}$$

with the 10×10 matrix $S_{ij,kl} = (S^{ij,kl})^*$. As shown in the last chapter, the quantities P^i_μ and $Q_\mu{}^i{}_j$ can be understood as the vielbein and composite connection defined on the scalar manifold; it can be obtained from the following relation

$$\Sigma^{-1}D_\mu\Sigma = \left(\begin{array}{c|c} \frac{1}{2}Q_\mu{}^i{}_j - \frac{1}{6}\delta^i{}_j Q_\mu{}^k{}_k & -\frac{1}{\sqrt{2}}P^i_\mu \\ \hline -\frac{1}{\sqrt{2}}P_{\mu i} & \frac{1}{3}Q_\mu{}^k{}_k \end{array}\right) \tag{2.7}$$

Explicitly, we can write

$$P^i_\mu = -\sqrt{2}e_1\left(\delta^i{}_j - e_2\phi^i\phi_j\right)D_\mu\phi^j \tag{2.8}$$

$$Q_\mu{}^i{}_j = 2e_2\phi^i \overleftrightarrow{D}_\mu\phi_j + \frac{1}{2}\left(e_1^2\delta^i{}_j - 2e_2^2\phi^i\phi_j\right)\phi^k \overleftrightarrow{D}_\mu\phi_k \tag{2.9}$$

with the gauge covariant derivative given by

$$D_\mu\phi_i = \partial_\mu\phi_i - gA_\mu^{ij}\phi_j \quad \text{and} \quad D_\mu\phi^i = (D_\mu\phi_i)^* \tag{2.10}$$

We now come to the gauge field part. The (anti) self-dual field strength tensors are defined as

$$F^{+ij}_{\mu\nu} = \frac{1}{2}\left(F^{ij}_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma ij}\right) \quad \text{and} \quad F^{-}_{\mu\nu ij} = \frac{1}{2}\left(F^{ij}_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma ij}\right) \tag{2.11}$$

with the gauge field strengths defined by

$$F^{ij}_{\mu\nu} = 2\partial_{[\mu}A^{ij}_{\nu]} - 2gA^{ik}_{[\mu}A^{kj}_{\nu]} \tag{2.12}$$

with summation over k. We also note that $(F^{+ij}_{\mu\nu})^* = F^{-}_{\mu\nu ij}$.

The matrix $S^{ij,kl}$ is defined by the relation

$$\left(\delta^{ij}_{kl} + \frac{1}{2}\epsilon^{ijklp}\phi_p\right)S^{kl,mn} = \delta^{ij}_{mn} \tag{2.13}$$

with summation over repeated indices. The explicit form of $S^{ij,kl}$ is first derived in [61] in which black hole attractors in ungauged $N = 5$ supergravity have been studied. In our notation, this matrix reads

$$S^{ij,kl} = \frac{1}{1-(\phi_k\phi_k)^2}\left[\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijklm}\phi_m - 2\delta_{[i[k}\phi_{l]}\phi_{j]}\right] \tag{2.14}$$

with summation over repeated indices.

Finally, the scalar potential is given by

$$V = -g^2\left[2 + 4e_1^2 - \frac{1}{2}e_1^4\left(|\phi|^2 - (\phi_i\phi_i)(\phi^j\phi^j)\right)\right] \tag{2.15}$$

To calculate the holographic solution, we also need supersymmetry transformation rules for fermions. In the chiral notation, the fermionic fields are subject to the chirality projection

$$\gamma_5\psi^i_\mu = \psi^i_\mu, \quad \gamma_5\chi = -\chi, \quad \gamma_5\chi^{ijk} = \chi^{ijk} \tag{2.16}$$

with

$$\gamma_5 \psi_{\mu i} = -\psi_{\mu i}, \quad \gamma_5 \chi_{ijk} = -\chi_{ijk} \tag{2.17}$$

The corresponding supersymmetry transformations read

$$\begin{aligned}
\delta\psi_{\mu i} &= 2\mathcal{D}_\mu \epsilon_i - Q_\mu{}^j{}_i \epsilon_j - \frac{1}{2\sqrt{2}} \gamma^{\nu\rho} \gamma_\mu G^+_{\nu\rho kl} C_{ij}{}^{kl} \epsilon^j + \sqrt{2} g \gamma_\mu S_{ij} \epsilon^j, && (2.18)\\
\delta\chi_{ijk} &= \epsilon_{ijklm} P^m_\mu \gamma^\mu \epsilon^l + \frac{3}{2} G^+_{\mu\nu rs} \gamma^{\mu\nu} C_{[ij}{}^{rs} \epsilon_{k]} - 2g N^l{}_{ijk} \epsilon_l, && (2.19)\\
\delta\chi &= -P_{\mu i} \gamma^\mu \epsilon^i - 2g N^i \epsilon_i && (2.20)
\end{aligned}$$

in which

$$\begin{aligned}
S^{ij} &= e_1 \delta^{ij} + \frac{1}{2} e_2^2 \left[|\phi|^2 \left(\phi^i \phi_j + \phi_i \phi^j \right) - 2(\phi_k)^2 \phi^i \phi^j \right], && (2.21)\\
N_l{}^{ijk} &= e_1 \epsilon^{ijklm} \phi_m + e_1 e_2 \epsilon^{ijklmn} \phi_m \phi^n \phi_l + 3 e_1^2 \delta^{ijk}_{lmn} \phi_m \phi^n, && (2.22)\\
N^i &= -e_1^2 \phi_i - e_1 e_2 (\phi_k)^2 \phi^i, && (2.23)\\
C^{ij}{}_{kl} &= \frac{1}{e_1} \delta^{ij}_{kl} - 2 \frac{e_2}{e_1} \delta^{[i}{}_{[k} \phi^{j]} \phi_{l]} && (2.24)
\end{aligned}$$

The first three matrices are known as the fermion-shift matrices related to gravitini, 3-form fermions, and a singlet fermion. We have already included the space-time covariant derivative in $\mathcal{D}_\mu$. We emphasize that raising and lowering of $SU(5)$ indices $i, j, k, ...$ correspond to taking complex conjugate. The field strengths $G^+_{\mu\nu ij}$ are obtained from $F^+_{\mu\nu ij}$ by dressing with scalars

$$G^+_{\mu\nu ij} = S^{ij,kl} F^+_{\mu\nu kl} \tag{2.25}$$

in order to have a manifest $SU(5)$ covariant object.

In addition, the scalar potential can be written in term of the fermion-shift matrices as

$$V = -\frac{1}{5} g^2 \left(6 S^{ij} S_{ij} - \frac{1}{3} N_l{}^{ijk} N^l{}_{ijk} - 2 N^i N_i \right) \tag{2.26}$$

due to the supersymmetric Ward identity.

There are two kind of AdS_4 vacua appeared in scalar potential, one with maximal $N = 5$ supersymmetry and the other one with completely broken supsersymmetry. These two vacua are given respectively by

$$\phi^i = 0, \quad V_0 = -6g^2, \quad L = \frac{1}{\sqrt{2} g} \tag{2.27}$$

and

$$\phi^i = 0 \quad \text{for} \quad i = 1,2,3, \quad \phi^4 = -i\phi^5 = \sqrt{\tfrac{2}{5}}$$
$$V_0 = -14g^2, \quad L = \tfrac{\sqrt{3}}{\sqrt{14}g} \tag{2.28}$$

where V_0 is the cosmological constant. The supersymmetric critical point preserves the full $SO(5)$ gauge symmetry while the non-supersymmetric one is only invariant under $SO(3) \subset SO(5)$. The AdS_4 radius L is related to the cosmological constant by

$$L^2 = -\frac{3}{V_0} \tag{2.29}$$

where we have taken $g > 0$ for definiteness.

Chapter 3

Holographic RG flows

We begin with the holographic RG flow solutions in the form of supersymmetric domain wall solutions interpolating between the supersymmetric AdS_4 vacuum in the UV and singular geometries in the IR. There is an excellent review on this topic, see [62].

The metric ansatz takes of the form

$$ds^2 = e^{2A(r)}dx_{1,2}^2 + dr^2 \tag{3.1}$$

with $dx_{1,2}^2$ being the flat metric on three-dimensional Minkowski space-time. Along with these solutions, no gauge fields are turned on, and scalar fields depend only on the radial coordinate r to preserve the Poincare invariance on the embedded three-dimensional Minkowski space-time. We also use Majorana representation for gamma matrices in which all γ^μ are real, but γ_5 is purely imaginary from now on.

Before we take a close look at each solution with different residual symmetries, we note here some technical calculations which will be used in the following subsections.

The non-vanishing vielbein one-forms followed from the ansatz (3.1) are given by

$$e^{\hat{\mu}} = e^A dx^\mu, \quad e^{\hat{r}} = dr \tag{3.2}$$

for $\mu = 0, 1, 2$. The non-vanishing spin connections are then obtained by using the structure equation[1]

$$\omega^{\hat{\mu}\hat{r}} = A' e^A dx^\mu = A' e^{\hat{\mu}} \tag{3.3}$$

where $'$ denotes r-derivatives.

[1]see Appendix A for brief reviews.

Since we are here looking for the supersymmetric solutions interpolating between supersymmetric vacuum, we demand that supersymmetry variation of the fermionic fields along the solutions vanish. We first consider the case of gravitini along $\mu = 0, 1, 2$ directions.

$$\begin{aligned} 0 &= 2\mathcal{D}_\mu \epsilon_i + \sqrt{2} g \gamma_x S_{ij} \epsilon^j \\ &= \frac{1}{2} e^{\hat{\mu}}_\mu \omega_{\hat{\mu}}{}^{ab} \gamma_{ab} \epsilon_i + \sqrt{2} g e^{\hat{\mu}}_\mu \gamma_{\hat{\mu}} S_{ij} \epsilon^j \\ &= A' \gamma_{\hat{\mu}\hat{r}} \epsilon_i + \sqrt{2} g S_{ij} \gamma_{\hat{\mu}} \epsilon^j . \end{aligned}$$

To obtain non-trivial solutions to this equation, we impose the projector

$$\gamma_{\hat{r}} \epsilon_i = e^{i\Lambda} e^i \tag{3.4}$$

with Λ being a real function of r. It should be noted here that this projector relates two chiralities of Killing spinors ϵ^i, the solutions then preserve only half of the original supersymmetry or ten supercharges for this case. Since the fermion-shift matrix S_{ij} is symmetric, we can always diagonalize it into a diagonal matrix written as

$$S_{ij} = \frac{\mathcal{W}}{g\sqrt{2}} \delta_{ij} \tag{3.5}$$

in which we have introduced the "superpotential" $\mathcal{W}$ for convenience. Inserting back, we then get

$$0 = A' e^{i\Lambda} \epsilon^i + \mathcal{W} \epsilon^i . \tag{3.6}$$

This equation leads to

$$0 = e^{i\Lambda} A' + \mathcal{W}, \tag{3.7}$$

or, in particular,

$$A' = \pm W = \pm |\mathcal{W}| \quad \text{and} \quad e^{i\Lambda} = \mp \frac{\mathcal{W}}{W}. \tag{3.8}$$

We will choose an upper sign for definiteness. We next consider the gravitini's variation along r-direction

$$\begin{aligned} 0 &= 2\partial_r \epsilon_i + \mathcal{W} \gamma_r \epsilon^i \\ &= 2\partial_r \epsilon_i + \mathcal{W} e^{\hat{r}}_r e^{-i\Lambda} \epsilon_i \end{aligned}$$

where we have used the expressions of vielbein and projector defined earlier. Combining with equation (3.7), we find

$$\partial_r \epsilon_i = \frac{A}{2} \epsilon_i \tag{3.9}$$

Solution to this equation is the usual form of the Killing spinors for supersymmetric domain wall solutions,

$$\epsilon^i = e^{\frac{A}{2}} \epsilon^i_{(0)}, \tag{3.10}$$

for constant spinors $\epsilon^i_{(0)}$ satisfying $\gamma_{\hat{r}} \epsilon_{i(0)} = \epsilon^i_{(0)}$.

Hence, what we get from the analysis of the supersymmetry variations of gravitino are

1. the projector (3.4) used to relate different chiralities of spinors resulting in the half number of supersymmetries,
2. the Killing spinor equation (3.7) used to determine the relation between warped factor A and scalars ϕ_i,
3. the solutions of Killing spinors (3.10) satisfying the Killing spinor equations.

We now move to consider the supersymmetry variation of 3-form fermions and a singlet fermion, applying the projector (3.4) gives

$$\delta\chi_{ijk} = 0 \quad \rightarrow \quad 0 = \left(\epsilon_{ijklm} P^m_r e^{-i\Lambda} - 2g N^l{}_{ijk}\right) \epsilon_l \tag{3.11}$$

$$\delta\chi = 0 \quad \rightarrow \quad 0 = \left(-P_{ri} e^{-i\Lambda} - 2g N^i\right) \epsilon_i \tag{3.12}$$

which are the rest Killing spinor equations. It clearly be seen that these two equations contain purely scalar fields, it then be the first-order differential equations, used for determining the form of scalar fields, also known as the BPS equations.

In addition, solutions of scalars taken from equations (3.11) and (3.12) can be checked that it satisfy the field equations. Recall first that, any symmetry transformation of a field equation also consistently yields a combination of field equations of transformed fields. As the equations (3.11) and (3.12) are direct consequences of supersymmetry transformation, the solutions to these equations consistently satisfy field equations if and only if the fermionic field equations, involving χ and χ_{ijk}, are satisfied which is the case for BPS solutions. However, some chosen set of scalars may transform inappropriately under gauge symmetry. This would yield an inconsistency in the BPS equations. Nevertheless, choosing a set of scalars to be singlet under residual symmetry can avoid this case together with a particular form of BPS equations introduced in [63],

$$(\phi^I)' = g^{IJ} \frac{\partial W}{\partial \phi^J} \tag{3.13}$$

$$A' = W \tag{3.14}$$

with scalar potential written as

$$V = -3W^2 + g^{IJ}\frac{\partial W}{\partial \phi^I}\frac{\partial W}{\partial \phi^J}. \tag{3.15}$$

We have denoted ϕ^I as a set of singlet scalars and g^{IJ} as the inverse of metric on the scalar manifold.

We end this section by mentioning the criterion given in [64] to classify physical and unphysical singularities (if exist) in domain wall solutions. This conjecture states that

> Large curvatures in geometries of the form of domain wall solutions are allowed only if the scalar potential is bounded above in the solutions. (3.16)

In particular, the behavior of solutions at the singular geometries with scalar potential $V \to \infty$ would give no sensible non-conformal field theory in the IR.

3.1 RG flows with $SO(4)$ symmetry

We first consider the simplest case, solutions with the largest symmetry contained within $SO(5)$ gauge symmetry with non-vanishing scalars, an $SO(4)$ symmetry. By further calculating complicated solutions, we should obtain this class of solutions as a common solution with consistent truncation. With the parametrization given in chapter 2, we can easily see that only one complex scalar is an $SO(4) \subset SO(5)$ singlet. We will choose this singlet, specifically ϕ^5, to take the form

$$\phi^5 = \tanh\varphi e^{i\zeta} \tag{3.17}$$

with real scalars $\varphi \in [0, \infty)$ and $\zeta \in [0, 2\pi)$. According to the expression of the fermion-shift matrices, we obtain the non-vanishing components

$$S^{ij} = (g\cosh\varphi)\,\delta^{ij} \tag{3.18}$$

$$N_5 = -e^{i\zeta}\sinh\varphi \tag{3.19}$$

$$N^1{}_{234} = -e^{i\zeta}\sinh\varphi \tag{3.20}$$

in which $N^1{}_{234}$ is only an example of the matrix $N^l{}_{ijk}$ from several identical components up to sign factor. In this case, every eigenvalues of S_{ij} coincide, we then get the superpotential

$$\mathcal{W} = \sqrt{2}g\cosh\varphi \tag{3.21}$$

This coincidence of $\mathcal{W}$ along every Killing spinors ϵ_i gives us a clue that the following solutions must preserve maximal $N = 5$ supersymmetry.

According to the supersymmetric Ward identity, we then find the scalar potential from fermion-shift matrices as

$$V = -2g^2\left(2 + \cosh 2\varphi\right) \tag{3.22}$$

Since $\mathcal{W}$ is real, we can therefore check whether this choice of singlet scalars is consistent with the field equations by using the real superpotential $W = |\mathcal{W}|$ to determine the scalar potential and see whether it is identical with our result from supersymmetric Ward identity. To do this, we first calculate the vielbein on scalar manifold,

$$P^5_r = -\sqrt{2}e^{i\zeta}\left(\varphi' + i\cosh\varphi\sinh\varphi\zeta'\right); \quad \text{otherwise vanish,} \tag{3.23}$$

and then use it to write down scalar Lagrangian,

$$\begin{aligned} e^{-1}\mathcal{L}_{scal} &= -\frac{1}{2}P^5_r P^r_5 - V & (3.24)\\ &= -(\varphi')^2 - \cosh^2\varphi\sinh^2\varphi(\zeta')^2 - V. & (3.25)\end{aligned}$$

The metric defined on scalar manifold is readily verified, its non-vanishing inverse components then read

$$g^{\varphi\varphi} = 1 \quad \text{and} \quad g^{\zeta\zeta} = \frac{4}{\sinh^2 2\varphi} \tag{3.26}$$

With all of these, the scalar potential calculated by using (3.15) thus takes of the form

$$\begin{aligned} V &= -3W^2 + g^{IJ}\frac{\partial W}{\partial\phi^I}\frac{\partial W}{\partial\phi^J}\\ &= -3W^2 + \left(\frac{\partial W}{\partial\varphi}\right)^2 + \frac{4}{\sinh^2 2\varphi}\left(\frac{\partial W}{\partial\zeta}\right)^2 \\ &= -2g^2\left(2+\cosh 2\varphi\right)\end{aligned} \tag{3.27}$$

in which we have labeled $\phi^I = \{\varphi, \zeta\}$. This result is exactly identical to the result derived from the supersymmetric Ward identity; our choice of singlet scalars is therefore consistent with the field equations. For the reader who prefer a more explicit calculation indicating a consistence of the field equations, we will comeback to show this after we obtain the explicit form of solutions.

Graphically, the scalar potential is shown in figure 3.1 where we have set gauge coupling $g = 1$ for simplicity. Note here that, from figure 3.1, we can guess that solutions should roll our scalar φ from the trivial critical point to ∞ which exhibits evidence of singularity.

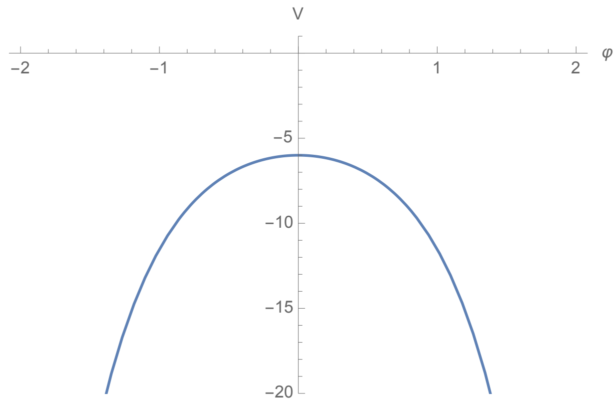


Figure 3.1: The scalar potential with $SO(4)$ symmetry. It can be seen directly that the only supersymmetric critical point in the case of $SO(4)$ symmetry is at $\varphi = 0$ which gives a negative cosmological constant as desired.

Next, we now move to finding a set of BPS equations. This can be done by considering the Killing spinor equations coming from gravitini, 3-form spinors, and a singlet spinor. The first one is readily read off from the expression of superpotential $\mathcal{W}$. Since $\mathcal{W}$ is real and phase $e^{i\Lambda} = 1$, we then get

$$A' = \sqrt{2} g \cosh\varphi. \tag{3.28}$$

Another two Killing spinor equations, $\delta\chi_{ijk} = 0$ and $\delta\chi = 0$, yield the almost identical BPS equations

$$\varphi' + i\cosh\varphi\sinh\varphi\zeta' = -\sqrt{2}e^{i\Lambda} g\sinh\varphi \quad \text{and} \tag{3.29}$$

$$\varphi' + i\cosh\varphi\sinh\varphi\zeta' = -\sqrt{2}e^{-i\Lambda} g\sinh\varphi \tag{3.30}$$

The difference is on the sign of the exponent of the phase; however, it does not affect our equations because of the vanishing of the phase as mentioned before. Nevertheless, this is not the case of Janus solutions in which even though there are the same BPS equations, but $e^{i\Lambda} \neq 1$ resulting in a more discussion on this equations. We will comeback to this later.

Using the reality of scalars, we find the complete set of BPS equations as

$$\begin{aligned} A' &= \sqrt{2} g\cosh\varphi \\ \varphi' &= -\sqrt{2} g\sinh\varphi \\ \zeta' &= 0 \end{aligned}$$

which are also known as flow equations. Before we solve the above equations, it should be noted that, as mentioned in the beginning of chapter, the above equations can be rewritten in terms of derivatives of superpotential W. Provided inverse of the scalar metric, we can write the flow equations together with the equation of A' as

$$A' = W = \sqrt{2}g\cosh\varphi \tag{3.31}$$
$$\varphi' = -\frac{\partial W}{\partial \varphi} = -\sqrt{2}g\sinh\varphi \tag{3.32}$$
$$\zeta' = -\frac{4}{\sinh^2 2\varphi}\frac{\partial W}{\partial \zeta} = 0 \tag{3.33}$$

therefore ensuring the consistency with the field equations.

The equation (3.33) simply yields the constant of pseudoscalar $\zeta = \zeta_0$. Because ζ does not appear in other BPS equations and scalar potential, ζ_0 can be any real constant value. Subsequently, it is straightforward to solve the equations (3.31) and (3.32). We first consider equation (3.31) divided by equation (3.32),

$$\frac{dA}{d\varphi} = -\frac{1}{\tanh\varphi}$$

which gives a relation between warped factor A and scalar φ,

$$A = C_1 - \ln\sinh\varphi$$

Substitute back into the equation (3.31), we find

$$\tanh\frac{\varphi}{2} = e^{-\sqrt{2}g(r-r_0)} \tag{3.34}$$

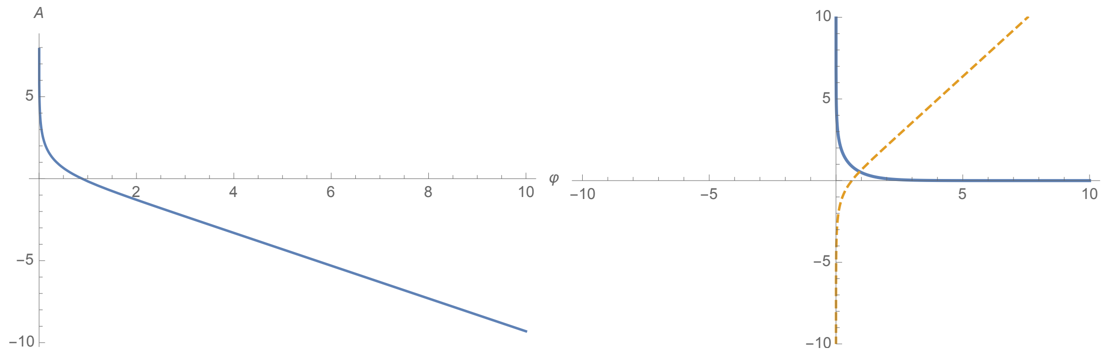
We will neglect the integration constant C_1 by rescalling coordinates on $dx^2_{1,2}$. The solutions are shown graphically in figure 3.2.

At the asymptotic limit $r \to \infty$, we find

$$\varphi \sim e^{-\sqrt{2}gr} \sim e^{-\frac{r}{L}} \quad \text{and} \quad A \sim \sqrt{2}gr \sim \frac{r}{L}. \tag{3.35}$$

This is the $N = 5$ supersymmetric AdS_4 configuration as desired.

On the other side of the solution, we expect that it should be another supersymmetric AdS_4 configuration, so the solution interpolates between supersymmetric critical points. Unfortunately, this is not the case, since there is only one supersymmetric critical point.



(a) A relation between warped factor A and scalar φ.

(b) A solution of A (dashed) and φ (thick) along the radial coordinate.

Figure 3.2: A Domain wall solution from the $N = 5$ AdS_4 critical point as $r \to \infty$ to a singularity geometry in $r_0 = 0$.

Nevertheless, there exhibit a singular geometry instead. To see this, we consider the radial coordinate as $r \to (r_0)_+$ at which the solution becomes

$$\varphi \sim -\ln\left(1 - e^{-\sqrt{2}g(r-r_0)}\right) \sim \infty \quad \text{and} \quad A \sim -\ln\left(\frac{1}{2(e^{-\sqrt{2}g(r-r_0)} - 1)}\right) \sim -\infty. \tag{3.36}$$

Accordingly, scalar potentials are of the form

$$V \sim -\left(\frac{g}{e^{-\sqrt{2}g(r-r_0)} - 1}\right)^2 \sim -\infty. \tag{3.37}$$

According to the criterion given in [64], the singularity is then physically acceptable. Therefore the above solution holographically describes an RG flow from the $N = 5$ SCFT in the UV to an $N = 5$ non-conformal field theory in the IR. The flow breaks conformal symmetry but preserves the full $N = 5$ Poincare supersymmetry in three dimensions. We identify this flow with the mass deformation pointed out in [13] in which the explicit form of relevant mass terms have also been given. The deformation preserves $N = 5$ supersymmetry but breaks the $SO(5)$ R-symmetry to an $SO(4)$ subgroup in agreement with the supergravity result obtained here.

Finally, we will end this section by showing that our solution is compatible with the field equations. We begin with the bosonic Lagrangian underlying this supersymmetric domain wall.

$$\mathcal{L} = e^{3A}\left[-3\left(2A'^2 + A''\right) - \frac{1}{2}\left(\varphi'^2 + \sinh\varphi\cosh\varphi\zeta'^2\right) + 2g^2\left(2 + \cosh 2\varphi\right)\right] \tag{3.38}$$

By using Euler-Lagrange equation, Einstein field equation in $x_{1,2}$-direction, Einstein field equation in r-direction, scalar field equation for φ, and scalar field equation for ζ are respectively given by

$$
\begin{aligned}
3A'^2 + 2A'' &= 2g^2\left(2+\cosh 2\varphi\right) - \varphi'^2 - \cosh\varphi\sinh\varphi\zeta'^2 && (3.39)\\
3A'^2 &= 2g^2\left(2+\cosh 2\varphi\right) + \varphi'^2 + \cosh\varphi\sinh\varphi\zeta'^2 && (3.40)\\
0 &= 4g^2\sinh^2 2\varphi + 6A'\varphi' - \cosh 2\varphi\zeta'^2 + 2\varphi'' && (3.41)\\
0 &= 2\cosh 2\varphi\varphi'\zeta' + \sinh 2\varphi\left(A'\zeta' + \zeta''\right) && (3.42)
\end{aligned}
$$

Inserted BPS equations (3.31), (3.32), and (3.33), it is easily to verify that the field equations are satisfied.

3.2 RG flows with $SO(3)$ symmetry

In this section, we will repeat the analysis with a smaller residual symmetry $SO(3) \subset SO(5)$. There are two complex scalars which are $SO(3)$ singlets. Particularly, we choose them to be ϕ^4 and ϕ^5. These scalars are parametrized as follows

$$\phi^4 = \tanh\varphi\cos\vartheta e^{i\zeta_1} \quad \text{and} \quad \phi^5 = \tanh\varphi\sin\vartheta e^{i\zeta_2} \tag{3.43}$$

We first consider the fermion-shift matrix S^{ij} to predict number of unbroken supersymmetries. It turns out that S^{ij} has non-vanishing off-diagonal components along the $\epsilon^{4,5}$ directions, i.e.

$$
\begin{aligned}
S^{ij} &= g\cosh\varphi\delta^{ij} \qquad \text{for} \qquad i,j=1,2,3 && (3.44)\\
S^{45} &= S^{54} = ig\sin(\zeta_1-\zeta_2)\sin 4\vartheta\sinh^4\frac{\varphi}{2} && (3.45)\\
S^{44} &= g\Big[\cosh\varphi - \cos^2\vartheta\left(\cosh\varphi-1\right)^2\left(\cos^2\vartheta + e^{2i(\zeta_1-\zeta_2)}\sin^2\vartheta\right)\\
&\quad +4\cos^2\vartheta\sinh^4\frac{\varphi}{2}\Big] && (3.46)\\
S^{55} &= g\Big[\cosh\varphi - \sin^2\vartheta\left(\cosh\varphi-1\right)^2\left(e^{-2i(\zeta_1-\zeta_2)}\cos^2\vartheta + \sin^2\vartheta\right)\\
&\quad +4\sin^2\vartheta\sinh^4\frac{\varphi}{2}\Big] && (3.47)
\end{aligned}
$$

It can be observed that there is no term involving ζ_1 or ζ_2 individually. This means that superpotential inherited from either one of these components would depends only on difference of pseudoscalars, $\zeta_1 - \zeta_2$, so we expect that consistent solutions must relate only difference $\zeta_1 - \zeta_2$. Consequently, we redefine our scalars as

$$\phi^4 = \tanh\varphi\cos\vartheta e^{i\zeta} \quad \text{and} \quad \phi^5 = \tanh\varphi\sin\vartheta e^{i(\zeta-\eta)} \tag{3.48}$$

in which $\zeta_1 = \zeta$ and $\zeta_2 = \zeta - \eta$.

Accordingly, the eigenvalues of S^{ij} lead to three different values of superpotential

$$\mathcal{W}_D = \sqrt{2}g\cosh\varphi \tag{3.49}$$

$$\mathcal{W}_\pm = \frac{g}{4\sqrt{2}}\Big[2\left(3+\cos 2\eta\right)\cosh\varphi + \left(3+\cosh 2\varphi\right)\sin^2\eta - 8\sinh^4\frac{\varphi}{2}\left(\cos 4\vartheta\sin^2\eta \pm i\Gamma\right)\Big] \tag{3.50}$$

with

$$\Gamma = \sin\eta\sin 2\vartheta\sqrt{3+\cos 2\eta + 2\cos 4\vartheta\sin^2\eta}\,. \tag{3.51}$$

Note here that the Killing spinors corresponding to $\mathcal{W}_\pm$ are not ϵ^4 and ϵ^5, but a linear combination between them.

Nevertheless, only one of these superpotential will play a role of true superpotential along the solutions. we then expect to find three inequivalent solutions with $SO(3)$ residual symmetry preserving $N = 3$, $N = 1$, and $N = 1$ supsersymmetries (if exist) for choosing $\mathcal{W}_D$, $\mathcal{W}_+$, and $\mathcal{W}_-$, respectively. Before that, we note here the scalar Lagrangian invariant under $SO(3)$ residual symmetry,

$$e^{-1}\mathcal{L}_{scal} = -\varphi'^2 - \sinh^2\varphi\Big[\vartheta'^2 + \cosh^2\varphi\left(\zeta'^2 - 2\sin^2\vartheta\eta'\zeta'\right) + \frac{1}{4}\sin^2\vartheta\left(3+\cosh 2\varphi - 2\cos 2\vartheta\sinh^2\varphi\right)\eta'^2\Big] - V \tag{3.52}$$

where scalar potential V depends on the choice of unbroken supersymmetry. The inverse of scalar metric reads

$$g^{IJ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sinh^2\varphi} & 0 & 0 \\ 0 & 0 & \frac{4}{\sin^2 2\vartheta\sinh^2\varphi} & \frac{1}{\sinh^2\varphi\cos^2\vartheta} \\ 0 & 0 & \frac{1}{\sinh^2\varphi\cos^2\vartheta} & \frac{1}{\sinh^2\varphi\cos^2\vartheta} - \frac{1}{\cosh^2\varphi} \end{pmatrix} \tag{3.53}$$

in which we have labelled $\phi^I \in (\varphi, \vartheta, \eta, \zeta)$

We now consider the first possibility where we take $\mathcal{W}_D$ to be the true superpotential. This amounts to set $\epsilon^{4,5} = 0$ implying the solutions to preserve $N = 3$ supersymmetry. There is an interesting feature coming from the Killing spinor equations from $\delta\chi_{ijk}$ for $i, j, k = 1, 2, 3$ resulting in algebraic constraints of the form

$$\sin\eta = 0 \quad \rightarrow \quad \eta = n\pi \tag{3.54}$$

for an integer n. With this, the rest of the Killing spinor equations are given by

$$\begin{aligned}
A' &= W_D = \sqrt{2}g\cosh\varphi, \\
\varphi' &= g^{\varphi\varphi}\frac{\partial W_D}{\partial\varphi} = -\sqrt{2}g\sinh\varphi, \\
\vartheta' &= g^{\vartheta\vartheta}\frac{\partial W_D}{\partial\vartheta} = 0, \\
\eta' &= g^{\eta\eta}\frac{\partial W_D}{\partial\eta} + g^{\eta\zeta}\frac{\partial W_D}{\partial\zeta} = 0, \qquad (3.55)\\
\zeta' &= g^{\zeta\eta}\frac{\partial W_D}{\partial\eta} + g^{\zeta\zeta}\frac{\partial W_D}{\partial\zeta} = 0,
\end{aligned} \tag{3.56}$$

in which they are consistent with the field equations as shown. We obtain the same solution as in the $SO(4)$ case up to some field redefinition. Therefore, keeping $\epsilon^{1,2,3}$ to be unbroken supersymmetry necessarily leads to a truncation of $N = 5$ supersymmetric solutions solved in the previous section.

Next, we will consider another possibility which is setting $\epsilon^{1,2,3} = 0$. Consequently, the constraint (3.54) is not needed, η is not fixed by any multiple of constant. We first consider the real superpotential $W_\pm = |\mathcal{W}_\pm|$ obtained from turning on $\epsilon^\pm$. Since equation (3.51) shows that Γ must be real due to the positive definite value in the square root, $\mathcal{W}_\pm$ thus yield the same modulus resulting in the same real superpotential $W_\pm = W$. Consistency with the field equation then requires that both $\epsilon_\pm$ should yield the same solutions. The scalar potential can then be written as

$$\begin{aligned}
V &= -3W^2 + \left(\frac{\partial W}{\partial\varphi}\right)^2 + \frac{1}{\sinh^2\varphi}\left(\frac{\partial W}{\partial\vartheta}\right)^2 + \frac{4}{\sin^2 2\vartheta\sinh^2\varphi}\left(\frac{\partial W}{\partial\eta}\right)^2 \\
&= \frac{1}{2}g^2\left(-8 - 4\cosh 2\varphi + \sin^2\eta\sin^2 2\vartheta\sinh^4\varphi\right)
\end{aligned} \tag{3.57}$$

where we have omitted the term depending of ζ because it does not appear in (3.50). Graphically, an example of scalar potential is shown in fig 3.3. We thus expect that we

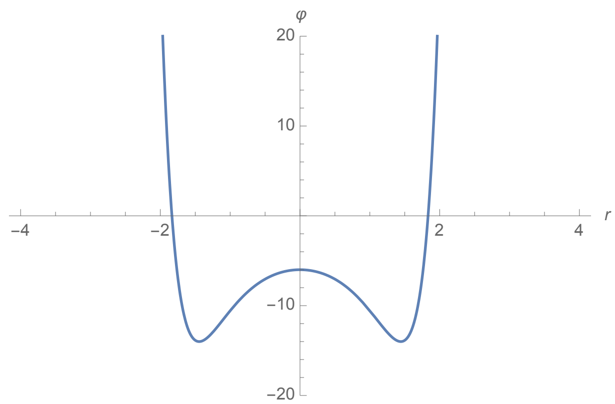


Figure 3.3: The scalar potential with $SO(3)$ symmetry with $\vartheta = \pi/4, \eta\pi/2$. the supersymmetric critical point is located at trivial point $\varphi = 0$ while another two critical points shown above are the non-supersymmetric critical points mentioned in the begining of this chapter.

will obtain two equivalent $N = 1$ solutions depending on either one of $\epsilon_\pm$ be an unbroken supersymmetry, well this is not the case.

We will pause here for a moment and discuss an important consequence of $\epsilon_\pm$. We first show the explicit form of $\epsilon_\pm$

$$\epsilon_\pm = \epsilon_5 \pm \left(\frac{\sin 2\eta \sin^2 2\vartheta - \Gamma}{\sin \eta \sin 4\vartheta} \right) \epsilon_4 \tag{3.58}$$

with Γ given by (3.51). Recall that the Killing spinor equation coming from $\delta\psi^\pm_\mu = 0$ for $\mu = 0, 1, 2$ takes the form

$$0 \;=\; A' \gamma_{\hat{r}} \epsilon_\pm + \mathcal{W}_\pm \epsilon^\pm. \tag{3.59}$$

Imposing projection condition $\gamma_{\hat{r}}\epsilon_\pm = e^{i\Lambda}\epsilon^\pm$ would simply demand that $\mathcal{W}_+ = \mathcal{W}_-$ which obviously contradicts to our direct calculation before. To fix this problem, we can require one of $\epsilon_\pm$ to vanish, or take some tea and look for another projector; we choose the later one. We observe that two superpotentials are complex conjugate to each other as to give the same modulus, i.e. $\mathcal{W}_\pm = (\mathcal{W}_\mp)^*$. We then impose the following projectors

$$\gamma_{\hat{r}}\epsilon_\pm = e^{\pm i\Lambda} e^\pm \tag{3.60}$$

which give us that $\epsilon_\pm$ must live in different representations of $SO(4)$ residual symmetry. Therefore, for general non-vanishing $\epsilon_\pm$, the solutions will preserve $N = 2$ supersymmetry.

The BPS equations are computed and checked to be consistent with the field equations; they are given as follows

$$\begin{aligned} A' &= W = \frac{g}{8}\Big[67+\cosh 4\varphi - 16\cos 2\eta\,(3+4\cosh\varphi)\sinh^4\frac{\varphi}{2} \\ &\quad +\cosh 2\varphi\left(60-16\cos 2\eta\sinh^4\frac{\varphi}{2}\right)-16\cos 4\vartheta\sin^2\eta\sinh^4\varphi\Big]^{1/2}, \qquad (3.61)\\ \varphi' &= -\frac{\partial W}{\partial\varphi}=\frac{g^2}{32W}\left[8\cosh\varphi\left(\cos 2\eta+2\cos 4\vartheta\sin^2\eta\right)\sinh^3\varphi\right.\\ &\quad \left.-30\sinh 2\varphi-\sinh 4\varphi\right], \qquad (3.62)\\ \vartheta' &= -\frac{1}{\sinh^2\varphi}\frac{\partial W}{\partial\vartheta}=-\frac{g^2}{2W}\sin^2\eta\sin 4\vartheta\sinh^2\varphi, \qquad (3.63)\\ \eta' &= -\frac{4}{\sin^2 2\vartheta\sinh^2\varphi}\frac{\partial W}{\partial\eta}=-\frac{g^2}{W}\sin 2\eta\sinh^2\varphi, \qquad (3.64)\\ \zeta' &= -\frac{1}{\sinh^2\varphi\cos^2\vartheta}\frac{\partial W}{\partial\eta}=-\frac{g^2}{W}\sin 2\eta\sin^2 2\vartheta\sinh^2\varphi. \qquad (3.65) \end{aligned}$$

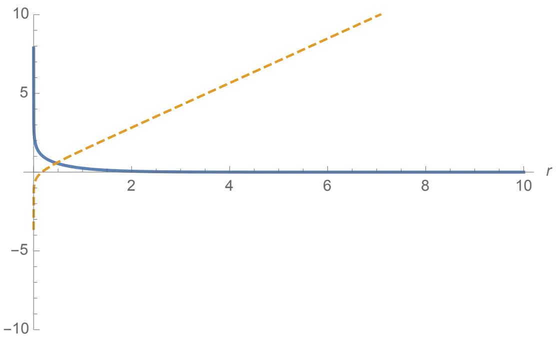
Accordingly, we will study each limiting case of this solution to classify whether which case yields a physical solution. First, It is straightforward to see that, as $\eta\to n\pi$ or $\zeta\to n\pi/2$, we recover the $N=5$ solutions as it should be. For the non-trivial cases, we will have a separate discussion for 2 different flows in which one can be solved analytically while the other cannot.

For flow I, we have $\vartheta=\pi/4$ and $\eta=\pi/2$, as to cancel the BPS equations for ϑ' and η'. The complete set of BPS equations become

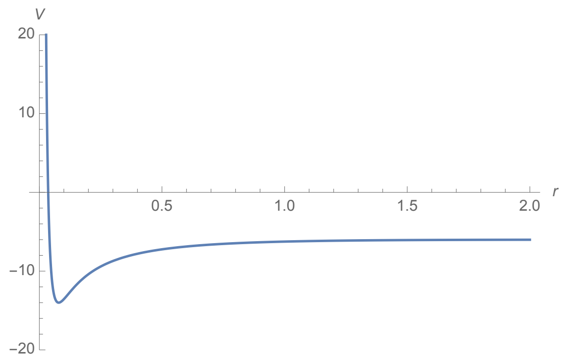
$$\begin{aligned} A' &= W=\frac{g}{2\sqrt{2}}\,(3+\cosh 2\varphi) \qquad (3.66)\\ \varphi' &= -\frac{\partial W}{\partial\varphi}=-\sqrt{2}g\cosh\varphi\sinh\varphi \qquad (3.67)\\ \vartheta' &= -\frac{1}{\sinh^2\varphi}\frac{\partial W}{\partial\vartheta}=0 \qquad (3.68)\\ \eta' &= -\frac{4}{\sinh^2\varphi}\frac{\partial W}{\partial\eta}=0 \qquad (3.69)\\ \zeta' &= 0 \qquad (3.70) \end{aligned}$$

which are consistent with the field equations. Noted that, along this flow, scalar potential takes the form

$$V=\frac{1}{2}g^2\left(-8-4\cosh 2\varphi+\sinh^4\varphi\right). \qquad (3.71)$$



(a) A solution of A (dashed) and φ (thick) along the radial coordinate.



(b) The behavior of the scalar potential along flow I.

Figure 3.4: Flow I: A Domain wall solution from the $N = 5$ AdS_4 critical point as $r \to \infty$ to a singularity geometry in $r_0 = 0$ with $\vartheta = \pi/4$ and $\eta = \pi/2$.

It can be seen that this solution can be obtained analytically; the solution are of the following form

$$\tanh\varphi = e^{-\sqrt{2}g(r-r_0)} \tag{3.72}$$

$$A = \frac{1}{2}\ln\cosh\varphi - \ln\sinh\varphi \tag{3.73}$$

where they are presented graphically in figure 3.4a. Although this solution is very similar to the $SO(4)$ case, it should be emphasized that this solution only preserves $N = 2$ supersymmetry and breaks $SO(5)$ to $SO(3)$ symmetry. As $r \to \infty$, we find the solution at asymptotic behavior,

$$\varphi \sim e^{-\sqrt{2}gr} \sim e^{-\frac{r}{L}} \quad \text{and} \quad A \sim \sqrt{2}gr \sim \frac{r}{L} \tag{3.74}$$

which is the $N = 5$ supersymmetric AdS_4 configuration as desired. Furthermore, this solution also exhibit a singular geometry. To see this, we consider the region as $r \to r_0$ at which the solution becomes

$$\varphi \sim -\frac{1}{2}\ln\left(1 - e^{-\sqrt{2}g(r-r_0)}\right) \quad \text{and} \quad A \sim \frac{1}{4}\ln\left(1 - e^{-\sqrt{2}g(r-r_0)}\right). \tag{3.75}$$

Near the singularity, scalar potential becomes

$$V \sim \frac{1}{32}\left(\frac{g}{e^{-\sqrt{2}g(r-r_0)} - 1}\right)^2 \sim \infty \tag{3.76}$$

which, according to the critetion given in [64], is an unphysical singularity.

For flow II, we consider the case of $\vartheta \neq \pi/4$ and $\eta \neq \pi/2$. Due to complicated forms of the BPS equations, we are not able to solve these equations in an analytic form; however, we can find some relations among them. By combining equations (3.63) and (3.64), we obtain

$$\cot 2\vartheta = C_1 \cos\eta \tag{3.77}$$

with an integration constant C_1. Similarly, combining equations (3.62) and (3.63) gives

$$\begin{aligned} 2\sqrt{2}\text{sech}^2\varphi &= 32(1+C_1^2)C_2\sqrt{(1+\cos 2\eta)(2+C_1^2+C_1^2\cos 2\eta)} \\ &\quad -3-4C_1^2\cos\eta-\cos 2\eta \end{aligned} \tag{3.78}$$

with another integration constant C_2. In order to interpret the physical meaning of C_1 and C_2, we impose the boundary condition at the AdS_4 configuration as

$$\varphi_0 \sim 0, \quad \vartheta \sim \vartheta_0, \quad \eta \sim \eta_0. \tag{3.79}$$

Combined equations (3.77) and (3.78) and inserted the boundary condition at the AdS_4 configuration, we find expressions of integration constants C_1 and C_2 in terms of initial value ϑ_0 and η_0,

$$C_1 = \frac{1}{\cos\eta_0 \tan 2\vartheta_0}, \tag{3.80}$$

$$C_2 = \frac{3+2\sqrt{2}+4C_1^2\cos\eta_0+\cos 2\eta_0}{32(1+C_1^2)\sqrt{(1+\cos 2\eta_0)(2+C_1^2+C_1^2\cos 2\eta_0)}}. \tag{3.81}$$

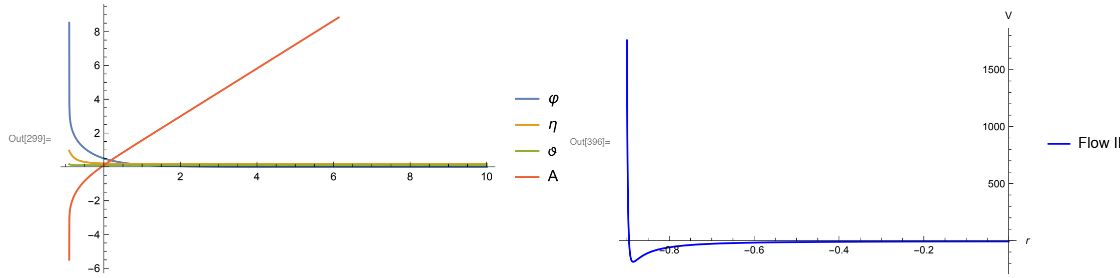
It should be emphasized that these expressions are not valid in the case of $\vartheta_0 = \pi/4$ or $\eta_0 = \pi/2$ due to the vanishing of BPS equations for η and ϑ.

The complete solution of flow II can be obtained only numerically. Example of these flows are given in figure 3.5a in which we can see that scalar φ goes to 0 as $r \to \infty$ which exhibit AdS_4 configuration as desired. On the other hand, we find a singular geometry as $\varphi \to \infty$. To see that whether this singularity is physical, we have to consider the behavior of scalar potential, given in (3.71), near this singularity. For $\varphi \to \infty$, we find that

$$V \sim \sin^2\eta \sin^2\vartheta e^{4\varphi} \to \infty \tag{3.82}$$

unless $\sin\vartheta = 0$ or $\sin\eta = 0$.

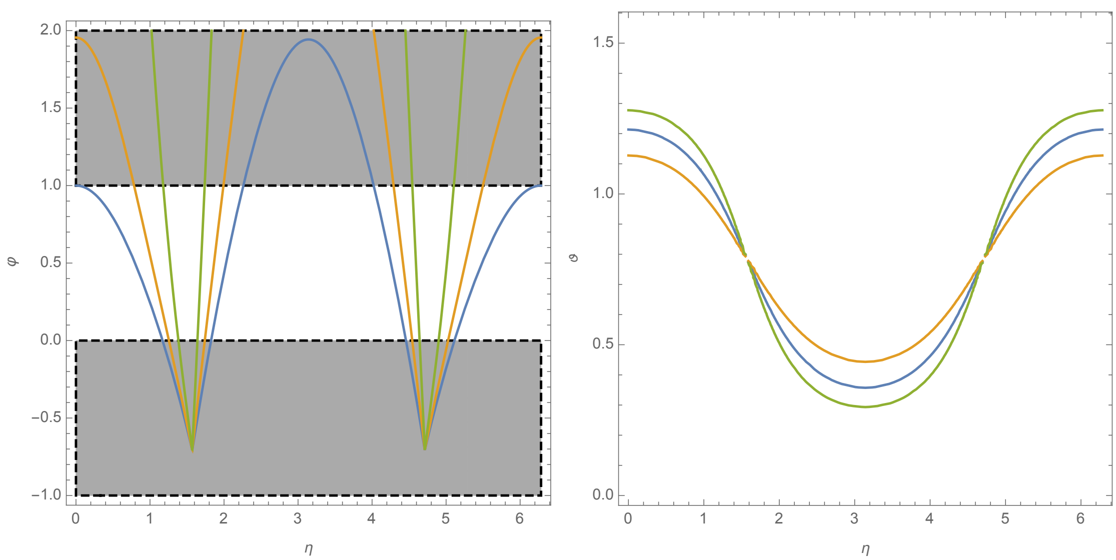
Therefore, we have to look for the value of pseudoscalars ϑ and η near the singularity. This can be done by using equations (3.77) and (3.78) together with the boundary condition



(a) Flow II: An RG flow solution with $\vartheta \neq 0$ and $\varphi \to \infty$ in the IR.

(b) The behavior of the scalar potential along flow II.

Figure 3.5: An RG flow from the $N = 5$ AdS_4 critical point as $r \to \infty$ to a non-conformal field theory in the IR with $\vartheta \neq 0$ and $\varphi \to \infty$ in the IR.



(a) A relation of $\text{sech}^2\varphi$ (vertical) and η (horizontal) along flow II.

(b) A relation of ϑ (vertical) and η (horizontal) along flow II.

Figure 3.6: Several examples of the relations between scalars φ, ϑ, and η along flow II. The blue, orange, and green curves represent the flows with $(\eta, \vartheta) = (0, \pi/3)$, $(\pi/4, \pi/3)$, and $(\pi/4 + \pi/8, \pi/3)$, respectively.

sech$\varphi = 0$. Numerically, the relations between φ, ϑ, and η are given in figures 3.6. From which, the shaded region is a forbidden region due to the condition that scalar φ must be real.

It can readily be seen that flow II continuosly connects the scalar function sech$^2\varphi$ taken the value of 1 at the AdS_4 configuration to 0 at the singular geometry. With all these, near the singularity, $\varphi \sim \infty$ and sech$^2\varphi \sim 0$, we find that $\eta \neq 0$ and $\vartheta \neq 0$ and then imply $V \sim \infty$. Consequently, we can conclude that flow II gives an unphysical singularity.

Finally, it would be interesting to find an uplift (if exists) of this solution to string/M-theory and check whether the singularities are acceptable. If this is the case, identifying the analog of non-vanishing η and ϑ in the dual $N = 5$ SCFT that breaks $N = 5$ supersymmetry to $N = 2$ also deserves further study.

3.3 Comment on general supersymmetric domain wall solutions

In this section, we will provide an analysis to explain a universal behavior of the singlet scalars under residual symmetry $SO(n)$ for $1 < n < 5$. It is motivated by the algebraic relation taken schematically as

$$\sin(\zeta_i - \zeta_j) = 0 \quad \text{for} \quad i, j = 1, 2, ..., 5 - n \tag{3.83}$$

where the Killing spinors are ϵ^i projected on $SO(n)$ subspace. It occurs previously in the case of $SO(3)$ domain wall solution. This kind of constraint implies that the phase of all complex scalar fields are equal up to some multiple of π. Interestingly, the presence of pseudoscalars ζ_i obeying the equation (3.83) also leads to the BPS equations given by

$$\zeta_i' = 0 \tag{3.84}$$

which is exemplified by the case of $SO(3)$ domain wall solution with $\epsilon^{4,5} = 0$.

With these clues, we will try to describe this feature analytically from which, in the end, it will provide that every case of domain wall solutions with residual symmetry $SO(n)$ and unbroken supersymmetries living on $SO(n)$ subspace always effectively reduce to the $SO(4)$ case.

Since there are only five complex scalars in $N = 5$ gauged supergravity, we can generalize the results obtained in the previous cases to the full $SU(5,1)/U(5)$ scalar coset. We first consider solutions with a residual symmetry $SO(n)$ for $1 < n < 5$. For $n = 5$, no scalars can be turned on because there is no $SO(5)$ singlet among the five scalars.

To proceed further, we recall that the conditions $\delta\chi^{ijk} = 0$ for vanishing gauge fields can be written as

$$\delta\chi^{ijk} = -\epsilon^{ijklm}\gamma^\mu P_{\mu m}\epsilon_l - 2gN_l{}^{ijk}\epsilon^l = 0 \tag{3.85}$$

with

$$N_l{}^{ijk} = e_1\epsilon^{ijklm}\phi_m + e_1e_2\epsilon^{ijkmn}\phi_m\phi^n\phi_l + 3e_1^2\delta^{ijk}_{lmn}\phi_m\phi^n\,. \tag{3.86}$$

It turns out that some of these conditions do not involve derivatives of scalars from P^i_μ. In particular, this can happen when indices l and i are equal among other possibilities due to the appearance of levi-civita symbol. We then write $\phi^m = \varphi_m e^{i\zeta_m}$ and consider the conditions $\delta\chi^{ijk} = 0$, for $l = i$, which reduce to

$$e_1e_2\epsilon^{ljkmn}\varphi_m\varphi_n\varphi_l e^{i(-\zeta_m+\zeta_n-\zeta_l)} + 3e_1^2\delta^{ljk}_{lmn}\varphi_m\varphi_n e^{i(-\zeta_m+\zeta_n)} = 0 \tag{3.87}$$

without summing over l. By antisymmetrizing the products of φ_m's, we arrive at the result

$$e_1e_2\epsilon^{ljkmn}\varphi_l\varphi_n\varphi_m e^{-i\zeta_l}\sin(\zeta_n - \zeta_m) + 6e_1^2\varphi_j\varphi_k\sin(\zeta_j - \zeta_k) = 0\,. \tag{3.88}$$

Since the two terms on the left hand side are independent of each other, this condition implies

$$\sin(\zeta_i - \zeta_j) = 0 \tag{3.89}$$

which gives the previously obtained result $\zeta_i = \zeta_j + n\pi$.

By splitting indices $i, j, \ldots = 1, 2, \ldots, 5$ into $\hat{i}, \hat{j}, \ldots = 1, 2, \ldots, n$ and $\tilde{i}, \tilde{j}, \ldots = n + 1, \ldots, 5$ with scalars $\varphi^{\tilde{i}}$ and $\phi^{\hat{i}}$ being respectively singlets and non-singlets of $SO(n)$, we can summarize possible cases as follow.

- For $n = 4$, there is only one $SO(4)$ singlet scalar, and in this case $N_l{}^{ljk}$ automatically vanish.
- For $1 < n < 4$, there are $5 - n$ singlet scalars denoted by $\phi^{\tilde{i}}$. The relevant non-vanishing components of $N_l{}^{ijk}$ are $N_{\hat{l}}{}^{\tilde{l}\tilde{j}\tilde{k}}$ which lead to the conditions $\sin(\zeta_{\tilde{i}} - \zeta_{\tilde{j}}) = 0$. Accordingly, we need to set $\zeta_{\tilde{i}} = \zeta_{\tilde{j}} + m\pi$ or $\epsilon_{\hat{i}} = 0$. In the former case, all the phases are equivalent up to an additive constant $m\pi$ and lead to the tensor S^{ij} proportional to the identity matrix. The latter case gives $N = 5 - n$ supersymmetric solutions with the corresponding Killing spinors $\epsilon^{\tilde{i}}$.

In particular, this result implies that domain wall solutions with all five scalars non-vanishing are possible only when all the complex phases of the scalars are equal up to an additive constant $m\pi$. In addition, for scalar fields of the form

$$\phi^i = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)e^{i\zeta} \tag{3.90}$$

with $m = 0$ for convenience, we can verify from the definition (2.14) that the S^{ij} tensor is real and independent of ζ. This leads to the BPS equation $\zeta' = 0$ according to which ζ can be set to zero.

Furthermore, by using the parametrization of the form

$$\begin{aligned}\varphi_1 &= \tanh\varphi\cos\xi_1, \qquad \varphi_2 = \tanh\varphi\sin\xi_1\cos\xi_2, \qquad \varphi_3 = \tanh\varphi\sin\xi_1\sin\xi_2\cos\xi_3,\\ \varphi_4 &= \tanh\varphi\sin\xi_1\sin\xi_2\sin\xi_3\cos\xi_4, \qquad \varphi_5 = \tanh\varphi\sin\xi_1\sin\xi_2\sin\xi_3\sin\xi_4,\end{aligned} \tag{3.91}$$

we readily find

$$S^{ij} = g\cosh\varphi\delta^{ij} \tag{3.92}$$

with the scalar potential

$$V = -2g^2(2+\cosh 2\varphi),. \tag{3.93}$$

The one-form scalar Lagrangian can be written as

$$e^{-1}\mathcal{L}_{scal} = -d\varphi^2 - \sinh^2\varphi(\cosh^2\varphi d\zeta^2 + d\Omega^2) \tag{3.94}$$

in which $d\Omega^2$ describes a line element on S^4 parametrized by $\{\xi_i\}$.

Therefore, the resulting BPS equations will give

$$\xi_i' = -g^{\xi_i\xi_i}\frac{\partial W}{\partial \xi_i} = 0 \quad \text{for all} \quad i = 1,2,3,4, \tag{3.95}$$

$$\varphi' = -\frac{\partial W}{\partial\varphi} = -\sqrt{2}g\sinh\varphi \tag{3.96}$$

$$A' = W = \sqrt{2}g\cosh\varphi \tag{3.97}$$

$$\zeta' = 0. \tag{3.98}$$

Thus, the solution effectively reduces to that of the $SO(4)$ case. We can then conclude that the most general supersymmetric domain wall solutions of $N = 5$ gauged supergravity can only involve non-vanishing real scalars with $SO(4)$ symmetry.

We end this section by emphasizing that this discussion here only works for the solutions with unbroken supersymmetry ϵ^i living on the $SO(n)$ subspace. Oppositely, this is not the case of $N = 2$ domain wall solutions with $SO(3)$ symmetry, since, in that case, we worked with the unbroken supersymmetry living on $SO(5)/SO(3)$ subspace, specifically along $\epsilon^\pm$.

Chapter 4

Supersymmetric Janus solutions

We now move to supersymmetric Janus solutions in the form of AdS_3-sliced domain wall solution. The metric anstaz takes of the form

$$ds^2 = e^{2A(r)}\left(e^{\frac{2\xi}{l}}dx_{1,1}^2 + d\xi^2\right) + dr^2 \tag{4.1}$$

with $dx_{1,1}^2$ being the flat metric on two-dimensional Minkowski space. The two-dimensional conformal defects live on the boundary of (1+1)-Minkowski space, namely $\xi \to \infty$. As shown in [44], this metric must obey the boundary conditions, as $r \to \pm\infty$,

$$A = \ln\left(\frac{L}{l}\cosh\frac{r}{L}\right). \tag{4.2}$$

Therefore, this solution connects two similar AdS_4 configurations. As usual, we will take some time here describing some technical calculation used in the rest of this chapter.

We begin with the vielbein one-forms defined over space-time,

$$e^{\hat{\mu}} = e^A e^{\xi/l}dx^\mu, \quad e^{\hat{\xi}} = e^A d\xi, \quad \text{and} \quad e^{\hat{r}} = dr \tag{4.3}$$

for $\mu = 0, 1$. The spin connection are then of the form

$$\omega^{\hat{\mu}\hat{r}} = A'e^{\hat{\mu}}, \quad \omega^{\hat{\mu}\hat{\xi}} = \frac{e^{-A}}{l}e^{\hat{\mu}}, \quad \text{and} \quad \omega^{\hat{\xi}\hat{r}} = A'e^{\hat{\xi}}. \tag{4.4}$$

In this case, the Killing spinor equation coming from $\delta\psi^i_\mu$ for $\mu = 0, 1$-direction gives

$$\begin{aligned}
0 &= 2\mathcal{D}_\mu \epsilon_i + \sqrt{2} g \gamma_\mu S_{ij} \epsilon^j \\
&= \frac{1}{2} e^{\hat{\mu}}_\mu \omega_{\hat{\mu}}{}^{\hat{\mu}\hat{r}} \gamma_{\hat{\mu}\hat{r}} \epsilon_i + \frac{1}{2} e^{\hat{\mu}}_\mu \omega_{\hat{\mu}}{}^{\hat{\mu}\hat{\xi}} \gamma_{\hat{\mu}\hat{\xi}} + \sqrt{2} g e^{\hat{\mu}}_\mu S_{ij} \gamma_{\hat{\mu}} \epsilon^j \\
&= A' \gamma_{\hat{\mu}\hat{r}} \epsilon_i + \frac{e^{-A}}{l} \gamma_{\hat{\mu}\hat{\xi}} \epsilon_i + \sqrt{2} g S_{ij} \gamma_{\hat{\mu}} \epsilon^j \qquad (4.5)
\end{aligned}$$

We will impose the same $\gamma_{\hat{r}}$ projection and an additional $\gamma_{\hat{\xi}}$ projection, respectively,

$$\gamma_{\hat{r}} \epsilon_i = e^{i\Lambda} e^i \quad \text{and} \quad \gamma_{\hat{\xi}} \epsilon_i = i\kappa e^{i\Lambda} \epsilon^i \qquad (4.6)$$

with $\kappa^2 = 1$ due to the normalization of clifford algebra $\gamma^2_{\hat{\xi}} = 1$. With these two independent projections, we then expect that our solutions preserve $\frac{1}{4}$-supersymmetry, or particularly five supersymmetry on conformal defect.

Despite from the phase $e^{i\Lambda}$, we can understand the interpretation of projection $\gamma_{\hat{\xi}}$ as follows. Consider the highest rank gamma matrix, or the so-called gamma-5 $\gamma_5 = i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{\xi}}\gamma^{\hat{r}}$, acting on the Killing spinors

$$\gamma^5 \epsilon_i = -\epsilon_i \quad \text{and} \quad \gamma^5 \epsilon^i = \epsilon^i \qquad (4.7)$$

Substitution with the two projections, we then have

$$\gamma^{\hat{0}}\gamma^{\hat{1}} \epsilon_i = -\kappa \epsilon_i \quad \text{and} \quad \gamma^{\hat{0}}\gamma^{\hat{1}} \epsilon^i = -\kappa \epsilon^i \qquad (4.8)$$

Operationally, $\gamma_{\hat{0}}\gamma_{\hat{1}}$ is a highest-rank gamma matrix in the $(1+1)-$dimensional space which can be used to determine the chirality of spinors. Thus, it clearly be seen that we have the chiral supersymmetric theory in which κ determines the chirality of Killing spinors on conformal defects.

We continue our calculation (4.5) by using the projectors (4.6) such that

$$0 = \left(A' e^{i\Lambda} + \frac{i\kappa}{l} e^{-A} e^{i\Lambda} + \mathcal{W} \right) \epsilon^i$$

where, as in the previous chapter, we have defined superpotential via the eigenvalues of $\sqrt{2} S^{ij}$. For a nontrivial solution of ϵ^i, we therefore obtain

$$A' + \frac{i\kappa}{l} e^{-A} = -e^{-i\Lambda} \mathcal{W}. \qquad (4.9)$$

To find a general solution of ϵ_i, we first consider $\delta\psi^i_\xi = 0$ such that

$$\begin{aligned} 0 &= 2\partial_\xi \epsilon_i + \frac{1}{2} e^{\hat{\xi}}_\xi \omega_{\hat{\xi}}{}^{\hat{\xi}\hat{r}} \gamma_{\hat{\xi}\hat{r}} \epsilon_i + \mathcal{W} e^{\hat{\xi}}_\xi \gamma_{\hat{\xi}} \epsilon^i \\ &= 2\partial_\xi \epsilon_i - i\kappa e^A A' \epsilon_i - i\kappa e^A e^{-i\Lambda} \mathcal{W} \epsilon_i \\ &= 2\partial_\xi \epsilon_i - i\kappa e^A A' \epsilon_i + i\kappa e^A \left(A' + \frac{i\kappa}{l} e^{-A} \right) \epsilon_i \\ &= 2\partial_\xi \epsilon_i - \frac{\epsilon_i}{l} \end{aligned}$$

which finally gives

$$\epsilon_i = e^{\frac{\xi}{2l}} \varepsilon_i \tag{4.10}$$

where ε_i does depends solely on r. By repeating the same calculation as in the case of domain wall solution, we finally obtain

$$\epsilon_i = e^{\frac{A}{2} + \frac{\xi}{2l} + \frac{i\Lambda}{2}} \varepsilon_i^{(0)} \tag{4.11}$$

in which the constant spinor $\varepsilon_i^{(0)}$ satisfy

$$\gamma_{\hat{r}} \varepsilon_i^{(0)} = \varepsilon^{(0)i} \quad \text{and} \quad \gamma_{\hat{\xi}} \varepsilon_i^{(0)} = i\kappa \varepsilon^{(0)i}. \tag{4.12}$$

For $\delta\chi_{ijk} = 0$ and $\delta\chi = 0$, these two equation involve only $\gamma_{\hat{r}}$ as our scalars still depend only on radial coordinate r, so we obtain the same Killing spinor equations for scalar fields as in the case of domain wall solutions; so we can use Killing spinor equations obtained from the domain wall's case.

We again emphasize that ϵ^i in difference representation can yield different phase $e^{i\Lambda}$. We finally note that, as $l \to \infty$, the metric ansatz reduces to the form of domain wall solutions, so it can be used to check the consistency.

4.1 Janus solutions with $SO(4)$ symmetry

We now begin with supersymmetric Janus solutions with $SO(4)$ symmetry under which ϵ_i transform as $\mathbf{4} + \mathbf{1}$. We choose the parametrization of scalars as in the case of $SO(4)$ domain wall solution, i.e.

$$\phi^5 = \tanh \varphi e^{i\zeta}. \tag{4.13}$$

The superpotential takes the form

$$\mathcal{W} = \sqrt{2}g \cosh\varphi, \tag{4.14}$$

and the phase are then of the form

$$\begin{aligned} e^{i\Lambda} &= -\frac{\sqrt{2}g\cosh\varphi}{A' + \frac{i\kappa}{l}e^{-A}} \\ &= -\frac{A'}{\sqrt{2}g\cosh\varphi} - \frac{i\kappa}{l}\frac{e^{-A}}{\sqrt{2}g\cosh\varphi} \end{aligned} \tag{4.15}$$

According to the case of $SO(4)$ domain wall, the Killing spinor equations coming from $\delta\chi_{ijk}$ and $\delta\chi$, (3.29) and (3.30) respectively, now yield the different equations for this case. In other words, we obtain a requirement that

$$\mathrm{Im}(e^{i\Lambda}) = -\frac{\kappa}{l}\frac{e^{-A}}{\sqrt{2}g\cosh\varphi} = 0 \tag{4.16}$$

which contradicts to our assumption for Janus solutions, but it is automatically satisfied in the domain wall limit. To solve this problem, we take a step back and look at the origin of equations (3.29) and (3.30). It shows up that these equations are actually given by

$$(\varphi' + i\cosh\varphi\sinh\varphi\zeta')\,\gamma_{\hat{r}}\epsilon^{\hat{i}} = \left(-\sqrt{2}e^{i\Lambda}g\sinh\varphi\right)\epsilon_{\hat{i}} \quad \text{for} \quad \hat{i} = 1,2,3,4\text{and} \tag{4.17}$$

$$(\varphi' + i\cosh\varphi\sinh\varphi\zeta')\,\gamma_{\hat{r}}\epsilon_5 = \left(-\sqrt{2}e^{-i\Lambda}g\sinh\varphi\right)\epsilon^5. \tag{4.18}$$

Hence, we must require that the projectors are of the form

$$\begin{aligned} \gamma_{\hat{r}}\epsilon_{\hat{i}} &= e^{i\Lambda}\epsilon^{\hat{i}}, \quad \gamma_{\hat{\xi}}\epsilon_{\hat{i}} = i\kappa e^{i\Lambda}\epsilon^{\hat{i}} \quad \text{for} \quad \hat{i} = 1,2,3,4 \quad \text{and} \\ \gamma_{\hat{r}}\epsilon_5 &= e^{-i\Lambda}\epsilon^5, \quad \gamma_{\hat{\xi}}\epsilon_5 = -i\kappa e^{-i\Lambda}\epsilon^5 \end{aligned} \tag{4.19}$$

In particular, ϵ^i transform under $SO(4)$ as $\mathbf{4}+\mathbf{1}$ as claimed in the beginning. By inserting the phase (4.15) into equations (4.17) and (4.18) with projectors (4.19), the complete set of BPS equations are therefore given by

$$\varphi' = -\left(\frac{A'}{W}\right)\frac{\partial W}{\partial\varphi} + \frac{2}{\sinh 2\varphi}\left(\frac{\kappa e^{-A}}{Wl}\right)\frac{\partial W}{\partial\zeta} = -A'\tanh\varphi \tag{4.20}$$

$$\zeta' = -\frac{4}{\sinh^2 2\varphi}\left(\frac{A'}{W}\right)\frac{\partial W}{\partial\zeta} - \frac{2}{\sinh 2\varphi}\left(\frac{\kappa e^{-A}}{Wl}\right)\frac{\partial W}{\partial\varphi} = -\frac{\kappa e^{-A}}{l\cosh^2\varphi} \tag{4.21}$$

$$A'^2 = W^2 - \frac{e^{-2A}}{l^2} = 2g^2\cosh^2\varphi - \frac{e^{-2A}}{l^2} \tag{4.22}$$

in which writing the BPS equations in the above form ensures the consistency with the field equations; however, the general prescription of doing this is still not well-defined as opposed to the case of supersymmetric domain wall solutions. So direct check with the field equations is still needed to confirm. As it is an important step, we already check its consistency.

Next, we will solve the above BPS equations. It can be seen that, as opposed to the $SO(4)$ domain wall solution, pseudoscalar ζ is not constant along with the flow. Furthermore, it can easily be seen that in the domain wall limit $l \to \infty$, the equations become our $SO(4)$ domain wall solution found previously.

To obtain solutions, we first consider the relation between warped factor A and scalar φ from equation (4.20),

$$A = -\ln \sinh \varphi \tag{4.23}$$

where we have neglected an additive constant. Inserted back into equation (4.22), we find a differential equation determining φ as

$$\varphi'^2 = 2g^2 \sinh^2 \varphi - \frac{2g^2}{a^2} \frac{\sinh^4 \varphi}{\cosh^2 \varphi} \tag{4.24}$$

where $a \equiv \sqrt{2} gl$. Solution to this equation is given by

$$\sinh \varphi = \frac{a}{\sqrt{1-a^2}} \frac{1}{\cosh\left(\sqrt{2} g(r-r_0)\right)} \quad \text{for} \quad a < 1 \quad \text{and} \tag{4.25}$$

$$\sinh \varphi = \frac{a}{\sqrt{a^2-1}} \frac{1}{\sinh\left(\sqrt{2} g(r-r_0)\right)} \quad \text{for} \quad a > 1. \tag{4.26}$$

The warped factor are then expressed in terms of radial coordinate r as

$$e^A = \frac{\sqrt{1-a^2}}{a} \cosh\left(\sqrt{2} g(r-r_0)\right) \quad \text{for} \quad a < 1 \quad \text{and} \tag{4.27}$$

$$e^A = \frac{\sqrt{a^2-1}}{a} \sinh\left(\sqrt{2} g(r-r_0)\right) \quad \text{for} \quad a > 1. \tag{4.28}$$

Consequently, pseudoscalar ζ is given by

$$\tan(\zeta - \zeta_0) = -\kappa \sqrt{1-a^2} \sinh\left(\sqrt{2} g(r-r_0)\right) \quad \text{for} \quad a < 1 \quad \text{and} \tag{4.29}$$

$$\tan(\zeta - \zeta_0) = -\kappa \sqrt{a^2-1} \cosh\left(\sqrt{2} g(r-r_0)\right) \quad \text{for} \quad a > 1. \tag{4.30}$$

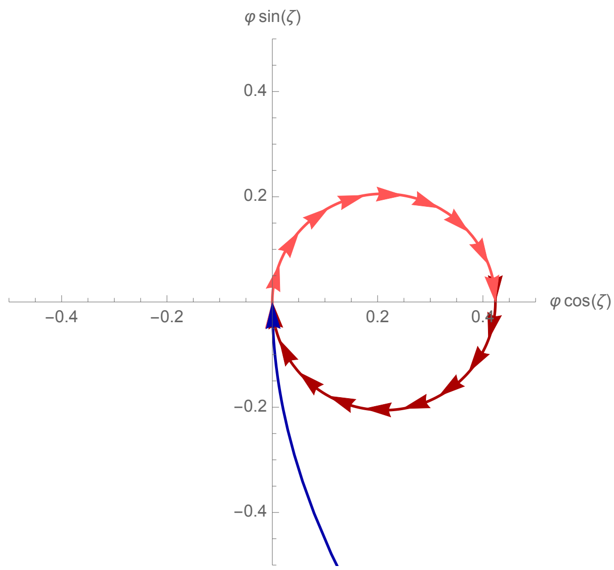
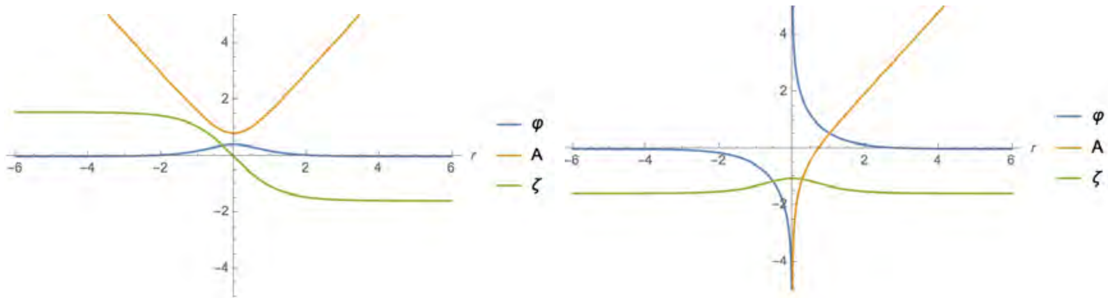


Figure 4.1: A space of Janus solution with examples of solution represented by the red curve and the blue curve. The directions along the flow determine an increasing of radial coordinate r. The supersymmetric AdS_4 critical point is located at the middle. The red curve represents a solution with $a = 0.4$ and $\zeta_0 = 0$ along which the lighter part and the darker part give the solution over regions $r < r_0$ and $r > r_0$, respectively. Furthermore, the blue curve represents a solution with $a = 2$ and $\zeta_0 = 0$.



(a) A solution along the radial coordinate with $a = 0.4$.

(b) A solution along the radial coordinate with $a = 2$.

Figure 4.2: Flow I: A Janus solution with $\kappa = 1$, $r_0 = 0$, and $\zeta_0 = 0$ from the $N = 5$ AdS_4 critical point as $r \to \infty$ to (a) the $N = 5$ AdS_4 critical point as $r \to -\infty$ for $a < 1$, and (b) a singularity geometry in $r_0 = 0$ with $a > 1$.

What we have obtained from solving BPS equations are two possible Janus solutions, exemplified graphically by figure 4.2a and 4.2b, separated by the parameter $a = \sqrt{2}gl$. A space of solution are given in figure 4.1 in the $(\varphi\cos\zeta, \varphi\sin\zeta)$-plane. It should be noted that these solutions are similar to the solutions given in [44] and [65] in which they are interested in $N = 8$ and $N = 3$ gauged supergravity, respectively. Noted that the chirality of Killing spinors on the defect, $\kappa = \pm$, affects the direction of pseudoscalar ζ along the flow.

For the solution with $a < 1$, as $r \to \infty$, the solution becomes, recall that $L = \frac{1}{\sqrt{2}g}$,

$$\varphi \sim \frac{2a}{\sqrt{1-a^2}} e^{-\frac{r}{L}}, \quad A \sim \ln\left(\frac{\sqrt{1-a^2}}{2a} e^{\frac{r}{L}}\right), \quad \zeta \sim \zeta_0 + \kappa\frac{\pi}{2} \tag{4.31}$$

which is an AdS_4 configuration with specific value of ζ as desired. Oppositely, as $r \to -\infty$, we obtain

$$\varphi \sim \frac{2a}{\sqrt{1-a^2}} e^{\frac{r}{L}}, \quad A \sim \ln\left(\frac{\sqrt{1-a^2}}{2a} e^{-\frac{r}{L}}\right), \quad \zeta \sim \zeta_0 - \kappa\frac{\pi}{2} \tag{4.32}$$

which also be AdS_4 configuration similarly. It can readily be seen that ζ is manifestly changed between two AdS_4 configurations by value of π. Although, this is just a rotation of complex scalar in complex plain in this point of view; however, there is an interesting physical meaning in a holographic point of view, see an example in [44]. To obtain this, we need to find an uplift solution to the higher dimensional original theory.

On the other hand, for $a > 1$, as $r \to \infty$, the solution takes the form

$$\varphi \sim \frac{2a}{\sqrt{a^2-1}} e^{-\frac{r}{L}}, \quad A \sim \ln\left(\frac{\sqrt{a^2-1}}{2a} e^{\frac{r}{L}}\right), \quad \zeta \sim \zeta_0 - \kappa\frac{\pi}{2}, \tag{4.33}$$

while near singularity, $r \to r_0$, the solution becomes

$$\begin{aligned} &\varphi \sim \ln\left(\frac{a}{\sqrt{a^2-1}} \frac{1}{e^{\sqrt{2}g(r-r_0)}-1}\right), \quad A \sim \ln\left(\frac{\sqrt{a^2-1}}{a}\left(e^{\sqrt{2}g(r-r_0)}-1\right)\right) \\ &\zeta \sim \zeta_0 - \arctan\sqrt{a^2-1} \end{aligned} \tag{4.34}$$

Finally, we end this section by giving a result on the different chirality of $\epsilon^{\hat{i}}$ and ϵ^5. As projector $\gamma_{\hat{\xi}}$ is directly related to chirality of Killing spinors, we then conclude that, on the two-dimensional defect, $\epsilon^{\hat{i}}$ and ϵ^5 have opposite chirality, i.e. two-dimensional defect preserves $N = (4, 1)$ or $N = (1, 4)$ supersymmetry depending on the values of $\kappa = 1$ or $\kappa = -1$, respectively.

4.2 Janus solutions with $SO(3)$ symmetry

In this section, we now move to consider Janus solutions with $SO(3)$ symmetry. We parametrize the corresponding singlet scalars as in the domain wall case

$$\phi^4 = \tanh\varphi \cos\vartheta e^{i\zeta} \quad \text{and} \quad \phi^5 = \tanh\varphi \sin\vartheta e^{i(\zeta-\eta)}. \tag{4.35}$$

Superpotential is given by either equations (3.49) or (3.50) depending on number of unbroken supersymmetry.

We begin with the former case in which we have Killing spinors $\epsilon^{1,2,3}$ being unbroken supersymmetries living on conformal defects. With the projectors (4.6) required to solve the Killing spinor equations, we then expect that the two-dimensional defect preserves $N = (p, q)$ supersymmetry with $p + q = 3$. As the same analysis in the $SO(3)$ domain wall case, the Killing spinor equations along $\delta\chi_{ijk}$ for $i, j, k = 1, 2, 3$ give the algebraic constraint among pseudoscalar resulting in $\eta = n\pi$ with an integer n. By solving the rest Killing spinor equations, we finally obtain the same $SO(4)$ Janus solution as in the previous section. Therefore, this solution is then just a truncated solution of $SO(4)$ Janus solution with $N = (4, 1)$ or $N = (1, 4)$.

We then move to a more complicated case, solutions describing two-dimensional conformal defects preserving $N = (p, q)$ supersymmetry with $p + q = 2$. To obtain this result, we must choose $\mathcal{W}_\pm$ to be our superpotential simultaneously. In the case of domain wall solution, this amounts to impose $\gamma_{\hat{r}}$ projectors differently on two corresponding Killing spinors $\epsilon^\pm$. So, we then expect that this feature would be inherited automatically to Janus solution, and fortunately this is the case. Explicitly, we impose the projectors as follows

$$\gamma_{\hat{r}}\epsilon_\pm = e^{\pm i\Lambda}\epsilon^\pm \quad \text{and} \quad \gamma_{\hat{\xi}}\epsilon_\pm = \pm i\kappa e^{\pm i\Lambda}e^\pm \tag{4.36}$$

where we should emphasize that imposing projectors differently reflects the fact that $\epsilon^\pm$ sit in different representations under $SO(3)$ residual symmetry. As in the previous analysis, the following solutions preserve $N = (1, 1)$ supersymmetry.

Unfortunately, turning on full four real scalars together with the nontrivial phase $e^{i\Lambda}$ give rise to far complicated BPS equations of which we presently cannot find the explicit form. Therefore, we will proceed by taking $\eta = \frac{\pi}{2}$ for simplicity. In this case, superpotentials obtained from the eigenvalues of the fermion-shift matrix S^{ij} are given by

$$\mathcal{W}_\pm = \sqrt{2}g\left(\cosh^4\frac{\varphi}{2} - e^{\mp 4i\vartheta}\sinh^4\frac{\varphi}{2}\right) \tag{4.37}$$

corresponding to the simpler form of Killing spinors $\epsilon_\pm = \epsilon_4 \pm \epsilon_5$.

With all of these, $\delta\chi_{ijk} = 0$ and $\delta\chi = 0$ lead to a complete set of BPS equations as follows

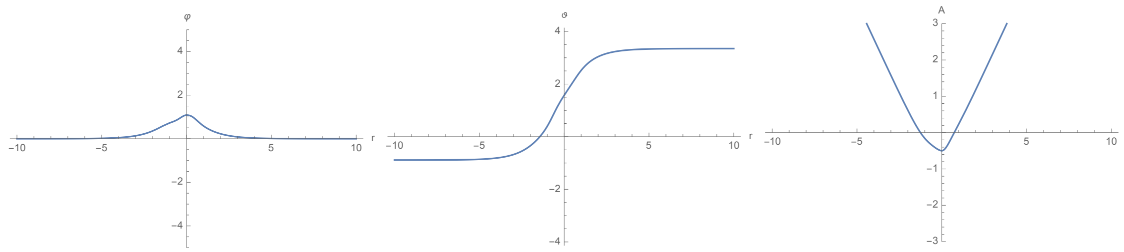
$$\begin{aligned}
\varphi' &= -\left(\frac{A'}{W}\right)\frac{\partial W}{\partial\varphi} + \left(\frac{\kappa e^{-A}}{W\ell}\right)\frac{1}{\sinh\varphi}\frac{\partial W}{\partial\vartheta} \\
&= \frac{g^2}{16W^2}\left[8\left(\frac{\kappa e^{-A}}{\ell}\right)\sin 4\vartheta\sinh^3\varphi\right. \\
&\qquad \left. + A'\left(8\cos 4\vartheta\cosh\varphi\sinh^3\varphi - 14\sinh 2\varphi - \sinh 4\varphi\right)\right],
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
\vartheta' &= -\frac{1}{\sinh^2\varphi}\left(\frac{A'}{W}\right)\frac{\partial W}{\partial\vartheta} - \left(\frac{\kappa e^{-A}}{W\ell}\right)\frac{1}{\sinh\varphi}\frac{\partial W}{\partial\varphi} \\
&= \frac{g^2}{16W^2\sinh\varphi}\left[-8A'\sin 4\vartheta\sinh^3\varphi\right. \\
&\qquad \left. + \left(\frac{\kappa e^{-A}}{\ell}\right)\left(8\cos 4\vartheta\cosh\varphi\sinh^3\varphi - 14\sinh 2\varphi - \sinh 4\varphi\right)\right],
\end{aligned} \tag{4.39}$$

$$0 = A'^2 - W^2 + \frac{e^{-2A}}{\ell^2} \tag{4.40}$$

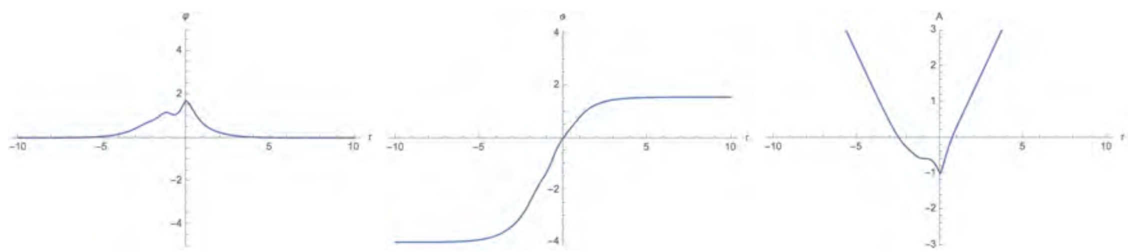
with real superpotential,

$$W \equiv |\mathcal{W}_\pm| = \frac{g}{4\sqrt{2}}\sqrt{35 + 28\cosh 2\varphi + \cosh 4\varphi - 8\cos 4\vartheta\sinh^4\varphi}\,. \tag{4.41}$$



(a) φ solution. (b) ϑ solution. (c) A solution.

Figure 4.3: A Janus solution with $SO(3)$ symmetry and $N = (1, 1)$ supersymmetry on the two-dimensional conformal defect within the $N = 5$ SCFT.



(a) φ solution. (b) ϑ solution. (c) A solution.

Figure 4.4: A Janus solution with $SO(3)$ symmetry and $N = (1, 1)$ supersymmetry on the two-dimensional conformal defect within the $N = 5$ SCFT.

Solutions to this set of BPS equations need numerically calculation. Several examples are given in figures 4.3 and 4.4.

Interestingly, it can be seen that these examples of $N = (1, 1)$ Janus solutions with $SO(3)$ symmetry take the similar profile as in the case of $N = (4, 1)$ or $N = (1, 4)$ Janus solution with $SO(4)$ symmetry. Graphically, we can immediately see that these solutions interpolate between fixed points with $\varphi \sim 0$, $A \sim r$, and $\vartheta \sim \vartheta_0$ in which pseudoscalar ϑ is changed by some definite value.

Chapter 5

Supersymmetric AdS_4 black holes

In this chapter, we consider supersymmetric AdS_4 black hole solutions by looking for solutions interpolating between AdS_4 and $AdS_2 \times \Sigma^2$ geometries. The former one is the asymptotic space-time at a large distance from the black holes while the latter one describes near horizon geometries with Σ^2 being two-dimensional Riemann surfaces.

In this work, we are only interested in the cases of Σ^2 being two-sphere (S^2) and a hyperbolic space (H^2).

We begin with the metric ansatz

$$ds^2 = -e^{2f(r)}dt^2 + dr^2 + e^{2h(r)}\left(d\theta^2 + F^2(\theta)d\phi^2\right) \tag{5.1}$$

with

$$F(\theta) \equiv \begin{cases} \sin\theta, & \Sigma_2 = S^2 \\ \sinh\theta, & \Sigma_2 = H^2 \end{cases}. \tag{5.2}$$

As mentioned above, at the large space-time distance away from a black hole, the metric should reproduce an AdS_4 configuration, as $r \to \infty$,

$$f = h = \frac{r}{L} \tag{5.3}$$

with L being AdS_4-radius. Furthermore, at the near horizon geometry, we expect to find an $AdS_2 \times \Sigma^2$ geometry depending on whether there exists $AdS_2 \times S^2$ or $AdS_2 \times H^2$ critical points. In particular, it set the boundary conditions, as $r \to -\infty$, as follows

$$\phi^{i\prime} = 0, \quad h' = 0, \quad f' = \frac{1}{L_{AdS_2}}. \tag{5.4}$$

As usual, we will collect sufficient technical calculations here. The vielbein one-forms are given by

$$\begin{aligned} e^{\hat{t}} &= e^{f}dt, \quad e^{\hat{r}} = dr \\ e^{\hat{\theta}} &= e^{h}d\theta, \quad e^{\hat{\phi}} = e^{h}Fd\phi. \end{aligned} \tag{5.5}$$

The spin connection are of the form

$$\begin{aligned} \omega^{\hat{t}\hat{r}} &= f'e^{\hat{t}}, \quad \omega^{\hat{\theta}\hat{r}} = h'e^{\hat{\theta}} \\ \omega^{\hat{\phi}\hat{r}} &= h'e^{\hat{\phi}}, \quad \omega^{\hat{\theta}\hat{\phi}} = \frac{F'}{F}e^{-h}e^{\hat{\phi}} \end{aligned} \tag{5.6}$$

in which $F'(\theta) \equiv \frac{dF}{d\theta}$.

The Killing spinor equation comming from $\delta\psi^i_{\hat{\phi}}$ is given by

$$\begin{aligned} 0 &= \frac{1}{2}\omega_{\hat{\phi}}{}^{ab}\gamma_{ab}\epsilon_i - 2gA_{\hat{\phi}i}{}^{j}\epsilon_j - \frac{1}{2\sqrt{2}}\gamma^{\nu\rho}\gamma_{\hat{\phi}}G^{+}_{\nu\rho kl}C_{ij}{}^{kl}\epsilon^j + \sqrt{2}g\gamma_{\hat{\phi}}S_{ij}\epsilon^j \\ &= \frac{F'}{F}e^{-h}\gamma_{\hat{\theta}\hat{\phi}}\epsilon_i - 2gA_{\hat{\phi}i}{}^{j}\epsilon_j + h'\gamma_{\hat{\phi}\hat{r}}\epsilon_i - \frac{1}{2\sqrt{2}}\gamma^{\nu\rho}\gamma_{\hat{\phi}}G^{+}_{\nu\rho kl}C_{ij}{}^{kl}\epsilon^j + \mathcal{W}\gamma_{\hat{\phi}}\epsilon^i. \end{aligned} \tag{5.7}$$

with our usual superpotential $\mathcal{W}$. Since the first term in the final line depends explicitly on θ together with the condition that ϵ_i should depend only on radial coordinate r, we then demand that this term must vanish in order to solve the equation. This can be done by requiring the form of gauge field's solution. Concretely, the one-form gauge fields ansatz take the following expression

$$A^{ij} = \left(-p^a\frac{F'}{F}e^{-h}e^{\hat{\phi}} + A^a_t e^{-f}e^{\hat{t}}\right)(T_a)^{ij} = \left(-p^aF'd\phi + A^a_t dt\right)(T_a)^{ij} \tag{5.8}$$

with A^{ij}_t, p^a, and $(T_a)^{ij}$ being arbitrary real functions depending on r, real constants, and gauge generators, respectively, and no summation on a. Consequently,

$$0 = \frac{F'}{F}e^{-h}\left(\gamma_{\hat{\theta}\hat{\phi}}\epsilon_i + 2gp^a(T_a)_i{}^j\epsilon_j\right). \tag{5.9}$$

To accomplish the above equation, we impose the simplest possible requirements which are

$$2gp^a = -1 \quad \text{for} \quad a = 1, 2, ..., \dim(G_{gauge}) \tag{5.10}$$

and the following projector on the Killing spinors

$$\gamma_{\hat{\theta}\hat{\phi}}\epsilon_i = (T_a)_i{}^j\epsilon_j \quad \text{and} \quad \gamma_{\hat{\theta}\hat{\phi}}\epsilon^i = (T_a)^i{}_j\epsilon^j. \tag{5.11}$$

This procedure, in which we cancel the spin connection on surface Σ^2, $\omega^{\hat{\theta}\hat{\phi}}$, by the gauge fields, is called a topological twist. The family of equations (5.10) are known as twist conditions.

Inserted the last result back on equation (5.7) and operated by $\gamma_{\hat{\phi}}$, we find

$$\begin{aligned} 0 &= h'\gamma_{\hat{r}}\epsilon_i - \frac{1}{\sqrt{2}}\gamma_{\hat{\phi}}\left(G^+_{\hat{\theta}\hat{\phi}kl}\gamma^{\hat{\theta}\hat{\phi}}\gamma_{\hat{\phi}} + G^+_{\hat{t}\hat{r}kl}\gamma^{\hat{t}\hat{r}}\gamma_{\hat{\phi}}\right) C_{ij}{}^{kl}\epsilon^j + \mathcal{W}\epsilon^i \\ &= h'\gamma_{\hat{r}}\epsilon_i + \frac{1}{\sqrt{2}}\left(G^+_{\hat{\theta}\hat{\phi}kl}\gamma^{\hat{\theta}\hat{\phi}} - G^+_{\hat{t}\hat{r}kl}\gamma^{\hat{t}\hat{r}}\right) C_{ij}{}^{kl}\epsilon^j + \mathcal{W}\epsilon^i. \end{aligned} \tag{5.12}$$

This equation can be simplified by noted that the chirality condition $\gamma_5\epsilon^i = i\gamma^{\hat{t}\hat{r}}\gamma^{\hat{\theta}\hat{\phi}}\epsilon^i = \epsilon^i$ implies

$$\gamma^{\hat{t}\hat{r}}\epsilon^i = i\gamma^{\hat{\theta}\hat{\phi}}\epsilon^i = i\gamma_{\hat{\theta}\hat{\phi}}\epsilon^i = i(T_a)^i{}_j\epsilon^j, \tag{5.13}$$

such that

$$\begin{aligned} 0 &= h'\gamma_{\hat{r}}\epsilon_i + \frac{1}{\sqrt{2}}\left(G^+_{\hat{\theta}\hat{\phi}kl} - iG^+_{\hat{t}\hat{r}kl}\right) C_{im}{}^{kl}(T_a)^m{}_j\epsilon^j + \mathcal{W}\epsilon^i \\ &= h'\gamma_{\hat{r}}\epsilon_i + (\mathcal{W}+\mathcal{Z})\epsilon^i. \end{aligned} \tag{5.14}$$

in which we have introduced the matrix

$$\mathcal{Z}_{ij} \equiv \frac{1}{\sqrt{2}}\left(G^+_{\hat{\theta}\hat{\phi}kl} - iG^+_{\hat{t}\hat{r}kl}\right) C_{im}{}^{kl}(T_a)^m{}_j \tag{5.15}$$

whose eigenvalues give the "central charges" $\mathcal{Z}_{ij} = \mathcal{Z}\delta_{ij}$. By imposing our usual $\gamma_{\hat{r}}$ projectors, we arrive at the first BPS equation describing black holes

$$0 = h'e^{i\Lambda} + \mathcal{W} + \mathcal{Z} \tag{5.16}$$

which gives

$$h' = \pm|\mathcal{W}+\mathcal{Z}| \quad \text{and} \quad e^{i\Lambda} = \mp\frac{\mathcal{W}+\mathcal{Z}}{|\mathcal{W}+\mathcal{Z}|}. \tag{5.17}$$

Similarly, the equation $\delta\psi^i_{\hat{\theta}} = 0$ yield the identical equation. In the case of $\delta\psi^i_{\hat{t}} = 0$, we find

$$\begin{aligned} 0 &= f'\gamma_{\hat{t}\hat{r}}\epsilon_i - 2gA^a_t e^{-f}(T_a)_i{}^j\epsilon_j - \frac{1}{\sqrt{2}}\gamma_{\hat{t}}\left(G^+_{\hat{\theta}\hat{\phi}kl} - iG^+_{\hat{t}\hat{r}kl}\right) C_{im}{}^{kl}(T_a)^m{}_j\epsilon^j + \mathcal{W}\gamma_{\hat{t}}\epsilon^i \\ &= f'\gamma_{\hat{r}}\epsilon_i - 2gA^a_t e^{-f}(T_a)_i{}^j\gamma^{\hat{t}}\epsilon_j - \mathcal{Z}\epsilon^i + \mathcal{W}\epsilon^i \\ &= f'e^{i\Lambda}\epsilon^i + 2ige^{i\Lambda}A^a_t e^{-f}\epsilon^i - \mathcal{Z}\epsilon^i + \mathcal{W}\epsilon^i \end{aligned} \tag{5.18}$$

where we have used the expression of $\gamma^{\hat{t}}$ in terms of $\gamma^{\hat{t}\hat{r}}$ and $\gamma^{\hat{r}}$ as follows

$$\begin{aligned}
\gamma^{\hat{t}\hat{r}}\epsilon^i &= -i(T_a)^i{}_j\epsilon^j \\
\gamma^{\hat{t}}e^{-i\Lambda}\epsilon_i &= -i(T_a)^i{}_j\epsilon^j \\
\gamma^{\hat{t}}\epsilon_i &= -ie^{i\Lambda}(T_a)^i{}_j\epsilon_j.
\end{aligned} \tag{5.19}$$

We emphasize here that this is not an independent projector, so the total projectors we have imposed so far are $\gamma_{\hat{\theta}\hat{\phi}}$ and $\gamma_{\hat{r}}$; we then expect the number of unbroken supercharges to be at most $20/4 = 5$ supercharges. Furthermore, equation (5.18) also yield the BPS equations as follows

$$f' = -\text{Re}\left[e^{-i\Lambda}(\mathcal{W} - \mathcal{Z})\right] \tag{5.20}$$

$$A_t^a = -\frac{1}{2g}e^f\text{Im}\left[e^{-i\Lambda}(\mathcal{W} - \mathcal{Z})\right] \tag{5.21}$$

in which the second equation can be used to determine explicit form of A_t^a in terms of $\mathcal{Z}$ which are proportional to the gauge field strengths[1]. Therefore, we have to find explicit form of the gauge field strengths via other approaches. This can be done by considering both electric and magnetic charges of black hole solutions. By definition, the electric and magnetic charges are expressed as

$$q_{ij} = \frac{1}{Vol(\Sigma^2)}\int_{\Sigma^2} H_{ij} \quad \text{and} \quad p^{ij} = \frac{1}{Vol(\Sigma^2)}\int_{\Sigma^2} F^{ij} \tag{5.22}$$

with H_{ij} defined by

$$H_{ij} = \frac{\delta S_{gauge}}{\delta F^{ij}}. \tag{5.23}$$

S_{gauge} denotes the gauge field part of the gauged supergravity action, and $Vol(\Sigma^2)$ amounts to volume of two-dimensional space Σ^2, such as $vol(S^2) = 4\pi$.

In the present case, we can rewrite the gauge field part of the Lagrangian (2.6) as follows

$$\mathcal{L}_{gauge} = -\frac{1}{4}R_{ij,kl} * F^{ij} \wedge F^{kl} + \frac{1}{4}I_{ij,kl}F^{ij} \wedge F^{kl} \tag{5.24}$$

with

$$R_{ij,kl} = \text{Re}\left(2S^{ij,kl} - \delta^{ik}\delta^{jl}\right) \quad \text{and} \quad I_{ij,kl} = \text{Im}\left(2S^{ij,kl} - \delta^{ik}\delta^{jl}\right) \tag{5.25}$$

[1] Recall that gauge field strengths are of the form $F = dA + A \wedge A$. Therefore, the equation (5.21) can be seen as the flow equation of the time-component of gauge fields.

and $(*F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ denoted Hodge star operator. The form of H_{ij} are then given by

$$H_{ij} = -\frac{1}{2}R_{ij,kl} * F^{kl} + \frac{1}{2}I_{ij,kl}F^{kl}. \tag{5.26}$$

In order to find the expression of F^{ij} in terms of magnetic charges p^{ij} and electric charges q^{ij}, we need to identify $F^{ij}_{\theta\phi}$ and $H^{ij}_{\theta\phi}$, which are

$$\begin{aligned} p^{ij} &= \frac{1}{Vol(\Sigma^2)}\int_{\Sigma^2}\kappa p^{ij}F(\theta)d\theta\wedge d\phi \\ &= \frac{1}{Vol(\Sigma^2)}\int_{\Sigma^2}\kappa p^{ij}e^{-2h}e^{\hat{\theta}}\wedge e^{\hat{\phi}} \\ F^{ij}_{\hat{\theta}\hat{\phi}} &= \kappa p^{ij}e^{-2h} \end{aligned} \tag{5.27}$$

and, similarly, $H_{ij\hat{\theta}\hat{\phi}} = -q_{ij}e^{-2h}$.

By using the gauge fields ansatz, it immediately be seen that the non-vanishing components of field stregths are only $F^{ij}_{\theta\phi}$ and F^{ij}_{tr}. Combined equations (5.27) and (5.26), we finally find the expression of gauge field strengths in terms of electric and magnetic charges as, where we are working in local space-time indices for simplicity,

$$F^{ij}_{\hat{t}\hat{r}} = -\frac{1}{2}e^{-2h}R^{ij,kl}\left(\frac{1}{2}I_{kl,mn}\kappa p^{mn} + q_{kl}\right), \tag{5.28}$$

$$F^{ij}_{\hat{\theta}\hat{\phi}} = \kappa p^{ij}e^{-2h}. \tag{5.29}$$

This complete our determination of the consistent gauge fields required for supersymmetric AdS_4 black hole solutions. Furthermore, compared with the gauge fields ansatz, we can identify that

$$p^a(T_a)^{ij} = p^{ij} \tag{5.30}$$

are precisely magnetic charges.

We note here that the self-dual field strengths can also be written as

$$\begin{aligned} F^{+ij} &= \frac{1}{2}\left(F^{ij}_{\hat{t}\hat{r}} + i(*F^{ij})_{\hat{t}\hat{r}}\right)e^{\hat{t}}\wedge e^{\hat{r}} + \frac{1}{2}\left(F^{ij}_{\hat{\theta}\hat{\phi}} + i(*F^{ij})_{\hat{\theta}\hat{\phi}}\right)e^{\hat{\theta}}\wedge e^{\hat{\phi}} \\ &= \frac{1}{2}\frac{1}{2}\left(F^{ij}_{\hat{t}\hat{r}} + iF^{ij}_{\hat{\theta}\hat{\phi}}\right)e^{\hat{t}}\wedge e^{\hat{r}} + \frac{1}{2}\left(F^{ij}_{\hat{\theta}\hat{\phi}} - iF^{ij}_{\hat{t}\hat{r}}\right)e^{\hat{\theta}}\wedge e^{\hat{\phi}} \\ &= \frac{1}{2}\mathcal{F}^{ij}\left(e^{\hat{t}}\wedge e^{\hat{r}} - ie^{\hat{\theta}}\wedge e^{\hat{\phi}}\right) \end{aligned} \tag{5.31}$$

with

$$\mathcal{F}^{ij} \equiv F^{ij}_{\hat{t}\hat{r}} + iF^{ij}_{\hat{\theta}\hat{\phi}} = -\frac{1}{2}e^{-2h}R^{ijkl}\left(\frac{1}{2}I_{klmn}p^{mn} + q_{kl}\right) + i\kappa p^{ij}e^{-2h} \tag{5.32}$$

Therefore, central charge matrix can be written generally as

$$\mathcal{Z}_{ij} = -\frac{i}{\sqrt{2}}S^{klnp}\mathcal{F}_{np}C_{im}{}^{kl}(T_a)^m{}_j. \tag{5.33}$$

We now get back to compute the general solution of Killing spinors satisfying Killing spinor equations coming from $\delta\psi^i_r$. By using our defined central charges and superpotentials and relation given in (5.18), we find

$$\begin{aligned} 0 &= 2\partial_r\epsilon_i - (\mathcal{Z} - \mathcal{W})e^{-i\Lambda}\epsilon_i \\ &= 2\partial_r\epsilon_i - \left(f' + 2igA^a_t e^{-f}\right)\epsilon_i. \end{aligned} \tag{5.34}$$

Solution to this equation is given by

$$\epsilon_i = e^{\frac{f}{2} - ig\int drA^a_t e^{-f}}\varepsilon^{(0)}_i \tag{5.35}$$

in which, as in the previous cases, the constant spinors $\varepsilon^{(0)}_i$ satisfy

$$\gamma_{\hat{r}}\varepsilon^{(0)}_i = \varepsilon^{(0)i} \quad \text{and} \quad \gamma_{\hat{\theta}\hat{\phi}}\varepsilon^{(0)}_i = (T_a)_i{}^j\varepsilon^{(0)}_j. \tag{5.36}$$

Next, we will consider the rest of Killing spinor equations for determining the scalar equations. we begin with the Killing spinors equation coming from $\delta\chi$. Since fermion-shift matrix N^i has non-vanishing components only along broken supersymmetries, as we will see, the twist conditions always require the broken supersymmetries to vanish such that $\delta\chi = 0$ is identically satisfied without any information on scalars.

The second one is the Killing spinor equation coming from $\delta\chi_{ijk}$; the gauge dependent term takes the form

$$\begin{aligned} 0 &= \frac{3}{2}G^+_{\mu\nu rs}\gamma^{\mu\nu}C_{[ij}{}^{rs}\epsilon_{k]} + ... \\ &= 3\left(G^+_{\hat{\theta}\hat{\phi}rs}\gamma^{\hat{\theta}\hat{\phi}} + G^+_{\hat{t}\hat{r}rs}\gamma^{\hat{t}\hat{r}}\right)C_{[ij}{}^{rs}\epsilon_{k]} + ... \\ &= 3\left(G^+_{\hat{\theta}\hat{\phi}rs} - iG^+_{\hat{t}\hat{r}}\right)C_{[ij}{}^{rs}\epsilon_{k]}T_k{}^l\epsilon_l + ... \\ &= -3\sqrt{2}\mathcal{Z}_{[ij}\epsilon_{k]} + ... \end{aligned} \tag{5.37}$$

where it should be noted that $\gamma^{\hat{t}\hat{r}}\epsilon_i = -i(T_a)_i{}^j\epsilon_j$ with an extra minus sign as opposed to (5.13). With the usual remaining terms, we find the BPS equations determining scalars.

Therefore, we are now in a position ready to find supersymmetric AdS_4 black hole solutions. We end this section by emphasizing that to answer whether Σ^2 be either S^2 and/or H^2, or even neither of them, we have to look at the equation $h' = 0$ at the $AdS_2 \times \Sigma^2$ configuration. Generally, a real condition of the metric function h and positive definite value of gauge coupling g together with the twist condition (5.10) is sufficient information.

5.1 Black hole solutions with $SO(2) \times SO(2)$ twist

We first consider $SO(2) \times SO(2)$ twist by turning on $SO(2) \times SO(2)$ gauge fields. We will separately consider magnetic and dyonic solutions.

5.1.1 Magnetic solutions

We begin with the magnetic case. The $SO(2) \times SO(2)$ gauge fields are of the form

$$A^{12} = -p_1 F'(\theta) d\phi + A_t dt \quad \text{and} \quad A^{34} = -p_2 F'(\theta) d\phi + A_t dt \tag{5.38}$$

with the corresponding field strength tensors

$$F^{12} = \kappa p_1 F(\theta) d\theta \wedge d\phi \quad \text{and} \quad F^{34} = \kappa p_2 F(\theta) d\theta \wedge d\phi. \tag{5.39}$$

where the absence of (t, r) components are due to $I_{ij,kl} = 0$ for the following scalars. The parameter κ takes the value ± 1 corresponding to the space $\Sigma = S^2/H^2$, see the definition of function $F(\theta)$.

Among the five complex scalars ϕ^i, the $SO(2) \times SO(2)$ singlet coincides with the $SO(4)$ singlet $\phi^5 = \varphi + i\zeta$ which which we have parametrized differently compared to the previous cases for simplicity in the following calculation. The corresponding gauge generators are expressed as Pauli matrix $(T_{1,2})_i{}^j = (i\sigma_2)_i{}^j$ for each $SO(2)$ gauge group. With this, the twist conditions are of the form

$$2gp_1 = -1 \quad \text{and} \quad 2gp_2 = -1 \tag{5.40}$$

together with the following projector

$$\gamma_{\hat{\theta}\hat{\phi}}\epsilon_i = (i\sigma_2 \otimes \mathbb{I}_2)_i{}^j\epsilon_j. \tag{5.41}$$

Remember that we impose the $\gamma_{\hat{\theta}\hat{\phi}}$ to cancel the spin connection along Σ^2, so we have $\epsilon^5 = 0$ because we cannot perform twist along ϵ^5-direction, namely the equation (5.7) cannot be satisfied in the presence of the Killing spinor ϵ^5.

The twist conditions imply that $p_2 = p_1 = p$ which means the twist is performed by the $SO(2)_{\text{diag}} \subset SO(2) \times SO(2)$ gauge field. This is very similar to the solution with a universal twist in pure $N = 4$ gauged supergravity studied in [66].

The central charge matrix $\mathcal{Z}_{ij}$ is given by

$$\mathcal{Z}_{ij} = \mathcal{Z}\delta_{ij} = \frac{e^{-2h}}{\sqrt{2}} \frac{p\kappa(-1 + \varphi + i\zeta)}{\sqrt{1 - \varphi^2 - \zeta^2}} \delta_{ij} \tag{5.42}$$

with an emphasizing that $i, j = 1, 2, 3, 4$ in this section only. Superpotential is given by

$$\mathcal{W} = \frac{\sqrt{2}g}{\sqrt{1 - \varphi^2 - \zeta^2}} \tag{5.43}$$

where, by redefining $\varphi = \tanh\tilde{\varphi}\cos\tilde{\zeta}$ and $\zeta = \tanh\tilde{\varphi}\sin\tilde{\zeta}$, it is identified with the superpotential in the case of $SO(4)$ domain wall solution.

The explicit form of the time-component of the gauge fields is given by

$$A_t = -\frac{1}{2g} e^f \text{Im}\left(e^{-i\Lambda}(\mathcal{W} - \mathcal{Z})\right) = \frac{-2e^f \kappa p\zeta}{\sqrt{1 - \varphi^2 - \zeta^2}\sqrt{p^2\zeta^2 + (2ge^{2h} - \kappa p + \kappa p\varphi)^2}} \tag{5.44}$$

where, however, this equation would not involve any further BPS equations.

For the case of a real scalar field, we have $\zeta = 0$, both central charge and superpotential are real such that A_t vanishes identically. This is the simplest case of supersymmetric AdS_4 black hole solutions with twist $SO(2) \times SO(2)$. Consequently, this leads to $e^{i\Lambda} = \mp 1$, and as usual we will choose the upper sign to bring AdS_4 fixed point living in the limit $r \to \infty$. With the explicit formula given in the previous section, we find the complete set of BPS equations in the case of real scalar as follows

$$\varphi' = -\frac{1}{\sqrt{2}}\sqrt{1 - \varphi^2} e^{-2h} \left(2g\varphi e^{2h} - \kappa p\varphi + \kappa p\right), \tag{5.45}$$

$$h' = \frac{1}{\sqrt{2}} \frac{e^{-2h}}{\sqrt{1 - \varphi^2}} \left(2ge^{2h} - \kappa p + \kappa p\varphi\right), \tag{5.46}$$

$$f' = \frac{1}{\sqrt{2}} \frac{e^{-2h}}{\sqrt{1 - \varphi^2}} \left(2ge^{2h} + \kappa p - \kappa p\varphi\right), \tag{5.47}$$

$$\zeta' = 0. \tag{5.48}$$

Next, we will investigate whether there admit any $AdS_2 \times \Sigma^2$ configuration. To do this, we first consider the limit $r \to \infty$; the boundary conditions $h' = 0$ and $\varphi' = 0$ yield the following results

$$\varphi = -1 \quad \text{and} \quad h = \frac{1}{2} \ln \left(\frac{\kappa p}{g} \right). \tag{5.49}$$

In order for $AdS_2 \times \Sigma^2$ to exist, we must identify $\kappa = 1$ corresponding to $\Sigma^2 = S^2$ as we mentioned before. However, using this result in the condition $f' = 1/L_{AdS_2}$ gives

$$f' \sim \sqrt{2} g \sqrt{\frac{1-\varphi}{1+\varphi}} \sim \frac{2g}{\sqrt{1+\varphi}} \to \infty. \tag{5.50}$$

Therefore, no $AdS_2 \times S^2$ solutions exist in this case.

Then, we consider the case of a complex scalar field, namely $\zeta \neq 0$. In this case, central charge $\mathcal{Z}$ is complex function as shown above. As the same procedure, we find the following BPS equations

$$f' = \frac{-2\sqrt{2} g \kappa p \zeta}{\sqrt{1-\varphi^2-\zeta^2}\sqrt{p^2\zeta^2 + (2ge^{2h} - \kappa p + \kappa p \varphi)^2}}, \tag{5.51}$$

$$h' = |\mathcal{W} + \mathcal{Z}| = e^{-2h} \sqrt{\frac{p^2\zeta^2 + (2ge^{2h} - \kappa p + \kappa p \varphi)^2}{2(1-\varphi^2-\zeta^2)}}, \tag{5.52}$$

$$\varphi' = (1-\varphi^2-\zeta^2)^2 \frac{\partial}{\partial \varphi} |\mathcal{W} + \mathcal{Z}|, \tag{5.53}$$

$$\zeta' = (1-\varphi^2-\zeta^2)^2 \frac{\partial}{\partial \zeta} |\mathcal{W} + \mathcal{Z}| \tag{5.54}$$

where we did not give the explicit form along the φ' and ζ' equations; additionally, writing φ' and ζ' as these form ensure the consistency with the field equations. However, by considering the boundary conditions at $AdS_2 \times \Sigma^2$ configuration, we find no solutions exist in these equations.

Finally, we note that one would hope for the pure gauge solutions, i.e. all scalars vanish, to admit $AdS_2 \times \Sigma^2$ fixed points. However setting $\varphi = \zeta = 0$ is inconsistent with the BPS equations (5.53) and (5.54). Therefore, we conclude that there are no magnetic AdS_4 black holes with $SO(2) \times SO(2)$ symmetry in $N = 5$ gauged supergravity with $SO(5)$ gauge group.

5.1.2 Dyonic solutions

In this section, we will enhance our last result to have a non-vanishing electric charges. This amounts to the dyonic AdS_4 black holes with $SO(2) \times SO(2)$ symmetry. We begin with the expression of scalar fields parametrized as in the magnetic case

$$\phi^5 = \varphi + i\zeta \quad \text{and} \quad \phi^{1,2,3,4} = 0. \tag{5.55}$$

Recall that having non-vanishing electric charges q^{ij} does not affect the twist procedure; we are still able to cancel the spin connection along Σ^2 by turning on $SO(2)_{\text{diag}} \subset SO(2) \times SO(2)$ gauge fields. The twist condition is given by

$$2gp = -1 \tag{5.56}$$

with the gauge field strength tensors given by

$$F^{12} = F^{34} = \kappa p e^{-2h} e^{\hat{\theta}} \wedge e^{\hat{\phi}} + 2qe^{-2h}\frac{1+\varphi - i\zeta}{-1+\varphi - i\zeta} e^{\hat{t}} \wedge e^{\hat{r}}. \tag{5.57}$$

The corresponding projector takes the form

$$\gamma_{\hat{\theta}\hat{\phi}}\epsilon_i = (i\sigma_2 \otimes \mathbb{I}_2)_i{}^j \epsilon_j. \tag{5.58}$$

The expression of $SO(2)_{\text{diag}}$ gauge fields are given as in the magnetic case, i.e.

$$A^{12} = A^{34} = -pF'(\theta)d\theta + A_t dt \tag{5.59}$$

with the time-component taken as a non-linear function of scalars

$$A_t = \frac{2e^f(2q + 2q\varphi - \kappa p\zeta)}{\sqrt{1-\varphi^2-\zeta^2}\sqrt{(2q+2q\varphi-\kappa p\zeta)^2 + (2ge^{2h} - \kappa p + \kappa p\varphi + 2q\zeta)^2}}. \tag{5.60}$$

With the explicit form of the superpotential and the central charge given by

$$\mathcal{W} = \frac{\sqrt{2}g}{\sqrt{1-\varphi^2-\zeta^2}}, \tag{5.61}$$

$$\text{and} \quad \mathcal{Z} = -\frac{1}{\sqrt{2}}e^{-2h}\frac{\kappa p + 2iq + (2iq - \kappa p)(\varphi + i\zeta)}{\sqrt{1-\varphi^2-\zeta^2}}, \tag{5.62}$$

we find the following BPS equations

$$f' = \frac{2\sqrt{2}g[2q(1+\varphi)-\kappa p\zeta]}{\sqrt{1-\varphi^2-\zeta^2}\sqrt{(2q+2q\varphi-\kappa p\zeta)^2+(2ge^{2h}-\kappa p+\kappa p\varphi+2q\zeta)^2}}, \quad (5.63)$$

$$h' = |\mathcal{W}+\mathcal{Z}| = e^{-2h}\sqrt{\frac{(2q+2q\varphi-\kappa p\zeta)^2+(2ge^{2h}-\kappa p+\kappa p\varphi+2q\zeta)^2}{2(1-\varphi^2-\zeta^2)}}, \quad (5.64)$$

$$\varphi' = (1-\varphi^2-\zeta^2)^2\frac{\partial}{\partial\varphi}|\mathcal{W}+\mathcal{Z}|, \quad (5.65)$$

$$\zeta' = (1-\varphi^2-\zeta^2)^2\frac{\partial}{\partial\zeta}|\mathcal{W}+\mathcal{Z}|\,. \quad (5.66)$$

Setting $q=0$, we recover the BPS equations for magnetic solutions. However, even generalizing the solution to cover both magnetic and electic charges does not lead to any consistent $AdS_2\times\Sigma^2$ configuration. Therefore, we conclude that there are no AdS_4 black holes both magnetically and electrically with $SO(2)\times SO(2)$ symmetry in $N=5$ gauged supergravity with $SO(5)$ gauge group.

5.2 Black hole solutions with $SO(2)$ twist

We now consider $AdS_2\times\Sigma^2$ solutions with $SO(2)$ twist by turning on only A^{12}_μ gauge field with both pure magnetic and dyonic charges. The same analysis as in the $SO(2)\times SO(2)$ can be repeated with $F^{34}_{\mu\nu}=0$. However, in this case, we will come back to use the parametrization of scalar fields as

$$\begin{aligned}\phi^1 &= \phi^2 = 0\\ \phi^3 &= \tanh\varphi\cos\vartheta e^{i\zeta_1}\\ \phi^4 &= \tanh\varphi\sin\vartheta\cos\xi e^{i\zeta_2}\\ \phi^5 &= \tanh\varphi\sin\vartheta\sin\xi e^{i\zeta_3}.\end{aligned} \quad (5.67)$$

In this case, the supersymmetry corresponding to $\epsilon^{3,4,5}$ is broken since it is not possible to perform the twist along these directions. We will accordingly set $\epsilon^{3,4,5}=0$ from now on. With this, $\delta\chi=0$ conditions are identically satisfied as in the $SO(2)\times SO(2)$ case.

As occur in the case of supersymmetric domain wall solutions, the Killing spinor equations coming from $\delta\chi_{ijk}$ give rise to the conditions

$$\zeta_i = \zeta_j + n\pi, \quad i\neq j, \quad (5.68)$$

for an integer n. However, we cannot use the same trick to avoid these constraints as analysed in the case of $N = 2$ domain wall solutions with $SO(3)$ symmetry because, due to the vanishing of $\epsilon^{3,4,5}$ implied by the twist condition. Therefore, in order to obtain the supersymmetric solution, we set

$$\zeta_5 = \zeta, \quad \zeta_4 = \zeta + m\pi, \quad z_3 = \zeta + n\pi \tag{5.69}$$

for integers m and n. It turns out that

$$\zeta' = 0, \tag{5.70}$$

so ζ is an arbitrary constant and will set to be zero for convenience. We finally end up with the analysis of real scalars φ, ϑ, and ξ. Fortunately, it is worth to consider the Killing spinor equations coming from $\delta\psi^i_{\hat{\theta}}$ and $\delta\psi^i_{\hat{\phi}}$ which yield

$$e^{i\Lambda}h' = \frac{1}{\sqrt{2}}e^{-2h}(2ge^{2h} - \kappa p - 2iq)\cosh\varphi. \tag{5.71}$$

It can be easily seen that, in order for $_2 \times \Sigma^2$ fixed points to exist, i.e. $h' = 0$, we need $q = 0$. Therefore, the black hole solutions (if exist) must be purely magnetic.

For $q = 0$, we have the real superpotential and central charges given by

$$\mathcal{W} = \sqrt{2}g\cosh\varphi \tag{5.72}$$

$$\mathcal{Z} = -\frac{\kappa p e^{-2h}}{\sqrt{2}}\cosh\varphi. \tag{5.73}$$

The complete set of BPS equations is of the following form

$$f' = \frac{1}{\sqrt{2}}(2g + \kappa p e^{-2h})\cosh\varphi, \tag{5.74}$$

$$h' = |\mathcal{W} + \mathcal{Z}| = \frac{1}{\sqrt{2}}(2g - \kappa p e^{-2h})\cosh\varphi, \tag{5.75}$$

$$\varphi' = -\frac{\partial|\mathcal{W} + \mathcal{Z}|}{\partial\varphi} = -\frac{1}{\sqrt{2}}(2g - \kappa p e^{-2h})\sinh\varphi, \tag{5.76}$$

$$\vartheta' = -\frac{1}{\sinh^2\varphi\sin^2\xi}\frac{\partial|\mathcal{W} + \mathcal{Z}|}{\partial\vartheta} = 0, \tag{5.77}$$

$$\xi' = -\frac{1}{\sinh^2\varphi}\frac{\partial|\mathcal{W} + \mathcal{Z}|}{\partial\xi} = 0. \tag{5.78}$$

By considering boundary conditions at $AdS_2 \times \Sigma^2$ configuration, we find

$$\begin{aligned} h &= \frac{1}{2}\ln\left(\frac{\kappa p}{2g}\right), \quad L_{AdS_2} = \frac{1}{2\sqrt{2}g\cosh\varphi_0} \\ \varphi &= \varphi_0, \quad \vartheta = \vartheta_0, \quad \xi = \xi_0 \end{aligned} \tag{5.79}$$

with φ_0, ϑ, and ξ any constant. By the twist condition, we find that the AdS_2 fixed point exists only for $\kappa = -1$ giving rise to an $AdS_2 \times H^2$ geometry.

Unlike the previous case with $SO(2) \times SO(2)$ twist, it is possible to truncate all scalars yielding the pure gauge solution. The solutions are then of the form

$$f = 2\sqrt{2}gr - \frac{1}{2}\ln\left(\frac{e^{2\sqrt{2}g(r-r_0)} - p}{2g}\right), \tag{5.80}$$

$$h = \frac{1}{2}\ln\left(\frac{e^{2\sqrt{2}g(r-r_0)} - p}{2g}\right). \tag{5.81}$$

As $r \to \infty$, we find

$$f \sim h \sim \sqrt{2}gr \tag{5.82}$$

which gives AdS_4 configuration, while for $r \to -\infty$, the solution becomes

$$h \sim \frac{1}{2}\ln\left(-\frac{p}{2g}\right) \quad \text{and} \quad f \sim 2\sqrt{2}gr \tag{5.83}$$

which is the $AdS_2 \times H^2$ configuration as desired. Accordingly, the full solution interpolates between the supersymmetric AdS_4 and $AdS_2 \times H^2$ geometries given graphically in figure 5.1. Therefore, this solution describes a black hole in asymptotically AdS_4 space with $AdS_2 \times H^2$ near horizon geometry. From the holographic point of view, the solution describes twisted compactification of $N = 5$ $SCFT$ in three dimensions to superconformal quantum mechanics.

For the case of non-vanishing scalars, we also find an analytic solution. We first combine equations, (5.75) and (5.76) yielding the relations between h and scalar φ as

$$h = -\ln\sinh\varphi \tag{5.84}$$

where we have omitted an irrelevant integration constant. With equation (5.84), we can find the explicit value of φ at the $AdS_2 \times H^2$ geometry by substitue $h = \frac{1}{2}\ln\frac{-p}{2g}$ such that

$$\sinh\varphi_0 = 2g. \tag{5.85}$$

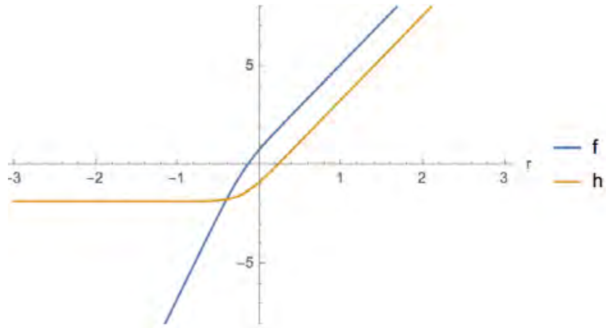


Figure 5.1: A black hole solution with $SO(2)$ twist and vanishing scalars.

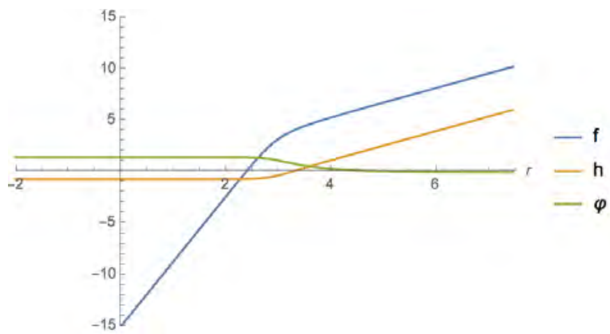


Figure 5.2: A black hole solution with $SO(2)$ twist along which scalars are non-vanishing.

Inserted (5.84) into (5.76), we obtain the differential equation for scalar φ,

$$\varphi' = -\frac{1}{\sqrt{2}}\left(2g + p\sinh^2\varphi\right)\sinh\varphi \tag{5.86}$$

which accordingly gives the rest of solution as follows

$$\begin{aligned} 2\sqrt{2}gr &= \frac{2}{\sqrt{1+4g^2}}\left(\tanh^{-1}\left(\frac{1-2g\tanh\frac{\varphi}{2}}{\sqrt{1+4g^2}}\right) + \tanh^{-1}\left(\frac{1+2g\tanh\frac{\varphi}{2}}{\sqrt{1+4g^2}}\right)\right) \\ &\quad -2\ln\tanh\frac{\varphi}{2}, \qquad (5.87) \\ f &= -\ln\sinh\varphi + \ln\left(-p + \frac{2g+p}{\cosh^2\varphi}\right). \qquad (5.88) \end{aligned}$$

An example of the solution with $g = 1$ is given graphically in figure 5.2.

Next, we consider the asymptotic behavior of this solution. From this solution, we can see that as $\varphi \to 0$,

$$\ln \varphi \sim -\sqrt{2}gr, \quad f \sim h \sim -\ln \varphi \sim \sqrt{2}gr \tag{5.89}$$

which precisely AdS_4 configuration as desired.

In order to have a flow to the $AdS_2 \times H^2$ fixed point, we consider the limit $\varphi \to \varphi_0 = \sinh^{-1} 2g$ resulting in

$$h \sim \frac{1}{2} \ln \left(-\frac{p}{2g} \right), \tag{5.90}$$

$$\varphi \sim \varphi_0 + Ce^{2\sqrt{2}gr\sqrt{\frac{p+2g}{p}}}, \tag{5.91}$$

$$f \sim 2\sqrt{2}gr\sqrt{\frac{p+2g}{p}}. \tag{5.92}$$

Therefore, the solution becomes the supersymmetric $AdS_2 \times H^2$ fixed point. This solution then describes an AdS_4 black hole with $AdS_2 \times H^2$ horizon.

Chapter 6

Conclusions and Discussions

In this senior project, we have studied supersymmetric solutions of $N = 5$ guaged supergravity in four dimensions with $SO(5)$ gauge group. For all scalars vanishing, the gauged supergravity admits an $N = 5$ supersymmetric AdS_4 vacuum dual to an $N = 5$ SCFT in the form of CSM theory in three dimensions. We have shown a required projector which turns out to be a condition reducing a number of supersymmetries and general form of BPS equations. For holographic RG flows describing mass deformations of the $N = 5$ SCFT to non-conformal field theories in the IR, we have found analytic solutions preserving $N = 5$ supersymmetry, but the $SO(5)$ R-symmetry is broken to $SO(4)$ subgroup. This is in agreement with the field theory result given in [13]. All of the IR singularities are physical by the criterion given in [64]. Accordingly, these solutions could be useful in the context of the AdS/CFT correspondence regarding the gravity dual of $N = 5$ CSM theory in three dimensions. For $SO(3)$ symmetric solutions perserving $N = 2$ supersymmetry, we have given numerical flow solutions, but the IR singularities turn out to be unphysical.

For supersymmetric Janus solutions describing two-dimensional conformal defects within the $N = 5$ SCFT, we have provided necessary projectors required for supersymmetric solutions; one of the projectors accounts for the chirality of Killing spinors on defects while the other reduces the number of supersymmetry. Similarly, BPS equations are also presented. Furthermore, we have studied solutions with $SO(4)$ and $SO(3)$ symmetries and $N = (4, 1)$ and $N = (1, 1)$ unbroken supersymmetries on the defects, respectively. The former can be found analytically and turns out to be the same as the solutions in $N = 8$ and $N = 3$ gauged supergravities given in [44] and [65]. This might suggest some universal property of the solution, and if this is indeed the case, there would be a universal surface defect in the dual three-dimensional SCFTs with $N = 3, 5, 8$ supersymmetries. Further investigation along this direction both in gauged supergravities and dual CSM theories might be worth considering. The $N = (1, 1)$ solution with $SO(3)$ symmetry appears to be new and can be only obtained numerically. Both of these solutions could be interesting in the holographic study of strongly coupled $N = 5$ SCFT in the presence of conformal defects.

We have also considered supersymmetric black holes in asymptotically AdS_4 space with $SO(2)\times SO(2)$ and $SO(2)$ twists. We have started with an analysis for twist procedure and ansatz for gauge fields. We then have considered projectors required for supersymmetric solutions and obtained corresponding BPS equations. It turns out that only the case of $SO(2)$ twist leads to a supersymmetric black hole preserving two supercharges with the horizon geometry $AdS_2 \times H^2$. In the dual $N = 5$ SCFT, the solution describes an RG flow across dimensions from three-dimensional SCFT to superconformal quantum mechanics. This could be used to compute microscopic entropy of the black hole. It is remarkable that we have found the analytic solution with a running scalar unlike most of the previous analytic solutions that only involve the metric. We accordingly hope our solution would be of particular interest in black hole physics and $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence.

It would be interesting to explicitly find an uplift of these solutions to M-theory via an orbifold compactification suggested in [13]. The uplifted solution could give rise to a complete holographic description of $N = 5$ CSM theory and possible deformations. In particular, the time component g_{00} of the resulting eleven-dimensional metric can be used to determine whether the aforementioned singular flow solutions are physically acceptable in M-theory by the criterion given in [67]. In this work, we have only considered gauged supergravity with the so-called electric $SO(5)$ gauge group. It could also be interesting to perform a similar study for other gauge groups such as non-compact and non-semisimple ones. In addition, working out the complete embedding tensor formalism of $N = 5$ gauged supergravity to incorporate magnetic and dyonic gaugings as initiated in [68] would be useful in various applications.

APPENDICES

Appendix A

Differential form and Vielbein formalism

In this appendix, we will show several ingredients used in this senior project in the context of differential form which is a manifest concept that appeared in differential geometry. This includes the definition of a differential form and their component expression, an integration over manifold, and a functional variation by a differential form. Furthermore, we will discuss the vielbein formalism (or sometimes is known as Tretrad formalism) which is used over and over in this senior project. These two topics are the vast topic containing deep concepts and details; however, we will not go into that. For readers who are interested in these topics in detail, there are several excellent General Relativity textbooks explaining these topics, for example, [69].

We begin with the definition of differential form. A differential p-form is a totally anti-symmetric tensor of type $(0, p)$, i.e. a multi-linear map of p-dual vectors, where the vector space of p-forms at a point x is denoted by Λ_x^p with $\dim\Lambda_x^p = \frac{n!}{p!(n-p)!}$.

We then define an outer product of a p-form and a q-form by the so-called wedge map $\wedge : \Lambda_x^p \times \Lambda_x^q \to \Lambda_x^{p+q}$. Therefore, we can use the wedge product to express any p-form in terms of p basis one-form dx^μ as

$$\omega = \frac{1}{p!}\omega_{\mu_1\mu_2...\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_p} \tag{A.1}$$

in which

$$\omega_{\mu_1\mu_2...\mu_p} = \omega_{[\mu_1\mu_2...\mu_p]}. \tag{A.2}$$

The prefactor $\frac{1}{p!}$ is introduced in order to set the weight of the components. This can be exemplified by the gauge field strength in the case of pure magnetically black hole with $SO(2) \times SO(2)$ twist in which, provided the non-vanishing components to be $F^{12}_{\theta\phi}$,

$$\begin{aligned} F^{12} &= \frac{1}{2} F^{12}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \\ &= \frac{1}{2}\left(F^{12}_{\theta\phi} d\theta \wedge d\phi + F^{12}_{\phi\theta} d\phi \wedge d\theta\right) \\ &= \frac{1}{2}\left(F^{12}_{\theta\phi} d\theta \wedge d\phi + F^{12}_{\theta\phi} d\theta \wedge d\phi\right) \\ &= F^{12}_{\theta\phi} d\theta \wedge d\phi \end{aligned} \tag{A.3}$$

which verify that $F^{12}_{\theta\phi} = \kappa p_1 F(\theta)$ in the equation (5.39).

Subsequently, we define an exterior derivative on a p-form to (p+1) form as $d : \Lambda^p_x \rightarrow \Lambda^{p+1}_x$, i.e.

$$\begin{aligned} d\omega &= \frac{1}{p!} \partial_{[\mu} \omega_{\mu_1 \mu_2 \ldots \mu_p]} dx^{\mu} \wedge dx^{\mu}_1 \wedge \ldots \wedge dx^{\mu}_p \\ &= \frac{(p+1)}{(p+1)!} \partial_{[\mu} \omega_{\mu_1 \mu_2 \ldots \mu_p]} dx^{\mu} \wedge dx^{\mu}_1 \wedge \ldots \wedge dx^{\mu}_p \end{aligned} \tag{A.4}$$

such that, in component form,

$$(d\omega)_{\mu\mu_1 \ldots \mu_p} = (p+1)\omega_{\mu_1 \mu_2 \ldots \mu_p}. \tag{A.5}$$

It should be noted that, up to now, there is no need of a metric structure. An example of this operator is the usual two-form field strength tensors written as a exterior derivative of a one-form gauge field, namely

$$dF = A. \tag{A.6}$$

Specifically, in the case of a pure magnetically black hole with $SO(2) \times SO(2)$ twist, given a one-form gauge field $A^{12} = -p_1 F'(\theta) d\phi$ results in

$$\begin{aligned} F^{12} &= dA^{12} \\ &= (1+1)\partial_{[\mu} A_{\nu]} dx^{\mu} \wedge dx^{\nu} \\ &= 2\frac{1}{2}\left(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\right) dx^{\mu} \wedge dx^{\nu} \\ &= \partial_{\theta}(-p_1 F'(\theta)) d\theta \wedge d\phi \\ &= -p_1 \kappa F(\theta) d\theta \wedge d\phi. \end{aligned} \tag{A.7}$$

which precisely equation (5.39).

Next, we define the Hodge-star operator or sometimes known as Hodge duality. Given M^n be an n-dimensional manifold. A Hodge-star operator is a operator taking a p-form into (n-p)-form by the use of levi-civita tensor $\epsilon_{\mu_1\mu_2...\mu_n}$, generally written as $\star : \Lambda^p_x \rightarrow \Lambda^{n-p}_x$. In particular, this operator takes of the following form

$$*\omega_p = \frac{1}{p!(n-p)!}\epsilon_{\mu_1\mu_2...\mu_{n-p}}{}^{\nu_1\nu_2...\nu_p}\omega_{\nu_1\nu_2...\nu_p}dx^{\mu_1}\wedge ... \wedge dx^{\mu_{n-p}} \tag{A.8}$$

such that, in component form

$$(*\omega_p)_{\mu_1\mu_2...\mu_{n-p}} = \frac{1}{p!}\epsilon_{\mu_1\mu_2...\mu_{n-p}}{}^{\nu_1\nu_2...\nu_p}\omega_{\nu_1\nu_2...\nu_p}. \tag{A.9}$$

It can readily be seen that the existence of Hodge-star operator requires a metric structure on a manifold in order to raising and lowering the indices of the levi-civita tensor. Furthermore, it should be noted that levi-civita tensor is not a usual levi-civita symbol denoted by $\varepsilon_{\mu_1\mu_2...\mu_n}$. The later does not transform as a tensor under global coordinate transformations while the former does. The relation between these two objects is

$$\epsilon_{\mu_1\mu_2...\mu_n} = \sqrt{-g}\varepsilon_{\mu_1\mu_2...\mu_n} \tag{A.10}$$

in which $\sqrt{-g} = \sqrt{-\det g}$.

Let $U \subset M^n$ be a subset arbitrary open region on manifold. we define the integral of a p-form ω over the region U by

$$\begin{aligned}\int_U \omega &= \int_U \omega_{\mu_1\mu_2...\mu_p}dx^{\mu_1}\wedge dx^{\mu_2}\wedge ... \wedge dx^{\mu_p} \\ &= \int_{\psi(U)} \omega_{\mu_1\mu_2...\mu_p}dx^{\mu_1}dx^{\mu_2}...dx^{\mu_p} \end{aligned}\tag{A.11}$$

where $\psi(U)$ denotes a chart over the region U. An example is given by an evaluation of electric and magnetic charges, (5.27).

It turns out that, in the case of integral over manifold, we can write a famous form of the integration by considering the integral of a Hodge-dual zero-form with an arbitrary scalar function f, i.e.

$$\begin{aligned}\int_{M^n} *\mathbb{I}f &= \int_{M^n} f\epsilon_{\mu_1\mu_2...\mu_n}dx^{\mu_1}\wedge dx^{\mu_2}\wedge ... \wedge dx^{\mu_n} \\ &= \int_{M^n} \sqrt{-g}f\varepsilon_{\mu_1\mu_2...\mu_n}dx^{\mu_1}\wedge dx^{\mu_2}\wedge ... \wedge dx^{\mu_n} \\ &= \int_{U(M^n)} \sqrt{-g}fd^nx \end{aligned}\tag{A.12}$$

in which, in four-dimension,

$$\varepsilon_{\mu_1\mu_2\mu_3\mu_4} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} \;=\; dt \wedge dx \wedge dy \wedge dz. \tag{A.13}$$

We note here some properties involving the integration and Hodge-star operator:

$$* (*\omega) = (-1)^{p(n-p)} g\omega, \tag{A.14}$$

$$\int *\omega \wedge \eta = \int *\eta \wedge \omega \quad \text{for} \quad \frac{n}{2} - \text{form} \quad \omega, \eta. \tag{A.15}$$

An example is given by the first term of gauge field Lagrangian introduced in (2.6). In particular, this term can be written as

$$\begin{aligned} -\frac{1}{4} R_{ij,kl} * F^{ij} \wedge F^{kl} &= -\frac{1}{4} R_{ij,kl} * (*(*F^{ij} \wedge F^{kl})) \\ &= -\frac{1}{4} R_{ij,kl} * (\epsilon^{\mu\nu\rho\sigma}(\frac{1}{2}\epsilon_{\mu\nu}{}^{\gamma\lambda} F^{ij}_{\gamma\lambda} F^{kl}_{\rho\sigma})\mathbb{I}) \\ &= -\frac{\sqrt{-g}}{8} R_{ij,kl} F^{\mu\nu ij} F^{kl}_{\mu\nu} d^4x. \end{aligned} \tag{A.16}$$

In the case of identity matrix $R_{ij,kl} = \delta_{ij,kl}$, we find the standard Maxwell Lagrangian.

We finally give a comment of functional variation of a differential form. This occur in the vicinity of definition of dual field strength tensor H_{ij}, (5.23). In order to find a functional derivative of the action with respect to the gauge field strength F^{ij}, we have to use the property (A.15). For instance, given a 4-form Lagrangian

$$L \;=\; -\frac{1}{4} R_{ij,kl} * F^{ij} \wedge F^{kl}, \tag{A.17}$$

we find

$$\begin{aligned} \delta S &= \int -\frac{1}{4} R_{ij,kl} \left(*\delta F^{ij} \wedge F^{kl} + *F^{ij} \wedge \delta F^{kl} \right) \\ &= \int -\frac{1}{4} R_{ij,kl} \left(*F^{kl} \wedge \delta F^{ij} + *F^{ij} \wedge \delta F^{kl} \right) \\ &= \int -\frac{1}{2} R_{ij,kl} * F^{kl} \wedge \delta F^{ij}. \end{aligned} \tag{A.18}$$

where we have used the symmetric property, $R_{ij,kl} = R_{kl,ij}$, to obtain the third line. Consequently, we can identify

$$\frac{\delta S}{\delta F^{ij}} = -\frac{1}{2} R_{ij,kl} * F^{kl}. \tag{A.19}$$

Next, we will give some aspects of vielbein formalism, and several practical examples are presented. This topic, as similar to a differential form, is a manifest mathematical branch providing many insights in the language of differential geometry. However, we will introduce this rich structure in a simple and sufficient, though effective, way in understanding a calculation used in this senior project.

We first consider a metric tensor $g_{\mu\nu}$ defined over (3+1)-dimensional space-time. With the symmetrical property of the metric, this object may be diagonalized by an orthogonal transformation, i.e. $(O^{-1})_\mu{}^a = O^a{}_\mu$ and

$$g_{\mu\nu} = O^a{}_\mu D_{ab} O^b{}_\nu, \tag{A.20}$$

with positive eigenvalues λ^a in $D_{ab} = \text{diag}(-\lambda^0, \lambda^1, \lambda^2, \lambda^3)$. This construction allow us to define the vielbein $e^a_\mu(x)$ as

$$e^a_\mu(x) \equiv \sqrt{\lambda^a(x)} O^a{}_\mu(x), \tag{A.21}$$

such that

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu \tag{A.22}$$

with η_{ab} being a flat space-time metric.

An advantage of using vielbein formalism is that, as we can see in the equation (A.22), vielbein contain information related to the global coordinate transformation, indicated by global indices $\mu, \nu = 0, 1, ..., 3$, and bypass into the local coordinate transformation, namely the local Lorentz transformation, indicated by the indices $a, b = 0, 1, ..., 3$ being local indices (or sometimes known as Lorentz indices). Therefore, given a well-known object defined over a flat space-time, e.g. Dirac spinor, we can find its globally counter-part living on the curved space-time, such as the gamma matrices $\gamma^\mu = e^\mu_a \gamma^a$ introduced in Appendix B.

Combine with the language of a differential form, we can construct a specific basis one-form in the space Λ^p_x as

$$e^a \equiv e^a_\mu dx^\mu \tag{A.23}$$

which are called local Lorentz basis of one-forms or vielbein one-forms.

Hence, components of p-form field written in the vielbein one-form are understood automatically to transform under local Lorentz transformation in tensor rank-p representation. For example, recall that a metric ansatz as AdS_4 black hole is of the form

$$ds^2 = -e^{2f(r)}dt^2 + dr^2 + e^{2h(r)}\left(d\theta^2 + F^2(\theta)d\phi^2\right), \tag{A.24}$$

we can readily find a complete set of one-form vielbein as follows

$$\begin{aligned} e^{\hat{t}} &= e^f dt, \quad e^{\hat{r}} = dr \\ e^{\hat{\theta}} &= e^h d\theta, \quad e^{\hat{\phi}} = e^h F d\phi. \end{aligned} \tag{A.25}$$

Furthermore, a two-form gauge field strengths given in equation (5.39) can be written in the local Lorentz basis as

$$\begin{aligned} F^{12} &= -p_1\kappa F(\theta)d\theta\wedge d\phi \\ &= -p_1\kappa e^{-2h}(e^h d\theta)\wedge(e^h F(\theta)d\phi) \\ &= -p_1\kappa e^{-2h}e^{\hat{\theta}}\wedge e^{\hat{\phi}}, \end{aligned} \tag{A.26}$$

such that $F^{12}_{\hat{\theta}\hat{\phi}} = -p_1\kappa e^{-2h}$. Another important example is the volume form used to construct the integration; it can be written in the local Lorentz basis as

$$\epsilon_{\mu_1\mu_2...\mu_n}dx^{\mu_1}\wedge dx^{\mu_2}\wedge ...\wedge dx^{\mu_n} = \epsilon_{a_1a_2...a_n}dx^{a_1}\wedge dx^{a_2}\wedge ...\wedge dx^{a_n} \tag{A.27}$$

in which $\epsilon_{a_1a_2...a_n}$ is precisely the levi-civita symbol, i.e. $\epsilon_{\hat{t}\hat{r}\hat{\theta}\hat{\phi}} = 1$. Thus, Hodge-star operation is much simpler in the local Lorentz basis.

According to the Lorentz transformation rule,

$$e'^a_\mu(x) = \Lambda^{-1a}{}_b(x)e^b_\mu(x), \tag{A.28}$$

we can see that the two-forms de^a constructed by acting exterior derivative on vielbein one-forms are not transform covariantly under the Lorentz transformation, i.e.

$$\begin{aligned} de'^a &= d(\Lambda^{-1a}{}_b e^b) \\ &= \Lambda^{-1a}{}_b de^b + d\Lambda^{-1a}{}_b\wedge e^b \end{aligned} \tag{A.29}$$

To cancel the second term which spoils the transformation property of two-forms, we introduce a one-form named spin connection $\omega^a{}_b$ which transform under Lorentz transformation as

$$\omega'^a{}_b = \Lambda^{-1a}{}_c d\Lambda^c{}_b + \Lambda^{-1a}{}_c\omega^c{}_d\Lambda^d{}_b, \tag{A.30}$$

such that we find two-form T^a satisfying

$$T^a \equiv de^a + \omega^a{}_b\wedge e^b. \tag{A.31}$$

The two-form T^a is called the torsion two-form which indeed transform covariantly under the Lorentz transformation, i.e. $T'^a = \Lambda^{-1a}{}_b T^b$. The equation (A.31) is known as the first Cartan structure equation.

It should be noted that the transformation rule of the spin connection (A.30) takes a similar form as the gauge transformation of Yang-Mills gauge fields for the gauge group $SO(3,1)$. Therefore, the spin connections are used as the gauge fields under gauging the

Lorentz group. This is the reason why spin connections appeared in the supersymmetry transformation rule.

In the absence of the torsion fields which is our case, we find the relation among the spin connection and vielbein as

$$de^a = e^b \wedge \omega^a{}_b \tag{A.32}$$

from which, provided a metric ansatz, we can identify the expressions of spin connection. For instance, given a metric ansatz (A.24) and corresponding one-form vielbeins (A.25), we consider the component along $a = \hat{\phi}$,

$$\begin{aligned} de^{\hat{\phi}} &= e^{\hat{t}} \wedge \omega^{\hat{\phi}}{}_{\hat{t}} + e^{\hat{r}} \wedge \omega^{\hat{\phi}}{}_{\hat{r}} + e^{\hat{\theta}} \wedge \omega^{\hat{\phi}}{}_{\hat{\theta}} \\ &= e^f dt \wedge \omega^{\hat{\phi}}{}_{\hat{t}} + dr \wedge \omega^{\hat{\phi}}{}_{\hat{r}} + e^h d\theta \wedge \omega^{\hat{\phi}}{}_{\hat{\theta}}. \end{aligned} \tag{A.33}$$

Compared with the direct calculation by exterior derivative,

$$de^{\hat{\phi}} = h' e^h F dr \wedge d\phi + e^h F' d\theta \wedge d\phi, \tag{A.34}$$

we can identify that

$$\omega^{\hat{\phi}}{}_{\hat{r}} = h' e^h F d\phi, \quad \omega^{\hat{\phi}}{}_{\hat{\theta}} = F' d\phi, \tag{A.35}$$

or, in the component form with the used of raising indices,

$$\omega_{\hat{\phi}}{}^{\hat{\phi}\hat{r}} = h', \quad \omega_{\hat{\phi}}{}^{\hat{\phi}\hat{\theta}} = \frac{F'}{F} e^{-h}. \tag{A.36}$$

Appendix B

γ-Matrices Manipulation

In this appendix, we note some useful properties of γ-matrices including examples used in this senior project. However, we will not get into a rigorous structure of Clifford algebra; readers, who are interested in more detail, can find this topic in several standard textbooks.

We begin with the definition of γ-matrices,

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{B.1}$$

where a, b label local space-time indices and η^{ab} denotes inverse of flat space-time metric, $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

It should be emphasized that the connection between true γ^a and the one labeled by global space-time indices γ^μ appearing in several times can be accomplished by contracting with vielbein, i.e.

$$\gamma^\mu = e^\mu_a \gamma^a. \tag{B.2}$$

With this, we can easily show that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{B.3}$$

However, working with local space-time indices is usually more practical than the global one.

Furthermore, we are also interested in the higher-rank gamma matrices which can be defined as

$$\gamma^{a_1 a_2 \ldots a_n} \equiv \gamma^{[a_1}\gamma^{a_2}\ldots\gamma^{a_n]} \tag{B.4}$$

in which $\gamma^{a_1 a_2 \ldots a_n}$ is known as rank-n gamma matrix. An example of this is the rank-2 gamma matrices γ^{ab} which are also generators of the Lorentz group and mostly appeared in the supersymmetry variation with non-vanishing gauge fields. In component form, we can see that, for example of $a_1 = x$ and $a_2 = y$,

$$\begin{aligned} \gamma^{xy} &= \gamma^{[x}\gamma^{y]} \\ &= \frac{1}{2}\left(\gamma^x\gamma^y - \gamma^y\gamma^x\right) \\ &= \frac{1}{2}\left(\gamma^x\gamma^y + \gamma^x\gamma^y\right) \\ &= \gamma^x\gamma^y \end{aligned} \tag{B.5}$$

and vanish when $x = y$. In the second line, we just substitute the definition of rank-2 gamma matrix, and then we use the Clifford algebra to get the third line.

In particular, we can use the higher-rank gamma matrices to construct the complete basis on the matrix space, i.e. a set of basis

$$\Gamma = \{\gamma^a, \gamma^{ab}, \gamma_5\gamma^a, \gamma_5\} \tag{B.6}$$

span the space of matrix. However, we will not go into detail of this topic.

With the calculation given in (B.5), we can show that, for instance,

$$\begin{aligned} \gamma^{\hat{t}}\gamma^{\hat{t}\hat{r}}\gamma_{\hat{t}} &= \gamma^{\hat{t}}\gamma^{\hat{t}}\gamma^{\hat{r}}\gamma_{\hat{t}} \\ &= -\gamma^{\hat{r}}\gamma_{\hat{t}} \\ &= \gamma^{\hat{r}}\gamma^{\hat{t}} \\ &= -\gamma^{\hat{t}\hat{r}}. \end{aligned} \tag{B.7}$$

This example shows how can we use the definition of rank-2 gamma matrix together with the Clifford algebra to compute the expression of the form $\gamma^{\hat{t}}\gamma^{\hat{t}\hat{r}}\gamma_{\hat{t}}$ which is an important step to obtain the BPS equation coming from $\delta\psi^i_{\hat{t}}$ in the presence of gauge field, for example (5.18).

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