# ทฤษฎีบทลิมิตเฉพาะที่สำหรับผลรวมของตัวแปรสุ่มแลตทิซค่าจำนวนเต็มที่เป็นอิสระต่อกัน

นายปุญญพัฒน์ คำหมู่

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตร์มหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2564

# LOCAL LIMIT THEOREMS FOR SUMS OF INDEPENDENT INTEGER-VALUED LATTICE RANDOM VARIABLES

Mr. Punyapat Kammoo

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2021

| | |
|---|---|
| Thesis Title | LOCAL LIMIT THEOREMS FOR SUMS OF INDEPENDENT INTEGER-VALUED LATTICE RANDOM VARIABLES |
| By | Mr. Punyapat Kammoo |
| Field of Study | Mathematics |
| Thesis Advisor | Professor Kritsana Neammanee, Ph.D. |
| | Assistant Professor Kittipong Laipaporn, Ph.D. |

---

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree.

. . . . . . . . . . . . . . . . . . . . . . . . . Dean of the Faculty of Science

(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

. . . . . . . . . . . . . . . . . . . . . . . . . Chairman

(Associate Professor Songkiat Sumetkijakan, Ph.D.)

. . . . . . . . . . . . . . . . . . . . . . . . . Thesis Advisor

(Professor Kritsana Neammanee, Ph.D.)

. . . . . . . . . . . . . . . . . . . . . . . . . Co-Advisor

(Assistant Professor Kittipong Laipaporn, Ph.D.)

. . . . . . . . . . . . . . . . . . . . . . . . . Examiner

(Raywat Tanadkithirun, Ph.D.)

. . . . . . . . . . . . . . . . . . . . . . . . . External Examiner

(Assistant Professor Dawud Thongtha, Ph.D.)

ปุญญพัฒน์ คำหมู่ : ทฤษฎีบทลิมิตเฉพาะที่สำหรับผลรวมของตัวแปรสุ่มแลตทิซค่าจำนวนเต็มที่เป็นอิสระต่อกัน (LOCAL LIMIT THEOREM FOR SUMS OF INDEPENDENT INTEGER-LATTICE RANDOM VARIABLES)
อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ศ.ดร. กฤษณะ เนียมมณี และ
อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : ผศ.ดร. กิตติพงษ์ ไหลภาภรณ์, 61 หน้า.

ให้ $X_1, X_2, \ldots, X_n$ เป็นตัวแปรสุ่มแลตทิซที่เป็นอิสระต่อกัน และให้ $S_n = \sum_{j=1}^{n} X_j$ ความน่าจะเป็นพื้นฐานที่น่าสนใจคือความน่าจะเป็นที่จุดใดจุดหนึ่ง นั่นคือ $P(S_n = k)$ โดยที่ $k \in \operatorname{Im} S_n$ ทฤษฎีบทลิมิตเฉพาะที่คือทฤษฎีบทที่รู้จักกันเป็นอย่างดี ซึ่งประมาณ $P(S_n = k)$ ด้วยฟังก์ชันความหนาแน่นของความน่าจะเป็นของการแจกแจงปกติ ในปี 2021 ศิริประภารัตน์และเนียมมณีให้ทฤษฎีบทลิมิตเฉพาะที่สำหรับผลรวมของตัวแปรสุ่มแลตทิซเต็มหน่วยในกรณีโมเมนต์ที่สามมีค่าจำกัด นั่นคือ $E|X_j|^3 < \infty$ สำหรับทุก $j = 1, 2, \ldots, n$ ในงานนี้เราปรับปรุงผลลัพธ์ของพวกเขาสำหรับกรณีทั่วไปของผลรวมนี้ ซึ่งสมมติให้ $E|X_j|^{2+\alpha} < \infty$ สำหรับทุก $j = 1, 2, \ldots, n$ โดยที่ $\alpha \in (0, 1]$ มากไปกว่านั้นเราได้ค่าคงที่ที่ชัดเจนของขอบเขตความคลาดเคลื่อนที่มีความซับซ้อนน้อยกว่าผลลัพธ์ก่อนหน้านี้ เทคนิคของเราคือวิธีการฟังก์ชันลักษณะเฉพาะ

| | |
|---|---|
| ภาควิชา : ....................................... | ลายมือชื่อนิสิต : ............................................. |
| สาขาวิชา : .................................... | ลายมือชื่อ อ.ที่ปรึกษาหลัก : ............................ |
| ปีการศึกษา : ................................ | ลายมือชื่อ อ.ที่ปรึกษาร่วม : ............................... |

# # 6370174023 : MAJOR MATHEMATICS

KEYWORDS : LOCAL LIMIT THEOREM / NORMAL DENSITY FUNCTION / LATTICE RANDOM VARIABLE / RATE OF CONVERGENCE / CHARACTERISTIC FUNCTION

PUNYAPAT KAMMOO : LOCAL LIMIT THEOREMS FOR SUMS OF INDEPENDENT INTEGER-VALUED LATTICE RANDOM VARIABLES.

ADVISOR : PROF. KRITSANA NEAMMANEE, Ph.D. and

CO-ADVISOR : ASST. PROF. KITTIPONG LAIPAPORN, Ph.D., 61 pp.

Let $X_1, X_2, \ldots, X_n$ be independent lattice random variables and let $S_n = \sum_{j=1}^{n} X_j$. One interesting fundamental probability is the probability at a particular point, i.e., $P(S_n = k)$ where $k \in \operatorname{Im} S_n$. The local limit theorem is the well-known theorem which estimates $P(S_n = k)$ by the probability density function of a normal distribution. In 2021, Siripraparat and Neammanee gave the local limit theorems for sums of integer-valued lattice random variables in the case of finite third moment, i.e., $E|X_j|^3 < \infty$ for all $j = 1, 2, \ldots, n$. In this work, we improve their results for generality of this sums which assumes $E|X_j|^{2+\alpha} < \infty$ for all $j = 1, 2, \ldots, n$ where $\alpha \in (0, 1]$. Moreover, we obtain explicit constants of the error bound which are less complicated than the previous result. Our technique is the characteristic function method.

| | |
|---|---|
| Department : ...Mathematics and..... | Student's Signature : ...................... |
| ...Computer Science... | Advisor's Signature : ...................... |
| Field of Study : ......Mathematics...... | Co-Advisor's Signature : ...................... |
| Academic Year : ............2021............ | |

# ACKNOWLEDGEMENTS

I would first like to thank my thesis advisor, Professor Dr. Kritsana Neammanee, for his invaluable advance and constant encouragement throughout the course of this thesis. His insightful feedback pushed me to sharpen my thinking and brought my work to a higher level. I would not have achieved this far and this thesis would not have been completed without all the support that I have always received from him. Besides my advisor, I would also like to thank my co-advisor, Assistant Professor Dr. Kittipong Laipaporn, for his valuable guidance throughout my thesis process. He additionally provided me with the tools that I needed to choose the right direction and successfully complete my thesis. Sincere thanks are also extended to Associate Professor Dr. Songkiat Sumetkijakan, the chairman, Dr. Raywat Tanadkithirun and Assistant Professor Dr. Dawud Thongtha, the committee members, for their comments and suggestions.

Special thanks go to the Development and Promotion of Science and Technology Talents Project (DPST) for financial supports.

Additionally, I would like to thank my family, my friends and those whose names are not mentions here but have greatly inspired and encouraged me throughout the period of this research.

# CONTENTS

# CHAPTER I

# INTRODUCTION

Let $X_1, X_2, \dots, X_n$ be independent integer-valued random variables with means $\mu_j$ and variances $\sigma_j^2$ for $j = 1, 2, \dots, n$. Then, let

$$S_n = \sum_{j=1}^{n} X_j, \qquad \mu = \sum_{j=1}^{n} \mu_j \quad \text{and} \quad \sigma^2 = \sum_{j=1}^{n} \sigma_j^2.$$

One interesting fundamental probability is the probability at a particular point, i.e.,

$$P(S_n = k)$$

for some $k \in \mathbb{Z}$. The local limit theorem describes how $P(S_n = k)$ approaches the normal density $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}$. There are two well-known techniques for deriving this theorem: the characteristic function method and the Bernoulli part extraction method. The characteristic function method is to estimate the characteristic function of a random variable. This method has been used in a number of studies such as ([1], [2], [3], [4] and [5]) in the case of bounded random variables and ([5],[6], [7], [8] and [9] ) in the case of lattice random variables. The extraction method of the Bernoulli part of a random variable is to extract an integer-valued random variable into sums of independent Bernoulli random variables. It was developed by McDonald ([10], 1979), for proving local limit theorems based on the assumption that the central limit theorem holds. However, twenty years before McDonald, there was a similar approach in research of Kolmogorov ([11]). In 2009, this idea was also developed for general random variables by Aizenmann, Germinet, Klein and Warzel ([12]).

If $P(X_j = 1) = p_j = 1 - P(X_j = 0)$, then $X_j$ is called *a Bernoulli random* *variable with parameter* $p_j$ and $S_n$ is said to be *a Poisson binomial random variable*. In addition, when $p_1 = p_2 = \cdots = p_n = p$, we call $S_n$ *a binomial random variable with parameter* $n$ *and* $p$ and use the notation $S_n \sim B(n,p)$. The first local limit theorem was proved by De Moivre and Laplace ([13], 1754) for a binomial random variable. We call $X$ *a lattice random variable with parameter* $(a,d)$, if the values of $X$ belong to $\mathcal{L}(a,d) = \{a + md : m \in \mathbb{Z}\}$ where $a$ and $d > 0$ are integers. In addition, $d$ is said to be *maximal*, if there are no other integer numbers $a'$ and $d' > d$ for which $P(X \in \mathcal{L}(a',d')) = 1$; we call $X$ *a maximal lattice random variable with parameter* $(a,d)$, if $X$ is a lattice random variable with parameter $(a,d)$ and $d$ is maximal. Observe that the Bernoulli random variable is a maximal lattice random variable with parameter $(0,1)$. In 1948, Gnedenko [14] generalized the local limit theorem of De Moivre and Laplace to identically distributed maximal lattice random variables having finite expectation and variance and showed that

$$\sup_{k\in\mathbb{Z}} \left| P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}} e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}} \right| \to 0$$

as $n \to \infty$. We denote

$$\Delta_n^{(a,d)} = \sup_{k\in\mathbb{Z}} \left| P(S_n = na + kd) - \frac{d}{\sigma\sqrt{2\pi}} e^{-\frac{(na+kd-\mu)^2}{2\sigma^2}} \right|.$$

In 1971, Ibragimov and Linnik ([17]) improved the result of Gnedenko by giving the rate of convergence $O\left(\frac{1}{n^\alpha}\right)$ where $0 < \alpha < \frac{1}{2}$. Their result is given by the following theorem.

**Theorem 1.1.** *([17], Theorem 4.5.3, p.138) Let* $X_1, X_2, \ldots, X_n$ *be independent identically distributed lattice random variables with parameter* $(a,d)$ *which have zero expectation and finite variance. Let* $F$ *denote the distribution function of* $X_1$. *In order that the property*

$$\Delta_n^{(a,d)} = O\left(\frac{1}{n^\alpha}\right), \quad \text{for } 0 < \alpha < \frac{1}{2}, \tag{1.1}$$

*holds, it is necessary and sufficient that the following conditions are satisfied:*

*(i)* $d$ *is maximal,*

*(ii)* $\int\limits_{|x|\geq u} x^2F(dx) = O\big(\frac{1}{u^{2\alpha}}\big)$ *as* $u \to \infty$*.*

A few years later, Petrov ([18], 1975) proved that if $E|X_1|^3 < \infty$, then (1.1) holds with $\alpha = \frac{1}{2}$. Moreover, for the case that $X_j$'s are non-identically distributed lattice random variables with parameter $(0, 1)$, Petrov gave the following theorem.

**Theorem 1.2.** *( [18], Theorem 6, p.197) Let* $X_1, X_2, \dots, X_n$ *be independent lattice random variables with parameter* $(0, 1)$ *satisfying the following conditions,*

*(i)* $\sigma^2 \to \infty$ *as* $n \to \infty$*,*

*(ii)* $\sum\limits_{j=1}^{n} E|X_j - \mu_j|^3 = O(\sigma^2)$*,*

*(iii)* $P(X_j = 0) \geq P(X_j = m)$ *for all* $j = 1, 2, \dots, n$ *and* $m \in \mathbb{Z}$*,*

*(iv) gcd*$\big\{m : \frac{1}{\log n} \sum\limits_{j=1}^{n} P(X_j = 0)P(X_j = m) \to \infty$ *as* $n \to \infty\big\} = 1.$

*Then,*

$$\Delta_n^{(0,1)} \leq \frac{C}{\sigma^2}.$$

The previous studies had not given explicit constants of the error bound until the twenty-first century. Giuliano and Weber ([15], 2017) gave the rate of convergence $O\left(\frac{1}{\sigma}\right)$ with explicit constants of the error bound in the case of non-identically distributed common lattice random variables with parameter $(a, d)$. Denote for a lattice random variable $X_j$ with parameter $(a, d)$,

$$\delta_j = \sum_{m\in\mathbb{Z}} P(X_j = a + md) \wedge P(X_j = a + (m + 1)d),$$

where $a \wedge b = \min(a, b)$. If $\delta_j > 0$, for all $j = 1, 2, \dots, n$, then they showed that

$$X_j \overset{\mathcal{D}}{=} V_j + \varepsilon_j L_j,$$

where $(V_j, \varepsilon_j)$, $L_j$, for $j = 1, 2, \ldots, n$ are mutually independent; $\varepsilon_j$, $L_j$, for $j = 1, 2, \ldots, n$, are independent Bernoulli random variables with $P(\varepsilon_j = 1) = 1 - P(\varepsilon_j = 0) = \vartheta_j$, $0 < \vartheta_j \leq \delta_j$ and $P(L_j = 0) = P(L_j = 1) = \frac{1}{2}$ (see [20], [21], [22] for more details). From these assumptions, Giuliano and Weber ([15]) illustrated the following theorem.

**Theorem 1.3.** *([15], Corollary 1.8, p.3274) Let $X_1, X_2, \ldots, X_n$ be independent lattice random variables with parameter $(a, d)$ such that $E|X_j|^2 < \infty$, for all $j = 1, 2, \ldots, n$. Let $W_n = \sum\limits_{j=1}^{n} V_j$, $B_n = \sum\limits_{j=1}^{n} \varepsilon_j$ and $S'_n = W_n + \frac{d}{2} B_n$. Suppose that $\frac{\log \Theta_n}{\Theta_n} \leq \frac{1}{14}$ where $\Theta_n = \sum\limits_{j=1}^{n} \vartheta_j$. Then, for all $k \in \mathbb{Z}$ such that*

$$\frac{(na + kd) - E(S_n)^2}{\sigma} \leq \left( \frac{\Theta_n}{14 \log \Theta_n} \right)^{\frac{1}{2}},$$

*we have*

$$\Delta_n^{(a,d)} \leq C_1 \left[ d \left( \frac{\log \Theta_n}{\sigma \Theta_n} \right)^{\frac{1}{2}} + \frac{H_n + \Theta_n^{-1}}{\sqrt{\Theta_n}} \right],$$

*where*

$C_1 = 2^{\frac{7}{2}} \max \left\{ \frac{8}{\sqrt{2\pi}}, C_0 \right\}$,

$C_0$ *is the constant such that* $\sup\limits_{z \in \{1,2\ldots,n\}} \left| P\left( B(n, \frac{1}{2}) = z \right) - \sqrt{\frac{2}{\pi n}} e^{-\frac{(2z-n)^2}{2n}} \right| \leq \frac{C_0}{n^{\frac{3}{2}}}$,

$H_n = \sup\limits_{x \in \mathbb{R}} \left| P\left( \frac{S'_n - E(S'_n)}{\sqrt{Var(S'_n)}} < x \right) - \Phi(x) \right|$ *and $\Phi(x)$ is the standard normal distribution.*

Three years later, Siripraparat and Neammanee ([9], 2021) had given the result for non-identically distributed random variables taking values in a common lattice $\mathcal{L}(a, d)$ as $d$ is maximal. In case of assuming finite third moment, they gave the rate of convergence $O\left(\frac{1}{\sigma^2}\right)$ with the explicit bound in the following theorem.

**Theorem 1.4.** *([9], Theorem 1.3, p. 4) Let $X_1, X_2, \ldots, X_n$ be independent maximal* *lattice random variables with parameter* $(a, d)$ *such that* $E|X_j|^3 < \infty$*, for all* $j = 1, 2, \ldots, n$*. Then,*

$$\Delta_n^{(a,d)} \leq \frac{1.7898d}{\sigma^4} \sum_{j=1}^{n} E|X_j - a|^3 + \frac{2.2075}{\tau\beta} e^{-\frac{\tau^2\beta}{\pi^2}}.$$

*where*

$$\beta = \sum_{j=1}^{n} \beta_j, \qquad \beta_j = 2 \sum_{m=-\infty}^{\infty} P(X_j = a + dm)P(X_j = a + d(m+1)) \text{ and}$$

$$\tau = \frac{1}{10\left(\sum\limits_{j=1}^{n} E\left|\frac{X_j - a}{d}\right|^3\right)^{\frac{1}{3}}}.$$

Notice that this result is less complicated than before and, thus, easier to use. Furthermore, in the case of $S_n$ being Poisson binomial, they gave the smaller constant of the error bound according to the following statement,

$$\sup_{k\in\{0,1,\ldots,n\}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{0.1194}{\sigma^2(1 - \frac{3}{4\sigma})^3} + \frac{0.0749}{\sigma^3} + \frac{0.2107}{\sigma^3(1 - \frac{3}{4\sigma})^6} + \left( \frac{0.4579}{\sqrt{\sigma}} + \frac{0.4725}{\sigma\sqrt{\sigma}} \right) e^{-\frac{3}{2}\sigma}. \tag{1.2}$$

Note that (1.2) improved the result of Zolotukhin et al. ([23], 2018) which gave the bound where $S_n \sim B(n, p)$ as follows:

$$\sup_{k\in\{0,1,\ldots,n\}} \left| P(S_n = k) - \frac{1}{\sqrt{2np(1-p)\pi}} e^{-\frac{(k-np)^2}{2np(1-p)}} \right| \leq \min\left\{ \frac{1}{\sqrt{2enp(1-p)}}, \frac{0.516}{np(1-p)} \right\}.$$

From the past until today, there are many works which investigate the local limit theorem for a generality of Bernoulli sums (see [9], [14], [15] and [16] for examples) including sums of lattice random variables explained above. The generalization of a binomial random variable is a Poisson binomial random variable which can be generalized to be the weighted sums of Bernoulli random variables. These sums are defined by $\tilde{S}_n = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ where $X_j$'s are independent Bernoulli random variables and $a_j$'s are any integers. On the other hand, the weighted sums of Bernoulli random variables are sums of random variables $X_j$'s satisfying $P(X_j = a_j) = 1 - P(X_j = 0) > 0$ for some integers $a_j$. Giuliano and Weber ([21], 2016) gave the local limit theorem for these sums and showed applications such as a probabilistic model for the Dickman function, a diophantine equation and Freiman–Pitman's probabilistic model of the partition function (see more details in [21]). Their result is given in the following statement.

**Theorem 1.5.** *[21] Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with parameters $p_1, p_2, \ldots, p_n$, respectively. Let $a_1, a_2, \ldots, a_n \in \mathbb{N}$ be such that*

$$a_i \neq a_j \tag{1.3}$$

*for any $i, j = 1, 2, \ldots, n$ such that $i \neq j$ and*

$$\tilde{S}_n = a_1X_1 + a_2X_2 + \cdots + a_nX_n$$

*with mean $\mu$ and variance $\sigma^2$. Then,*

$$\sup_{k\in\mathbb{Z}} \left| P(\tilde{S}_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{C}{\sum\limits_{j=1}^{n} p_j}$$

*for some unknown constant $C$.*

In this work, we investigate local limit theorems for sums of independent lattice integer-valued random variables without or with finite third moment assumption and also give explicit constants of the error bound. Our technique is the characteristic function method. First, we give the local limit theorems for sums of lattice random variable with or without finite third moment condition in Chapter 2. The followings are our main results.

**Theorem 1.6.** *Let $X_1, X_2, \ldots, X_n$ be independent lattice random variables with parameter $(a, d)$ such that $E|X_j|^{2+\alpha} < \infty$ for $j = 1, 2, \ldots, n$, where $0 < \alpha \leq 1$,* *and let* $S_n = \sum_{j=1}^n X_j$ *with mean* $\mu$ *and variance* $\sigma^2$*. Let* $\beta := \sum\limits_{j=1}^{n} \beta_j$ *where* $\beta_j := 2\sum\limits_{m=-\infty}^{\infty} p_{jm}p_{j(m+1)}$ *and* $p_{jm} := P(X_j = a + md)$*. If* $\sigma^2 > d^2$ *and* $\beta > 0$*, then*

$$\Delta_n^{(a,d)} \leq \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{\left(\sum\limits_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})d^{\frac{\alpha^2-\alpha+2}{2}}}{\sigma^4}\left(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{3-\alpha}{2}}$$
$$+ \frac{0.3184d^2}{\sigma^2\tau}e^{-\frac{\sigma^2\tau^2}{2d^2}} + \frac{1.5708}{\tau\beta}e^{-\frac{\tau^2\beta}{\pi^2}}$$

*where* $\tau = \frac{d}{3^{\frac{1}{\alpha}}\left(\sum\limits_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1}{2+\alpha}}}$.

*Furthermore, if* $X_1, X_2, \ldots, X_n$ *are identically distributed and* $\beta_1 > 0$*, then*

$$\Delta_n^{(a,d)} \leq \frac{C_1}{n^{\frac{1+\alpha}{2}}}$$

*where*

$$C_1 = \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1 - a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171 \cdot 3^{\frac{1}{\alpha}} d^{\frac{\alpha^2-\alpha+2}{2}} (E|X_1 - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4}$$
$$+ \frac{0.6368 \cdot 3^{\frac{3}{\alpha}} d(E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4} + \frac{15.5032 \cdot 3^{\frac{3}{\alpha}} (E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{d^3\beta_1^2}.$$

**Theorem 1.7.** *Let* $X_1, X_2, \ldots, X_n$ *be independent lattice random variables with parameter* $(a, d)$ *such that* $E|X_j|^{2+\alpha} < \infty$ *for* $j = 1, 2, \ldots, n$*, where* $0 < \alpha \leq 1$*, and let* $S_n = \sum_{j=1}^n X_j$ *with mean* $\mu$ *and variance* $\sigma^2$*. Let* $\upsilon := \min\limits_{1\leq j\leq n} \upsilon_j$ *where* $\upsilon_j := 2\sum\limits_{m=-\infty}^{\infty} p_{jm}p_{j(m+j)}$*. If* $\sigma^2 > d^2$ *and* $\upsilon_j > 0$ *for all* $j = 1, 2, \ldots, n$*, then*

$$\Delta_n^{(a,d)} \leq \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{\left(\sum\limits_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})d^{\frac{\alpha^2-\alpha+2}{2}}}{\sigma^4}\left(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{3-\alpha}{2}}$$
$$+ \frac{0.3184d^2}{\sigma^2\tau}e^{-\frac{\sigma^2\tau^2}{2d^2}} + \exp\left(-\frac{n\upsilon}{4}\min\left(1, \left(\frac{n\tau}{2\pi}\right)^2\right)\right)$$

*where* $\tau = \frac{d}{3^{\frac{1}{\alpha}}\left(\sum\limits_{j=1}^{n} E|X_j-a|^{2+\alpha}\right)^{\frac{1}{2+\alpha}}}$.

*Furthermore, if* $X_1, X_2, \ldots, X_n$ *are identically distributed and* $\upsilon_j > 0$ *for all* $j = 1, 2, \ldots, n$*, then for* $n \geq \left(\frac{2\pi \cdot 3^{\frac{1}{\alpha}}(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d}\right)^{\frac{2+\alpha}{1+\alpha}}$,

$$\Delta_n^{(a,d)} \leq \frac{C_2}{n^{\frac{1+\alpha}{2}}} + e^{-\frac{n\upsilon}{4}}$$

*where*

$$C_2 = \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1-a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171 \cdot 3^{\frac{1}{\alpha}} d^{\frac{\alpha^2-\alpha+2}{2}}(E|X_1-a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4} + \frac{0.6368 \cdot 3^{\frac{3}{\alpha}} d(E|X_1-a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4}.$$

Moreover, we consider in the case that the third moment of every $X_j$ is finite. From Theorem 1.6 and Theorem 1.7, we directly obtain Corollary 1.8 and Corollary 1.9, respectively.

**Corollary 1.8.** *Let* $X_1, X_2, \ldots, X_n$ *be independent lattice random variables with parameter* $(a, d)$ *such that* $E|X_j|^3 < \infty$ *and* $\beta > 0$ *and let* $S_n = \sum_{j=1}^{n} X_j$ *with mean* $\mu$ *and variance* $\sigma^2 > d^2$*. Then,*

$$\Delta_n^{(a,d)} \leq \frac{13.8513d}{\sigma^4}\sum_{j=1}^{n} E|X_j-a|^3 + \frac{0.0020d^3}{\sum\limits_{j=1}^{n} E|X_j-a|^3} + \frac{0.3184d^2}{\sigma^2\kappa}e^{-\frac{\sigma^2\kappa^2}{2d^2}} + \frac{1.5708}{\kappa\beta}e^{-\frac{\kappa^2\beta}{\pi^2}}$$

*where* $\kappa = \frac{d}{3\left(\sum\limits_{j=1}^{n} E|X_j-a|^3\right)^{\frac{1}{3}}}$.

*Furthermore, if* $X_1, X_2, \ldots, X_n$ *are identically distributed and* $\beta_1 > 0$*, then*

$$\Delta_n^{(a,d)} \leq \frac{C_3}{n}$$

*where*

$$C_3 = \frac{0.0020d^3}{E|X_1-a|^3} + \frac{31.0449dE|X_1-a|^3}{\sigma_1^4} + \frac{418.5864E|X_1-a|^3}{d^3\beta_1^2}.$$

**Corollary 1.9.** *Let $X_1, X_2, \ldots, X_n$ be independent lattice random variables with parameter $(a, d)$ such that $E|X_j|^3 < \infty$ and $\upsilon_j > 0$ for all $j = 1, 2, \ldots, n$ and let $S_n = \sum_{j=1}^{n} X_j$ with mean $\mu$ and variance $\sigma^2 > d^2$. Then,*

$$\Delta_n^{(a,d)} \leq \frac{13.8513d}{\sigma^4}\sum_{j=1}^{n} E|X_j - a|^3 + \frac{0.0020d^3}{\sum\limits_{j=1}^{n} E|X_j - a|^3} + \frac{0.3184d^2}{\sigma^2\kappa}e^{-\frac{\sigma^2\kappa^2}{2d^2}} + e^{-\frac{n\upsilon}{4}\min\left(1, \left(\frac{n\kappa}{2\pi}\right)^2\right)}$$

*where* $\kappa = \frac{d}{3\left(\sum\limits_{j=1}^{n} E|X_j - a|^3\right)^{\frac{1}{3}}}$.

*Furthermore, if $X_1, X_2, \ldots, X_n$ are identically distributed and $\upsilon_j > 0$ for all $j = 1, 2, \ldots, n$, then for $n \geq \left(\frac{6\pi(E|X_1 - a|^3)^{\frac{1}{3}}}{d}\right)^{\frac{3}{2}}$,*

$$\Delta_n^{(a,d)} \leq \frac{C_4}{n} + e^{-\frac{n\upsilon}{4}}$$

*where*

$$C_4 = \frac{0.0020d^3}{E|X_1 - a|^3} + \frac{31.0449dE|X_1 - a|^3}{\sigma_1^4}.$$

In Chapter 3, we improve the constants in Corollary 1.8 and Corollary 1.9 to obtain the following results.

**Theorem 1.10.** *Let $X_1, X_2, \ldots, X_n$ be independent lattice random variables with parameter $(a, d)$ such that $E|X_j|^3 < \infty$ and $\beta > 0$ and let $S_n = \sum_{j=1}^{n} X_j$ with mean $\mu$ and variance $\sigma^2 > d^2$. Then,*

$$\Delta_n^{(a,d)} \leq \frac{0.6607d}{\sigma^4}\sum_{j=1}^{n} E|X_j - a|^3 + \frac{0.3184d^2}{\sigma^2\kappa}e^{-\frac{\sigma^2\kappa^2}{2d^2}} + \frac{1.5708}{\kappa\beta}e^{-\frac{\kappa^2\beta}{\pi^2}}$$

*where* $\kappa = \frac{d}{3\left(\sum\limits_{j=1}^{n} E|X_j - a|^3\right)^{\frac{1}{3}}}$.

*Furthermore, if $X_1, X_2, \ldots, X_n$ are identically distributed and $\beta_1 > 0$, then*

$$\Delta_n^{(a,d)} \leq \frac{C_5}{n}$$

*where*

$$C_5 = \frac{17.8543dE|X_1 - a|^3}{\sigma_1^4} + \frac{418.5858E|X_1 - a|^3}{d^3\beta_1^2}.$$

**Theorem 1.11.** *Let $X_1, X_2, \ldots, X_n$ be independent lattice random variables with parameter $(a, d)$ such that $E|X_j|^3 < \infty$ and $\upsilon_j > 0$ for all $j = 1, 2, \ldots, n$ and let $S_n = \sum_{j=1}^n X_j$ with mean $\mu$ and variance $\sigma^2 > d^2$. Then,*

$$\Delta_n^{(a,d)} \leq \frac{0.6607d}{\sigma^4} \sum_{j=1}^{n} E|X_j - a|^3 + \frac{0.3184d^2}{\sigma^2\kappa} e^{-\frac{\sigma^2\kappa^2}{2d^2}} + e^{-\frac{n\upsilon}{4}\min\left(1, \left(\frac{n\kappa}{2\pi}\right)^2\right)}$$

*where $\kappa = \frac{d}{3\left(\sum_{j=1}^{n} E|X_j - a|^3\right)^{\frac{1}{3}}}$.*

*Furthermore, if $X_1, X_2, \ldots, X_n$ are identically distributed and $\upsilon_j > 0$ for all $j = 1, 2, \ldots, n$, then for $n \geq \left(\frac{6\pi(E|X_1-a|^3)^{\frac{1}{3}}}{d}\right)^{\frac{3}{2}}$,*

$$\Delta_n^{(a,d)} \leq \frac{C_6}{n} + e^{-\frac{n\upsilon}{6}}$$

*where*

$$C_6 = \frac{17.8543dE|X_1 - a|^3}{\sigma_1^4}.$$

Finally, we consider the local limit theorem for the general weighted sums of Bernoulli random variables when all weights are mutually distinct positive in Chapter 4. We generalize the condition (1.3) to

$$|j_m - j_l| \neq |k_r - k_s| \tag{1.4}$$

for any $j_m \neq j_l$ in $\operatorname{Im} X_j$, $k_r \neq k_s$ in $\operatorname{Im} X_k$ and $j \neq k$ and investigate the error of the local limit theorem for these independent sums. The following theorems are our results.

**Theorem 1.12.** *Let $X_1, X_2, \ldots, X_n$ be independent integer-valued random variables such that $E|X_j|^{2+\alpha} < \infty$, where $0 < \alpha \leq 1$, for all $j = 1, 2, \ldots, n$ and let $S_n = \sum_{j=1}^{n} X_j$ with mean $\mu$ and variance $\sigma^2 > d^2$. If the condition* (1.4) *holds, then*

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \\
&\quad \leq \frac{0.0020}{\left(\sum\limits_{j=1}^{n} E|X_j|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})}{\sigma^4} \left(\sum_{j=1}^{n} E|X_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}} \\
&\quad\quad + \frac{0.3184}{\sigma^2 \tau_\alpha} e^{-\frac{\sigma^2\tau_\alpha^2}{2}} + \frac{8}{n}
\end{aligned}$$

*where* $\tau_\alpha = \frac{1}{3^{\frac{1}{\alpha}}\left(\sum\limits_{j=1}^{n} E|X_j|^{2+\alpha}\right)^{\frac{1}{2+\alpha}}}$.

**Theorem 1.13.** *Let $X_1, X_2, \ldots, X_n$ be independent integer-valued random variables such that $E|X_j|^3 < \infty$ for all $j = 1, 2, \ldots, n$ and let $S_n = \sum_{j=1}^{n} X_j$ with mean $\mu$ and variance $\sigma^2 > d^2$. If the condition* (1.4) *holds, then*

$$\sup_{k\in\mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{0.6607}{\sigma^4} \sum_{j=1}^{n} E|X_j|^3 + \frac{0.3184}{\sigma^2\tau_1} e^{-\frac{\sigma^2\tau_1^2}{2}} + \frac{8}{n}.$$

*where* $\tau_1 = \frac{1}{3\left(\sum\limits_{j=1}^{n} E|X_j|^3\right)^{\frac{1}{3}}}$.

# CHAPTER II

# Local Limit Theorems for Lattice Random Variables without Assuming Finite Third Moment

Let $X$ be any integer-valued random variable. Then, $X$ is called a *lattice random variable with parameter* $(a, d)$ where $a$ and $d > 0$ are integers, if its values belong to $\mathcal{L}(a, d) = \{a + md \mid m \in \mathbb{Z}\}$. In addition, $d$ is said to be *maximal*, if there are no other numbers $a'$ and $d' > d$ such that $P(X \in \mathcal{L}(a', d')) = 1$; we call $X$ a *maximal lattice random variable with parameter* $(a, d)$, if $X$ is a lattice random variable with parameter $(a, d)$ and $d$ is maximal. In this chapter, we relax the third moment condition to find the local limit theorems for sums of independent integer-valued lattice random variables and also give explicit constants of the error bound.

Throughout this chapter, let $X_1, X_2, \ldots, X_n$ be independent integer-valued lattice random variables with parameter $(a, d)$, mean $\mu_j$ and variance $\sigma_j^2$ such that $E|X_j|^{2+\alpha} < \infty$, where $0 < \alpha \leq 1$, for all $j = 1, 2, \ldots, n$ and let

$$S_n = \sum_{j=1}^{n} X_j, \qquad \mu = \sum_{j=1}^{n} \mu_j, \quad \text{and} \quad \sigma^2 = \sum_{j=1}^{n} \sigma_j^2.$$

We use the ideas of Siripraparat and Neammanee [9], Giuliano and Weber [21] and Sunklodas [24] to obtain Theorem 2.1 and Theorem 2.2. These theorems are the local limit theorems in which the limit function is estimated by normal density function. We denote

$$\Delta_n^{(a,d)} = \sup_{k \in \mathbb{Z}} \left| P(S_n = na + kd) - \frac{d}{\sigma\sqrt{2\pi}} e^{-\frac{(na+kd-\mu)^2}{2\sigma^2}} \right|.$$

The following theorems and corollaries are our main results of this chapter. Let

$$\beta := \sum_{j=1}^{n} \beta_j$$

where $\beta_j := 2 \sum\limits_{m=-\infty}^{\infty} p_{jm} p_{j(m+1)}$ and $p_{jm} := P(X_j = a + md)$.

**Theorem 2.1.** *If $\sigma^2 > d^2$ and $\beta > 0$, then*

$$\Delta_n^{(a,d)} \le \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{\left(\sum\limits_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})d^{\frac{\alpha^2-\alpha+2}{2}}}{\sigma^4}\left(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{3-\alpha}{2}} + \frac{0.3184 d^2}{\sigma^2 \tau} e^{-\frac{\sigma^2\tau^2}{2d^2}} + \frac{1.5708}{\tau\beta} e^{-\frac{\tau^2\beta}{\pi^2}}$$

*where* $\tau = \dfrac{d}{3^{\frac{1}{\alpha}}\left(\sum\limits_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1}{2+\alpha}}}$.

*Furthermore, if $X_1, X_2, \ldots, X_n$ are identically distributed such that $\sigma^2 > d^2$ and $\beta_1 > 0$, then*

$$\Delta_n^{(a,d)} \le \frac{C_1}{n^{\frac{1+\alpha}{2}}}$$

*where*

$$C_1 = \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1 - a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})d^{\frac{\alpha^2-\alpha+2}{2}}(E|X_1 - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4} + \frac{0.6368(3^{\frac{3}{\alpha}})d(E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4} + \frac{15.5032(3^{\frac{3}{\alpha}})(E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{d^3\beta_1^2}.$$

Note that $\beta_j = 0$ if and only if $P(X_j = a + md)P(X_j = a + md + d) = 0$ for all $m \in \mathbb{Z}$. So, if $\beta_j = 0$, then we can find $d' > d$ such that $P(X_j \in \mathcal{L}(a, d')) = 1$ which implies that $d$ is not maximal. Hence, we can apply Theorem 2.1 when $d$ is maximal.

Let

$$\upsilon := \min_{1 \le j \le n} \upsilon_j$$

where $\upsilon_j := 2 \sum\limits_{m=-\infty}^{\infty} p_{jm} p_{j(m+j)}$.

**Theorem 2.2.** *If $\sigma^2 > d^2$ and $\upsilon_j > 0$ for all $j = 1, 2, \ldots, n$, then*

$$\Delta_n^{(a,d)} \leq \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{\left(\sum\limits_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}}) d^{\frac{\alpha^2-\alpha+2}{2}}}{\sigma^4} \left(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{3-\alpha}{2}}$$
$$+ \frac{0.3184 d^2}{\sigma^2 \tau} e^{-\frac{\sigma^2 \tau^2}{2d^2}} + \exp\left(-\frac{n\upsilon}{4} \min\left(1, \left(\frac{n\tau}{2\pi}\right)^2\right)\right)$$

*where* $\tau = \dfrac{d}{3^{\frac{1}{\alpha}} \left(\sum\limits_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1}{2+\alpha}}}$.

*Furthermore, if $X_1, X_2, \ldots, X_n$ are identically distributed such that $\sigma^2 > d^2$ and $\upsilon_j > 0$ for all $j = 1, 2, \ldots, n$, then for $n \geq \left(\frac{2\pi(3^{\frac{1}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d}\right)^{\frac{2+\alpha}{1+\alpha}}$,*

$$\Delta_n^{(a,d)} \leq \frac{C_2}{n^{\frac{1+\alpha}{2}}} + e^{-\frac{n\upsilon}{4}}$$

*where*

$$C_2 = \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1 - a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}}) d^{\frac{\alpha^2-\alpha+2}{2}} (E|X_1 - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4}$$
$$+ \frac{0.6368(3^{\frac{3}{\alpha}}) d (E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4}.$$

We directly obtain Corollary 2.3 and Corollary 2.4 from Theorem 2.1 and Theorem 2.2 in the case of $\alpha = 1$, respectively.

**Corollary 2.3.** *If $\sigma^2 > d^2$, $\beta > 0$ and $E|X_j|^3 < \infty$ for all $j = 1, 2, \ldots, n$, then*

$$\Delta_n^{(a,d)} \leq \frac{0.0020 d^3}{\sum\limits_{j=1}^{n} E|X_j - a|^3} + \frac{13.8513 d}{\sigma^4} \sum_{j=1}^{n} E|X_j - a|^3 + \frac{0.3184 d^2}{\sigma^2 \kappa} e^{-\frac{\sigma^2 \kappa^2}{2d^2}}$$
$$+ \frac{1.5708}{\kappa\beta} e^{-\frac{\kappa^2 \beta}{\pi^2}}$$

*where* $\kappa = \dfrac{d}{3\Big(\sum\limits_{j=1}^{n} E|X_j - a|^3\Big)^{\frac{1}{3}}}$.

*Furthermore, if* $X_1, X_2, \ldots, X_n$ *are identically distributed such that* $\sigma^2 > d^2$ *and* $\beta_1 > 0$*, then*

$$\Delta_n^{(a,d)} \leq \frac{C_3}{n}$$

*where*

$$C_3 = \frac{0.0020d^3}{E|X_1 - a|^3} + \frac{31.0449dE|X_1 - a|^3}{\sigma_1^4} + \frac{418.5864E|X_1 - a|^3}{d^3\beta_1^2}.$$

**Corollary 2.4.** *If* $\sigma^2 > d^2$*,* $\upsilon_j > 0$ *and* $E|X_j|^3 < \infty$ *for all* $j = 1, 2, \ldots, n$*, then*

$$\Delta_n^{(a,d)} \leq \frac{0.0020d^3}{\sum\limits_{j=1}^{n} E|X_j - a|^3} + \frac{13.8513d}{\sigma^4}\sum_{j=1}^{n} E|X_j - a|^3 + \frac{0.3184d^2}{\sigma^2\kappa}e^{-\frac{\sigma^2\kappa^2}{2d^2}} + e^{-\frac{n\upsilon}{4}\min\left(1, \left(\frac{n\kappa}{2\pi}\right)^2\right)}$$

*where* $\kappa = \dfrac{d}{3\Big(\sum\limits_{j=1}^{n} E|X_j - a|^3\Big)^{\frac{1}{3}}}$.

*Furthermore, if* $X_1, X_2, \ldots, X_n$ *are identically distributed such that* $\sigma^2 > d^2$ *and* $\upsilon_j > 0$ *for all* $j = 1, 2, \ldots, n$*, then for* $n \geq \left(\frac{6\pi(E|X_1 - a|^3)^{\frac{1}{3}}}{d}\right)^{\frac{3}{2}}$*,*

$$\Delta_n^{(a,d)} \leq \frac{C_4}{n} + e^{-\frac{n\upsilon}{4}}$$

*where*

$$C_4 = \frac{0.0020d^3}{E|X_1 - a|^3} + \frac{31.0449dE|X_1 - a|^3}{\sigma_1^4}.$$

We organize this chapter as follows. First, we give auxiliary results in Section 2.1. These results will be used to prove the main theorems in Section 2.2. Some examples will be given in Section 2.3.

## 2.1 Auxiliary results

Let $\psi_X$ be the characteristic function of a random variable $X$. The characteristic function is important in probability theory and statistics, especially in local limit theorem. In the study of local limit theorems, it is required to estimate bounds for modulus $|\psi_X|$ of a characteristic function $\psi_X$. The various bounds for $|\psi_X|$ play a key role in the investigation of the rate of convergence in the local limit theorem.

**Lemma 2.5.** *[9] Let $X$ be any integer-valued random variable. Then, for $t \in [0, \pi)$,*

$$|\psi_X(t)| \le e^{-\frac{1}{\pi^2}\beta_X t^2}$$

*where $\beta_X = 2 \sum\limits_{m=-\infty}^{\infty} P(X = m)P(X = m+1)$.*

In the following lemmas, we use an idea from ([24]) to give bounds of a characteristic function.

**Lemma 2.6.** *Let $X$ be any integer-valued random variable with mean $\mu_X$ and variance $\sigma_X^2$. If $E|X|^{2+\alpha} < \infty$ for some $0 < \alpha \le 1$, then, for all $|t| \le \left(\frac{1}{3E|X|^\alpha}\right)^{\frac{1}{\alpha}}$,*

$(i)$ $|\psi_X(t)| \ge \frac{1}{3}$ *and*

$(ii)$ *there exists a function $g_X$ such that $\psi_X(t) = \exp\left\{i\mu_X t - \frac{1}{2}\sigma_X^2 t^2 + \int\limits_0^t \frac{g_X(s)}{\psi_X(s)}\, ds\right\}$ and $\int\limits_0^t \left|\frac{g_X(s)}{\psi_X(s)}\right| ds \le 9E|X|^{2+\alpha}|t|^{2+\alpha}$.*

**Proof.** $(i)$ Using the fact that for $x \in \mathbb{R}$, $e^{ix} = 1 + 2^{1-\alpha}|x|^\alpha\Theta$ for some complex number $\Theta$ such that $|\Theta| \le 1$ ([24], p.359), we get that

$$Ee^{itX} = E(1 + \Theta_1 2^{1-\alpha}|tX|^\alpha) = 1 + 2^{1-\alpha}E(\Theta_1|X|^\alpha)|t|^\alpha \tag{2.1}$$

where $\Theta_1$ is a complex random variable such that $|\Theta_1| \le 1$. From this fact and the inequality $|z_1 + z_2| \ge |z_1| - |z_2|$ for complex numbers $z_1$ and $z_2$, we can see that

$$|Ee^{itX}| = \left|1 + 2^{1-\alpha}E(\Theta_1|X|^\alpha)|t|^\alpha\right|$$

$$\geq 1 - 2^{1-\alpha}E(|\Theta_1||X|^{\alpha})|t|^{\alpha}$$
$$\geq 1 - 2^{1-\alpha}E|X|^{\alpha}|t|^{\alpha} \tag{2.2}$$
$$\geq 1 - 2E|X|^{\alpha}|t|^{\alpha}.$$

Then, for all $|t| \leq \left(\frac{1}{3E|X|^{\alpha}}\right)^{\frac{1}{\alpha}}$, we have $|\psi_X(t)| = |Ee^{itX}| \geq \frac{1}{3}$.
$(ii)$ Let $t \in \mathbb{R}$ be such that $|t| \leq \left(\frac{1}{3E|X|^{\alpha}}\right)^{\frac{1}{\alpha}}$. Since

$$\psi_X(t) = Ee^{itX} = \sum_{m=-\infty}^{\infty} e^{itm}P(X=m),$$

we obtain

$$\psi_X'(t) = \sum_{m=-\infty}^{\infty} ime^{itm}P(X=m) = iE(Xe^{itX})$$

which implies that

$$\begin{aligned}\psi_X'(t) &= \frac{\psi_X(t)}{Ee^{itX}}\left(\frac{d}{dt}Ee^{itX}\right)\\ &= \left((i\mu_X - \sigma_X^2 t) - (i\mu_X - \sigma_X^2 t) + \frac{1}{Ee^{itX}}\left(\frac{d}{dt}Ee^{itX}\right)\right)\psi_X(t)\\ &= \left((i\mu_X - \sigma_X^2 t) + \frac{g_X(t)}{\psi_X(t)}\right)\psi_X(t)\end{aligned}$$

where

$$g_X(t) = -(i\mu_X - \sigma_X^2 t)Ee^{itX} + iE(Xe^{itX}).$$

Hence,

$$\frac{\psi_X'(t)}{\psi_X(t)} = i\mu_X - \sigma_X^2 t + \frac{g_X(t)}{\psi_X(t)}$$

and then

$$\ln\psi_X(t) = \int_0^t \frac{\psi_X'(s)}{\psi_X(s)}\,\mathrm{d}s = i\mu_X t - \frac{1}{2}\sigma_X^2 t^2 + \int_0^t \frac{g_X(s)}{\psi_X(s)}\,\mathrm{d}s,$$

which implies that

$$\psi_X(t) = \exp\left\{i\mu_X t - \frac{1}{2}\sigma_X^2 t^2 + \int_0^t \frac{g_X(s)}{\psi_X(s)}\,\mathrm{d}s\right\}. \tag{2.3}$$

From the fact that for $x \in \mathbb{R}$, $e^{ix} = 1 + ix + \frac{2^{1-\alpha}}{1+\alpha}|x|^{1+\alpha}\Theta$ for some complex number $\Theta$ such that $|\Theta| \leq 1$ ([24], p.359), we have that

$$Ee^{itX} = 1 + itEX + \frac{2^{1-\alpha}}{1+\alpha}E(\Theta_2|X|^{1+\alpha})|t|^{1+\alpha} \tag{2.4}$$

and
$$iE(Xe^{itX}) = i\mu_X - tEX^2 + \frac{2^{1-\alpha}}{1+\alpha}E(i\Theta_2|X|^{2+\alpha})|t|^{1+\alpha} \tag{2.5}$$

where $\Theta_2$ is a complex random variable such that $|\Theta_2| \leq 1$. From (2.1) and (2.4), we obtain that

$$\sigma_X^2 t(Ee^{itX}) = \sigma_X^2 t + 2^{1-\alpha}\sigma_X^2 E(\Theta_1|X|^{\alpha})|t|^{1+\alpha} \tag{2.6}$$

and
$$-i\mu_X(Ee^{itX}) = -i\mu_X + \mu_X^2 t - \frac{2^{1-\alpha}}{1+\alpha}\mu_X E(i\Theta_2|X|^{1+\alpha})|t|^{1+\alpha}. \tag{2.7}$$

Adding (2.5) - (2.7) and the fact that $\sigma_X^2 = EX^2 - \mu_X^2$, we have

$$\begin{aligned} g_X(t) &= -i\mu_X Ee^{itX} + \sigma_X^2 tEe^{itX} + iE(Xe^{itX}) \\ &= \frac{2^{1-\alpha}}{1+\alpha}\mu_X E(i\Theta_2|X|^{1+\alpha})|t|^{1+\alpha} + 2^{1-\alpha}\sigma_X^2 E(\Theta_1|X|^{\alpha})|t|^{1+\alpha} \\ &\quad + \frac{2^{1-\alpha}}{1+\alpha}E(i\Theta_2|X|^{2+\alpha})|t|^{1+\alpha}. \end{aligned} \tag{2.8}$$

According to the Lyapunov's inequality: $(E|X|^r)^{\frac{1}{r}} \leq (E|X|^s)^{\frac{1}{s}}$ where $0 < r \leq s$, we have that $E|X| \leq (E|X|^{2+\alpha})^{\frac{1}{2+\alpha}}$ and $E|X|^{1+\alpha} \leq (E|X|^{2+\alpha})^{\frac{1+\alpha}{2+\alpha}}$ which imply that

$$E|X|E|X|^{1+\alpha} \leq E|X|^{2+\alpha}.$$

We can use the same technique to show that

$$\sigma_X^2 E|X|^{\alpha} \leq EX^2 E|X|^{\alpha} \leq E|X|^{2+\alpha}.$$

From these facts and (2.8), we have

$$\begin{aligned}
\big|g_X(t)\big| &\le \frac{2^{1-\alpha}}{1+\alpha}E|X|E(|i\Theta_2||X|^{1+\alpha})|t|^{1+\alpha} + 2^{1-\alpha}\sigma_X^2E(|\Theta_1||X|^{\alpha})|t|^{1+\alpha}\\
&\quad + \frac{2^{1-\alpha}}{1+\alpha}E(|i\Theta_2||X|^{2+\alpha})|t|^{1+\alpha}\\
&\le \frac{2^{1-\alpha}}{1+\alpha}E|X|E(|X|^{1+\alpha})|t|^{1+\alpha} + 2^{1-\alpha}\sigma_X^2E(|X|^{\alpha})|t|^{1+\alpha}\\
&\quad + \frac{2^{1-\alpha}}{1+\alpha}E(|X|^{2+\alpha})|t|^{1+\alpha}\\
&\le \frac{2^{1-\alpha}}{1+\alpha}E(|X|^{2+\alpha})|t|^{1+\alpha} + 2^{1-\alpha}E(|X|^{2+\alpha})|t|^{1+\alpha}\\
&\quad + \frac{2^{1-\alpha}}{1+\alpha}E(|X|^{2+\alpha})|t|^{1+\alpha}\\
&= \left(2^{1-\alpha} + \frac{2^{2-\alpha}}{1+\alpha}\right)E|X|^{2+\alpha}|t|^{1+\alpha} \qquad (2.9)\\
&\le 6E|X|^{2+\alpha}|t|^{1+\alpha} \qquad (2.10)
\end{aligned}$$

where we use the fact that $2^{1-\alpha} + \frac{2^{2-\alpha}}{1+\alpha}$ is decreasing on $(0,1]$ in the last inequality. Hence, we can conclude from $(i)$ and (2.10) that for all $|t| \le \left(\frac{1}{3E|X|^{\alpha}}\right)^{\frac{1}{\alpha}}$, we have

$$\left|\frac{g_X(t)}{\psi_X(t)}\right| \le 18E|X|^{2+\alpha}|t|^{1+\alpha}$$

which implies that

$$\int_0^t \left|\frac{g_X(s)}{\psi_X(s)}\right| \mathrm{d}s \le \frac{18}{2+\alpha}E|X|^{2+\alpha}|t|^{2+\alpha} \le 9E|X|^{2+\alpha}|t|^{2+\alpha}. \qquad \square$$

**Lemma 2.7.** *Let $X_1, X_2, \ldots, X_n$ be independent integer-valued random variables and $W_n = \sum_{j=1}^{n} X_j$ with $E(W_n) = \mu_w$, $Var(W_n) = \sigma_w^2$ and the characteristic function $\psi_w$. Assume that $E|X_j|^{2+\alpha} < \infty$ for all $j = 1, 2, \ldots, n$, where $0 < \alpha \le 1$, and let $\tau_0 = \frac{1}{3^{\frac{1}{\alpha}}}\left(\frac{1}{\sum_{j=1}^{n} E|X_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}}$. Then,*

$$\left|\psi_w(t) - e^{it\mu_w - \frac{1}{2}\sigma_w^2 t^2}\right| \le 12.5606\sum_{j=1}^{n} E|X_j|^{2+\alpha}|t|^{2+\alpha}e^{-\frac{1}{2}\sigma_w^2 t^2}$$

*for all* $|t| \le \tau_0$.

**Proof.** From Lyapunov's inequality, we have

$$(E|X_l|^{\alpha})^{\frac{1}{\alpha}} \le (E|X_l|^{2+\alpha})^{\frac{1}{2+\alpha}}$$

which implies that

$$\left(\frac{1}{\sum\limits_{j=1}^{n} E|X_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}} \le \left(\frac{1}{E|X_l|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}} \le \left(\frac{1}{E|X_l|^{\alpha}}\right)^{\frac{1}{\alpha}}$$

for all $l = 1, 2, \ldots, n$. This provides that

$$\left(\frac{1}{3^{\frac{2+\alpha}{\alpha}} \sum\limits_{j=1}^{n} E|X_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}} \le \left(\frac{1}{3E|X_l|^{\alpha}}\right)^{\frac{1}{\alpha}}$$

for all $l = 1, 2, \ldots, n$. From this fact and Lemma 2.6, we have that for all $|t| \le \tau_0$,

$$\psi_{W}(t) = \exp\left\{i\mu_{W} t - \frac{1}{2}\sigma_{W}^2 t^2 + \sum_{j=1}^{n} G_j(t)\right\} \tag{2.11}$$

where $G_j(t) = \int\limits_0^t \frac{g_{X_j}(s)}{\psi_{X_j}(s)}\, \mathrm{d}s$ and $\left|G_j(t)\right| \le 9E|X_j|^{2+\alpha}|t|^{2+\alpha}$.

From (2.11) and the inequality $|e^z - 1| \le |z|e^{|z|}$ for a complex number $z$, we get that for all $|t| \le \tau_0$,

$$\begin{aligned}
\left|\psi_{W}(t) - e^{i\mu_W t - \frac{1}{2}\sigma_{W}^2 t^2}\right| &= \left|e^{i\mu_W t - \frac{1}{2}\sigma_{W}^2 t^2 + \sum\limits_{j=1}^{n} G_j(t)} - e^{i\mu_W t - \frac{1}{2}\sigma_{W}^2 t^2}\right| \\
&= \left|e^{i\mu_W t - \frac{1}{2}\sigma_{W}^2 t^2}\right| \left|e^{\sum\limits_{j=1}^{n} G_j(t)} - 1\right| \\
&\le \left|\sum_{j=1}^{n} G_j(t)\right| e^{-\frac{1}{2}\sigma_{W}^2 t^2 + \left|\sum\limits_{j=1}^{n} G_j(t)\right|} \\
&\le \sum_{j=1}^{n} \left|G_j(t)\right| e^{-\frac{1}{2}\sigma_{W}^2 t^2 + \sum\limits_{j=1}^{n} \left|G_j(t)\right|}
\end{aligned}$$

$$\leq 9\sum_{j=1}^{n} E|X_j|^{2+\alpha}|t|^{2+\alpha} \times \exp\Big\{-\frac{1}{2}\sigma_{w}^2 t^2 + 9\sum_{j=1}^{n} E|X_j|^{2+\alpha}|t|^{2+\alpha}\Big\}$$

$$\leq 9\sum_{j=1}^{n} E|X_j|^{2+\alpha}|t|^{2+\alpha} \times \exp\Big\{-\frac{1}{2}\sigma_{w}^2 t^2 + \frac{9}{3^{\frac{2+\alpha}{\alpha}}}\Big\}$$

$$\leq 9\sum_{j=1}^{n} E|X_j|^{2+\alpha}|t|^{2+\alpha} \times \exp\Big\{-\frac{1}{2}\sigma_{w}^2 t^2 + \frac{9}{3^3}\Big\}$$

$$\leq 12.5606\sum_{j=1}^{n} E|X_j|^{2+\alpha}|t|^{2+\alpha} e^{-\frac{1}{2}\sigma_{w}^2 t^2}. \qquad \square$$

**Lemma 2.8.** *Let $X_1, X_2, \ldots, X_n$ be independent integer-valued random variables and $W_n = \sum_{j=1}^{n} X_j$ with the characteristic function $\psi_w$. Then, for $|t| \leq \pi$ and $n \geq 2$,*

$$|\psi_w(t)| \leq \exp\Big(-\frac{n\upsilon_w}{4}\min\Big(1, \Big(\frac{nt}{2\pi}\Big)^2\Big)\Big)$$

*where $\upsilon_w = \min_{1\leq j\leq n} \upsilon_j$ and $\upsilon_j = 2\sum_{m=-\infty}^{\infty} P(X_j = m)P(X_j = m+j)$.*

**Proof.** Let $\psi_j$ be the characteristic function of $X_j$. Siripraparat and Neammanee ([9], p.6) showed that

$$\ln(|\psi_j(t)|) \leq -\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} P(X_j = m)P(X_j = l)\sin^2\Big((m-l)\frac{t}{2}\Big).$$

From this fact and the fact that

$$\sum_{j=1}^{n}\sin^2\Big(\frac{jt}{2}\Big) \geq \frac{n}{4}\min\Big(1, \Big(\frac{nt}{2\pi}\Big)^2\Big)$$

for $|t| \leq \pi$ and $n \geq 2$ ([25], p.399), we have

$$|\psi_w(t)| = \prod_{j=1}^{n} |\psi_j(t)|$$

$$\leq \prod_{j=1}^{n} \exp\Big(-2\sum_{m=-\infty}^{\infty} P(X_j = m)P(X_j = m+j)\sin^2\Big(\frac{jt}{2}\Big)\Big)$$

$$\leq \exp\left(-\sum_{j=1}^{n} \upsilon_j \sin^2\left(\frac{jt}{2}\right)\right)$$
$$\leq \exp\left(-\upsilon_{\scriptscriptstyle W} \sum_{j=1}^{n} \sin^2\left(\frac{jt}{2}\right)\right)$$
$$\leq \exp\left(-\frac{n\upsilon_{\scriptscriptstyle W}}{4} \min\left(1, \left(\frac{nt}{2\pi}\right)^2\right)\right). \qquad \square$$

## 2.2 Proof of main results

### 2.2.1 Proof of Theorem 2.1

**Proof.** First, we will prove the theorem in the case of $a = 0$ and $d = 1$. Let $Y_1, Y_2, \ldots, Y_n$ be independent integer-valued random variables and let

$$W_n = Y_1 + Y_2 + \cdots + Y_n$$

with $E(W_n) = \mu_{\scriptscriptstyle W}$, $Var(W_n) = \sigma_{\scriptscriptstyle W}^2 > 1$ and the characteristic function $\psi_{\scriptscriptstyle W}$. Suppose that $\beta_{\scriptscriptstyle W} := \sum_{j=1}^{n} \beta_{Y_j} > 0$ where $\beta_{Y_j} = 2 \sum_{m=-\infty}^{\infty} P(Y_j = m)P(Y_j = m+1) > 0$ for all $j = 1, 2, \ldots, n$ and let $\tau_0 = \frac{1}{3^{\frac{1}{\alpha}}}\left(\frac{1}{\sum_{j=1}^{n} E|Y_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}}$. Since $\sum_{j=1}^{n} E|Y_j|^{2+\alpha} \geq \sum_{j=1}^{n} E(Y_j)^2 \geq \sigma_{\scriptscriptstyle W}^2 > 1$, we have that

$$\tau_0 = \frac{1}{3^{\frac{1}{\alpha}}}\left(\frac{1}{\sum_{j=1}^{n} E|Y_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}} < \left(\frac{1}{\sum_{j=1}^{n} E|Y_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}} < 1.$$

From this fact and the fact that $P(W_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\psi_{\scriptscriptstyle W}(t)\, \mathrm{d}t$ ([26], p.511), we have

$$\left|P(W_n = k) - \frac{1}{\sigma_{\scriptscriptstyle W}\sqrt{2\pi}} e^{-\frac{(k-\mu_{\scriptscriptstyle W})^2}{2\sigma_{\scriptscriptstyle W}^2}}\right|$$
$$= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\psi_{\scriptscriptstyle W}(t)\, \mathrm{d}t - \frac{1}{\sigma_{\scriptscriptstyle W}\sqrt{2\pi}} e^{-\frac{(k-\mu_{\scriptscriptstyle W})^2}{2\sigma_{\scriptscriptstyle W}^2}}\right|$$

$$\leq \frac{1}{2\pi}\Big|\int\limits_{|t|<\tau_0} e^{-ikt}\psi_{W}(t)\,\mathrm{d}t - \int\limits_{|t|<\tau_0} e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t\Big|$$

$$+\frac{1}{2\pi}\Big|\int\limits_{|t|<\tau_0} e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t - \frac{\sqrt{2\pi}}{\sigma_W}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\Big|$$

$$+\frac{1}{2\pi}\Big|\int\limits_{\tau_0\leq|t|\leq\pi} e^{-ikt}\psi_{W}(t)\,\mathrm{d}t\Big|$$

$$:= |A|+|B|+|C| \tag{2.12}$$

where

$$A := \frac{1}{2\pi}\int\limits_{|t|<\tau_0} e^{-ikt}\psi_{W}(t)\,\mathrm{d}t - \int\limits_{|t|<\tau_0} e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t,$$

$$B := \frac{1}{2\pi}\int\limits_{|t|<\tau_0} e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t - \frac{\sqrt{2\pi}}{\sigma_W}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}},$$

$$\text{and}\quad C := \frac{1}{2\pi}\int\limits_{\tau_0\leq|t|\leq\pi} e^{-ikt}\psi_{W}(t)\,\mathrm{d}t.$$

From Lemma 2.7, we have

$$|A| \leq \frac{1}{2\pi}\int\limits_{|t|<\tau_0} |e^{-ikt}||\psi_{W}(t) - e^{it\mu_W-\frac{1}{2}\sigma_W^2t^2}|\,\mathrm{d}t$$

$$= \frac{1}{2\pi}\int\limits_{|t|<\tau_0} |\psi_{W}(t) - e^{it\mu_W-\frac{1}{2}\sigma_W^2t^2}|\,\mathrm{d}t$$

$$\leq \frac{12.5606}{\pi}\sum_{j=1}^{n} E|Y_j|^{2+\alpha}\int_0^{\tau_0} |t|^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t.$$

We let $\tilde{\tau_0} = \tau_0^{\frac{2+\alpha}{2}}$. Thus, $\tilde{\tau_0} < \tau_0$ since $\tau_0 < 1$. This implies that

$$\int_0^{\tau_0} |t|^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t = \int_0^{\tilde{\tau_0}} t^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t + \int_{\tilde{\tau_0}}^{\tau_0} t^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t.$$

From the fact that $3^{\frac{\alpha^2+7\alpha+6}{2\alpha}}$ is decreasing on $(0,1]$, we have that

$$\int\limits_0^{\tilde{\tau_0}} t^{2+\alpha}e^{-\frac{1}{2}\sigma_{_W}^2t^2}\ \mathrm{d}t \leq \int\limits_0^{\tilde{\tau_0}} t^{2+\alpha}\ \mathrm{d}t$$
$$= \frac{\tau_0^{\frac{(2+\alpha)(3+\alpha)}{2}}}{3+\alpha}$$
$$\leq \frac{1}{3^{\frac{\alpha^2+7\alpha+6}{2\alpha}}\left(\sum\limits_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{3+\alpha}{2}}}$$
$$\leq \frac{0.0005}{\left(\sum\limits_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{3+\alpha}{2}}}.$$

To bound $\int\limits_{\tilde{\tau_0}}^{\tau_0} t^{2+\alpha}e^{-\frac{1}{2}\sigma_{_W}^2t^2}\ \mathrm{d}t$, we use the facts that $3^{\frac{(2+\alpha)(1-\alpha)}{2\alpha}} = \frac{3^{\frac{1}{\alpha}}}{3^{\frac{\alpha+1}{2}}}$ and that $\frac{1}{3^{\frac{\alpha+1}{2}}}$ is decreasing on $(0,1]$ to obtain that

$$\int\limits_{\tilde{\tau_0}}^{\tau_0} t^{2+\alpha}e^{-\frac{1}{2}\sigma_{_W}^2t^2}\ \mathrm{d}t = \int\limits_{\tilde{\tau_0}}^{\tau_0} \frac{t^3}{t^{1-\alpha}}e^{-\frac{1}{2}\sigma_{_W}^2t^2}\ \mathrm{d}t$$
$$\leq \frac{1}{\tau_0^{\frac{(2+\alpha)(1-\alpha)}{2}}}\int\limits_{\tilde{\tau_0}}^{\tau_0} t^3e^{-\frac{1}{2}\sigma_{_W}^2t^2}\ \mathrm{d}t$$
$$\leq \frac{1}{\tau_0^{\frac{(2+\alpha)(1-\alpha)}{2}}}\int\limits_{0}^{\infty} t^3e^{-\frac{1}{2}\sigma_{_W}^2t^2}\ \mathrm{d}t$$
$$= 3^{\frac{(2+\alpha)(1-\alpha)}{2\alpha}}\left(\sum_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{1-\alpha}{2}}\left(\frac{2}{\sigma_{_W}^4}\right)$$
$$\leq \frac{1.1548(3^{\frac{1}{\alpha}})}{\sigma_{_W}^4}\left(\sum_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{1-\alpha}{2}}.$$

Hence,

$$|A| \leq \frac{12.5606}{\pi}\sum_{j=1}^{n} E|Y_j|^{2+\alpha}\Bigg(\int\limits_0^{\tilde{\tau_0}} t^{2+\alpha}e^{-\frac{1}{2}\sigma_{_W}^2t^2}\ \mathrm{d}t + \int\limits_{\tilde{\tau_0}}^{\tau_0} t^{2+\alpha}e^{-\frac{1}{2}\sigma_{_W}^2t^2}\ \mathrm{d}t\Bigg)$$

$$\leq \frac{0.0020}{\left(\sum\limits_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})}{\sigma_{_W}^4}\left(\sum_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}}. \tag{2.13}$$

By the fact that

$$\begin{aligned}
\int\limits_{|t|<\tau_0} e^{it(\mu_{_W}-k)-\frac{1}{2}\sigma_{_W}^2t^2}\,\mathrm{d}t &= \int\limits_{\mathbb{R}} e^{it(\mu_{_W}-k)-\frac{1}{2}\sigma_{_W}^2t^2}\,\mathrm{d}t - \int\limits_{|t|\geq\tau_0} e^{it(\mu_{_W}-k)-\frac{1}{2}\sigma_{_W}^2t^2}\,\mathrm{d}t \\
&= \frac{1}{\sigma_{_W}}\int\limits_{\mathbb{R}} e^{\frac{it(\mu_{_W}-k)}{\sigma_{_W}}-\frac{t^2}{2}}\,\mathrm{d}t - \frac{1}{\sigma_{_W}}\int\limits_{|t|\geq\sigma_{_W}\tau_0} e^{\frac{it(\mu_{_W}-k)}{\sigma_{_W}}-\frac{t^2}{2}}\,\mathrm{d}t \\
&= \frac{\sqrt{2\pi}}{\sigma_{_W}}e^{-\frac{(k-\mu_{_W})^2}{2\sigma_{_W}^2}} - \frac{1}{\sigma_{_W}}\int\limits_{|t|\geq\sigma_{_W}\tau_0} e^{\frac{it(\mu_{_W}-k)}{\sigma_{_W}}-\frac{t^2}{2}}\,\mathrm{d}t,
\end{aligned}$$

we have

$$\begin{aligned}
B &= \frac{1}{2\pi}\int\limits_{|t|<\tau_0} e^{it(\mu_{_W}-k)-\frac{1}{2}\sigma_{_W}^2t^2}\,\mathrm{d}t - \frac{1}{\sigma_{_W}\sqrt{2\pi}}e^{-\frac{(k-\mu_{_W})^2}{2\sigma_{_W}^2}} \\
&= -\frac{1}{2\pi\sigma_{_W}}\int\limits_{|t|\geq\sigma_{_W}\tau_0} e^{\frac{it(\mu_{_W}-k)}{\sigma_{_W}}-\frac{t^2}{2}}\,\mathrm{d}t
\end{aligned}$$

and hence,

$$\begin{aligned}
|B| &\leq \frac{1}{2\pi\sigma_{_W}}\int\limits_{|t|\geq\sigma_{_W}\tau_0} e^{-\frac{t^2}{2}}\,\mathrm{d}t \\
&\leq \frac{1}{\pi\sigma_{_W}}\int\limits_{\sigma_{_W}\tau_0}^{\infty} e^{-\frac{t^2}{2}}\,\mathrm{d}t \\
&\leq \frac{1}{\pi\sigma_{_W}^2\tau_0}\int\limits_{\sigma_{_W}\tau_0}^{\infty} te^{-\frac{t^2}{2}}\,\mathrm{d}t \\
&= \frac{0.3184}{\sigma_{_W}^2\tau_0}e^{\frac{-\sigma_{_W}^2\tau_0^2}{2}}.
\end{aligned} \tag{2.14}$$

By Lemma 2.5, we have

$$
\begin{aligned}
|C| &= \Big|\frac{1}{2\pi}\int\limits_{\tau_0\le|t|\le\pi} e^{-ikt}\psi_{w}(t)\ \mathrm{d}t\Big| \\
&\le \frac{1}{2\pi}\int\limits_{\tau_0\le|t|\le\pi} |\psi_{w}(t)|\ \mathrm{d}t \\
&= \frac{1}{\pi}\int\limits_{\tau_0}^{\pi} |\psi_{w}(t)|\ \mathrm{d}t \\
&\le \frac{1}{\pi}\int\limits_{\tau_0}^{\pi} e^{-\frac{1}{\pi^2}\beta_{w} t^2}\ \mathrm{d}t \\
&\le \frac{1}{\pi\tau_0}\int\limits_{\tau_0}^{\infty} te^{-\frac{1}{\pi^2}\beta_{w} t^2}\ \mathrm{d}t \\
&= \frac{1}{\pi\tau_0}\left(\frac{\pi^2 e^{-\frac{\tau_0^2\beta_{w}}{\pi^2}}}{2\beta_{w}}\right) \\
&\le \frac{1.5708}{\tau_0\beta_{w}} e^{-\frac{\tau_0^2\beta_{w}}{\pi^2}}.
\end{aligned}
\tag{2.15}
$$

From (2.12) - (2.15), we have

$$
\begin{aligned}
&\Big|P(W_n = k) - \frac{1}{\sigma_{w}\sqrt{2\pi}} e^{-\frac{(k-\mu_{w})^2}{2\sigma_{w}^2}}\Big| \\
&\le \frac{0.0020}{\left(\sum\limits_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})}{\sigma_{w}^4}\left(\sum_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}} \\
&\quad + \frac{0.3184}{\sigma_{w}^2\tau_0} e^{\frac{-\sigma_{w}^2\tau_0^2}{2}} + \frac{1.5708}{\tau_0\beta_{w}} e^{-\frac{\tau_0^2\beta_{w}}{\pi^2}}.
\end{aligned}
\tag{2.16}
$$

In general, let $X_1, X_2, \dots, X_n$ be independent lattice random variables with parameter $(a, d)$. For $j = 1, 2, \dots, n$, let $Y_j = \frac{X_j - a}{d}$ and $W_n = Y_1 + Y_2 + \cdots + Y_n$. Observe that $Y_1, Y_2, \dots, Y_n$ are independent integer-valued random variables and

$$
\mu_{w} = \frac{\mu - na}{d}, \qquad \sigma_{w}^2 = \frac{\sigma^2}{d^2}, \qquad P(Y_j = m) = P(X_j = a + dm), \tag{2.17}
$$

$$E|Y_j|^{2+\alpha} = \frac{E|X_j - a|^{2+\alpha}}{d^{2+\alpha}}, \qquad \tau_0 = \tau = \frac{d}{3^{\frac{1}{\alpha}}\left(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1}{2+\alpha}}}. \tag{2.18}$$

From (2.16) - (2.18), we have

$$\begin{aligned}
&\left|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}} e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right| \\
&\le \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{\left(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}}) d^{\frac{\alpha^2-\alpha+2}{2}}\left(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha}\right)^{\frac{3-\alpha}{2}}}{\sigma^4} \\
&\quad + \frac{0.3184 d^2}{\sigma^2\tau} e^{-\frac{\sigma^2\tau^2}{2d^2}} + \frac{1.5708}{\tau\beta} e^{-\frac{\tau^2\beta}{\pi^2}}.
\end{aligned}$$

From this fact and the fact that

$$\left|P(S_n = na + kd) - \frac{d}{\sigma\sqrt{2\pi}} e^{-\frac{(na+kd-\mu)^2}{2\sigma^2}}\right| = \left|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}} e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right|,$$

we have the conclusion of the theorem.

Furthermore, if $X_1, X_2, \ldots, X_n$ are identical, then

$$\mu = n\mu_1, \qquad \sigma = \sigma_1\sqrt{n}, \qquad \sum_{j=1}^{n} E|X_j - a|^{2+\alpha} = nE|X_1 - a|^{2+\alpha} \text{ and } \beta = n\beta_1$$

which imply that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}} e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \\
&\le \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1 - a|^{2+\alpha})^{\frac{1+\alpha}{2}} n^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}}) d^{\frac{\alpha^2-\alpha+2}{2}} (E|X_1 - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4 n^{\frac{1+\alpha}{2}}} \\
&\quad + \frac{0.3184(3^{\frac{1}{\alpha}}) d (E|X_1 - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2 n^{\frac{1+\alpha}{2+\alpha}}} \exp\left(\frac{-\sigma_1^2 n^{\frac{\alpha}{2+\alpha}}}{2(3^{\frac{2}{\alpha}})(E|X_1 - a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right) \\
&\quad + \frac{1.5708(3^{\frac{1}{\alpha}})(E|X_1 - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d\beta_1 n^{\frac{1+\alpha}{2+\alpha}}} \exp\left(\frac{-d^2\beta_1 n^{\frac{\alpha}{2+\alpha}}}{3^{\frac{2}{\alpha}}\pi^2 (E|X_1 - a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right).
\end{aligned} \tag{2.19}$$

Since $\frac{1+2\alpha}{2+\alpha} = \frac{2+\alpha}{2(2+\alpha)} + \frac{3\alpha}{2(2+\alpha)} \geq \frac{1+\alpha}{2}$ and $e^{-x} \leq \frac{1}{x}$ for a real number $x > 0$, we obtain that

$$\begin{aligned}
&\frac{0.3184(3^{\frac{1}{\alpha}})d(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2 n^{\frac{1+\alpha}{2+\alpha}}} \exp\left(\frac{-\sigma_1^2 n^{\frac{\alpha}{2+\alpha}}}{2(3^{\frac{2}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right) \\
&\leq \frac{0.3184(3^{\frac{1}{\alpha}})d(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2 n^{\frac{1+\alpha}{2+\alpha}}}\left(\frac{2(3^{\frac{2}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{2}{2+\alpha}}}{\sigma_1^2 n^{\frac{\alpha}{2+\alpha}}}\right) \\
&\leq \frac{0.6368(3^{\frac{3}{\alpha}})d(E|X_1-a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4 n^{\frac{1+\alpha}{2}}} \qquad (2.20)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1.5708(3^{\frac{1}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d\beta_1 n^{\frac{1+\alpha}{2+\alpha}}} \exp\left(\frac{-d^2\beta_1 n^{\frac{\alpha}{2+\alpha}}}{3^{\frac{2}{\alpha}}\pi^2(E|X_1-a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right) \\
&\leq \frac{1.5708(3^{\frac{1}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d\beta_1 n^{\frac{1+\alpha}{2+\alpha}}}\left(\frac{3^{\frac{2}{\alpha}}\pi^2(E|X_1-a|^{2+\alpha})^{\frac{2}{2+\alpha}}}{d^2\beta_1 n^{\frac{\alpha}{2+\alpha}}}\right) \\
&\leq \frac{15.5032(3^{\frac{3}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{d^3\beta_1^2 n^{\frac{1+\alpha}{2}}}. \qquad (2.21)
\end{aligned}$$

From (2.19) - (2.21), we have

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = na+kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \leq \frac{C_1}{n^{\frac{1+\alpha}{2}}}$$

where

$$\begin{aligned}
C_1 = &\frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1-a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})d^{\frac{\alpha^2-\alpha+2}{2}}(E|X_1-a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4} \\
&+ \frac{0.6368(3^{\frac{3}{\alpha}})d(E|X_1-a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4} + \frac{15.5032(3^{\frac{3}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{d^3\beta_1^2}. \qquad \square
\end{aligned}$$

### 2.2.2 Proof of Theorem 2.2

**Proof.** By the same reason of Theorem 2.1, it suffices to prove the theorem in case $a = 0$ and $d = 1$. Let $Y_1, Y_2, \ldots, Y_n$ be independent integer-valued random variables with the characteristic functions $\psi_{Y_j}$ and let

$$W_n = Y_1 + Y_2 + \cdots + Y_n$$

with $E(W_n) = \mu_{W}$, $Var(W_n) = \sigma_{W}^2 > 1$ and the characteristic function $\psi_{W}$. Suppose that $\upsilon_{Y_j} = 2 \sum\limits_{m=-\infty}^{\infty} P(Y_j = m)P(Y_j = m + j) > 0$ for all $j = 1, 2 \ldots, n$. From (2.12) - (2.14), we have

$$\begin{aligned}
&\left| P(W_n = k) - \frac{1}{\sigma_{W}\sqrt{2\pi}} e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}} \right| \\
&\qquad \leq \frac{0.0020}{\left( \sum\limits_{j=1}^{n} E|Y_j|^{2+\alpha} \right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})}{\sigma_{W}^4} \left( \sum_{j=1}^{n} E|Y_j|^{2+\alpha} \right)^{\frac{3-\alpha}{2}} \\
&\qquad\quad + \frac{0.3184}{\sigma_{W}^2 \tau_0} e^{\frac{-\sigma_{W}^2 \tau_0^2}{2}} + |C|. \qquad (2.22)
\end{aligned}$$

where $C = \frac{1}{2\pi} \int\limits_{\tau_0 \leq |t| \leq \pi} e^{-ikt} \psi_{W}(t) \, dt$. By Lemma 2.8 and the fact that $\min(1, x) \leq \min(1, y)$ if $x \leq y$, we obtain that

$$\begin{aligned}
|C| &\leq \frac{1}{2\pi} \int\limits_{\tau_0 \leq |t| \leq \pi} |\psi_{W}(t)| \, dt \\
&= \frac{1}{\pi} \int_{\tau_0}^{\pi} |\psi_{W}(t)| \, dt \\
&\leq \frac{1}{\pi} \int_{\tau_0}^{\pi} \exp\left( -\frac{n\upsilon_{W}}{4} \min\left(1, \left(\frac{nt}{2\pi}\right)^2\right)\right) dt \\
&\leq \frac{1}{\pi} \int_{\tau_0}^{\pi} \exp\left( -\frac{n\upsilon_{W}}{4} \min\left(1, \left(\frac{n\tau_0}{2\pi}\right)^2\right)\right) dt \\
&= \frac{\pi - \tau_0}{\pi} \exp\left( -\frac{n\upsilon_{W}}{4} \min\left(1, \left(\frac{n\tau_0}{2\pi}\right)^2\right)\right) \\
&\leq \exp\left( -\frac{n\upsilon_{W}}{4} \min\left(1, \left(\frac{n\tau_0}{2\pi}\right)^2\right)\right) \qquad (2.23)
\end{aligned}$$

where $\upsilon_{W} := \min_{1\le j\le n} \upsilon_{Y_j}$. From (2.22) and (2.23),

$$
\begin{aligned}
&\Big|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}} e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\Big| \\
&= \frac{0.0020}{\left(\sum_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})}{\sigma_W^4}\left(\sum_{j=1}^{n} E|Y_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}} + \frac{0.3184}{\sigma_W^2\tau_0} e^{\frac{-\sigma_W^2\tau_0^2}{2}} \\
&\quad + e^{-\frac{n\upsilon_W}{4}\min\left(1,\left(\frac{n\tau_0}{2\pi}\right)^2\right)}.
\end{aligned}
$$

Hence, we can use this fact and (2.17) - (2.18) to obtain the conclusion.

Furthermore, if $X_1, X_2, \ldots, X_n$ are identical and $\upsilon_j > 0$ for all $j = 1, 2, \ldots, n$, then

$$
\begin{aligned}
&\sup_{k\in\mathbb{Z}}\Big|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}} e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\Big| \\
&\le \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1-a|^{2+\alpha})^{\frac{1+\alpha}{2}} n^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}}) d^{\frac{\alpha^2-\alpha+2}{2}} (E|X_1-a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4 n^{\frac{1+\alpha}{2}}} \\
&\quad + \frac{0.3184(3^{\frac{1}{\alpha}}) d (E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2 n^{\frac{1+\alpha}{2+\alpha}}} \exp\left(\frac{-\sigma_1^2 n^{\frac{\alpha}{2+\alpha}}}{2(3^{\frac{2}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right) \\
&\quad + \exp\left(-\frac{n\upsilon}{4}\min\left(1, \left(\frac{n^{\frac{1+\alpha}{2+\alpha}} d}{2\pi(3^{\frac{1}{\alpha}})(E|X_j-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}\right)^2\right)\right).
\end{aligned}
$$

From (2.20) and $n \ge \left(\frac{2\pi(3^{\frac{1}{\alpha}})(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d}\right)^{\frac{2+\alpha}{1+\alpha}}$, we obtain that

$$
\sup_{k\in\mathbb{Z}}\Big|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}} e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\Big| \le \frac{C_2}{n^{\frac{1+\alpha}{2}}} + e^{-\frac{n\upsilon}{4}}
$$

where

$$
\begin{aligned}
C_2 &= \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1-a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}}) d^{\frac{\alpha^2-\alpha+2}{2}} (E|X_1-a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4} \\
&\quad + \frac{0.6368(3^{\frac{3}{\alpha}}) d (E|X_1-a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4}. \qquad \square
\end{aligned}
$$

## 2.3 Examples

In the following examples, we apply Theorem 2.1 and Theorem 2.2 with integer-valued lattice random variables without finite third moment but the $(2+\alpha)$-moment exists for some $\alpha \in (0,1)$.

**Example 2.9.** For $j = 1, 2, \dots, n$, let

$$P(X_j = 0) = P(X_j = 2) = \frac{9}{20} \text{ and } P(X_j = 2^k) = \frac{28}{5(2^{3k})} \text{ for integer } k \geq 2$$

and assume that $X_1, X_2, \dots, X_n$ are independent. Note that $X_1, X_2, \dots, X_n$ are maximal lattice random variables with parameter $(0,2)$ and for all $j = 1, 2, \dots, n$,

$$\begin{aligned}
\mu_j &= \frac{9}{10} + \frac{28}{5}\sum_{k=2}^{\infty}\frac{1}{2^{2k}} = \frac{41}{30} \\
\sigma_j^2 &= \frac{9}{5} + \frac{28}{5}\sum_{k=2}^{\infty}\frac{1}{2^{k}} - \Big(\frac{41}{30}\Big)^2 = \frac{2459}{900} \\
\beta_j &= 2[P(X_j = 0)P(X_j = 2) + P(X_j = 2)P(X_j = 4)] = \frac{387}{800} \\
E|X_j|^3 &= \frac{18}{5} + \frac{28}{5}\sum_{k=2}^{\infty} 1 = \infty
\end{aligned}$$

and for $\alpha \in (0,1)$,

$$\begin{aligned}
E|X_j|^{2+\alpha} &= 2^{2+\alpha}\Big(\frac{9}{20}\Big) + \frac{28}{5}\sum_{k=2}^{\infty}\frac{1}{2^{(1-\alpha)k}} \\
&= 2^{2+\alpha}\Big(\frac{9}{20}\Big) + \Big(\frac{28}{5}\Big)\Big(\frac{1}{2^{2(1-\alpha)} - 2^{1-\alpha}}\Big) \\
&< \infty.
\end{aligned}$$

Let

$$\Delta_n^{(0,2)} = \sup_{k\in\mathbb{Z}}\Big|P(S_n = 2k) - \sqrt{\frac{1800}{2459n\pi}}e^{-\frac{(60k-41n)^2}{2459n}}\Big|.$$

By Theorem 2.1, we have

$$\Delta_n^{(0,2)} \leq \frac{A_1}{n^{\frac{1+\alpha}{2}}} + \frac{A_2}{n^{\frac{1+\alpha}{2+\alpha}}} \exp\big(-A_3 n^{\frac{\alpha}{2+\alpha}}\big) + \frac{A_4}{n^{\frac{1+\alpha}{2+\alpha}}} \exp\big(-A_5 n^{\frac{\alpha}{2+\alpha}}\big)$$

where $A_j$'s are constants which depend on $\alpha$ and are given by

$$A_1 = \frac{0.0020(2^{\frac{(2+\alpha)(1+\alpha)}{2}})}{\big(E|X_1|^{2+\alpha}\big)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})(2^{\frac{\alpha^2-\alpha+2}{2}})(E|X_1|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4},$$

$$A_2 = \frac{0.6368(3^{\frac{1}{\alpha}})(E|X_1|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2},$$

$$A_3 = \frac{\sigma_1^2}{2(3^{\frac{2}{\alpha}})(E|X_1|^{2+\alpha})^{\frac{2}{2+\alpha}}},$$

$$A_4 = \frac{0.7854(3^{\frac{1}{\alpha}})(E|X_1|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\beta_1},$$

and $A_5 = \dfrac{4\beta_1}{(3^{\frac{2}{\alpha}})\pi^2(E|X_1|^{2+\alpha})^{\frac{2}{2+\alpha}}}$.

These are some examples of $A_j$'s.

| $\alpha$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|---|---|---|---|---|---|
| 0.1 | 815329.3725 | 30707.0472 | $7.8702 \cdot 10^{-11}$ | 231905.0017 | $1.1294 \cdot 10^{-11}$ |
| 0.2 | 3830.6039 | 131.7570 | $4.2748 \cdot 10^{-6}$ | 917.8183 | $6.1349 \cdot 10^{-7}$ |
| 0.3 | 709.0199 | 22.0903 | $1.5207 \cdot 10^{-4}$ | 153.8809 | $2.1824 \cdot 10^{-5}$ |
| 0.4 | 332.9755 | 9.2986 | $8.5826 \cdot 10^{-4}$ | 64.7743 | $1.2317 \cdot 10^{-4}$ |
| 0.5 | 230.5188 | 5.6875 | $2.2941 \cdot 10^{-3}$ | 39.6191 | $3.2924 \cdot 10^{-4}$ |
| 0.6 | 198.0421 | 4.2260 | $4.1553 \cdot 10^{-3}$ | 29.4383 | $5.9634 \cdot 10^{-4}$ |
| 0.7 | 199.3648 | 3.5533 | $5.8776 \cdot 10^{-3}$ | 24.7522 | $8.4351 \cdot 10^{-4}$ |
| 0.8 | 237.4348 | 3.3064 | $6.7879 \cdot 10^{-3}$ | 23.0327 | $9.4716 \cdot 10^{-4}$ |
| 0.9 | 384.3726 | 3.5196 | $5.9908 \cdot 10^{-3}$ | 24.5172 | $8.5976 \cdot 10^{-4}$ |

Observe that Theorem 2.2 cannot be applied to Example 2.9, since $v_5 = 2\sum_{m=-\infty}^{\infty} P(X_5 = 2m)P(X_5 = 2m+10)) = 0$.

$\square$

**Example 2.10.** Let $X_1, X_2, \ldots, X_n$ be independent random variables defined by

$$P(X_j = 0) = \frac{7}{8} - \frac{1}{(2j)^6 - (2j)^3},\ P(X_j = 2j) = \frac{1}{8} \text{ and } P(X_j = (2j)^k) = (2j)^{-3k}$$

for integer $k \geq 2$. We see that $X_1, X_2, \ldots, X_n$ are common lattice random variables with parameter $(0, 2)$ and

$$\begin{aligned}
\mu_j &= \frac{j}{4} + \frac{1}{16j^4 - 4j^2}, \\
\sigma_j^2 &= \frac{j^2}{2} + \frac{1}{4j^2 - 2j} - \mu_j^2, \\
E|X_j|^{2+\alpha} &= \frac{(2j)^{2+\alpha}}{8} + \frac{1}{(2j)^{2-2\alpha} - (2j)^{1-\alpha}}, \\
\text{and } E|X_j|^3 &= j^3 + \sum_{k=2}^{\infty} 1 = \infty.
\end{aligned}$$

These imply that

$$\sigma^2 \geq \frac{n^3}{48} \text{ and } \frac{n^3}{6} \leq \sum_{j=1}^{n} E|X_j|^{2+\alpha} \leq \left(\frac{2^{2+\alpha}}{8} + \frac{2^{1+2\alpha}}{48(2^{1-\alpha} - 1)}\right) n^{3+\alpha}.$$

Moreover, we have that $\upsilon_j = 2 \sum\limits_{m=-\infty}^{\infty} P(X_j = 2m)P(X_j = 2m + 2j)$ and then $\upsilon_1 = \frac{1}{4}\left(\frac{7}{8} - \frac{1}{(2)^6 - (2)^3}\right) + \frac{1}{4}\left(\frac{1}{2^6}\right) \leq \upsilon_j$ for all $j \geq 2$. Hence,

$$\upsilon = \min_{1 \leq j \leq n} \upsilon_j = \frac{1}{4}\left(\frac{7}{8} - \frac{1}{(2)^6 - (2)^3}\right) + \frac{1}{4}\left(\frac{1}{2^6}\right) = \frac{391}{1792}.$$

Let

$$\Delta_n^{(0,2)} = \sup_{k \in \mathbb{Z}} \left| P(S_n = 2k) - \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{(2k-\mu)^2}{2\sigma^2}} \right|.$$

By Theorem 2.2, we have

$$\Delta_n^{(0,2)} \leq \frac{B_1}{n^{\frac{3+3\alpha}{2}}} + \frac{B_2}{n^{\frac{\alpha^2+3}{2}}} + \frac{B_3}{n^3} \exp\left(-B_4 n^{\frac{\alpha}{2+\alpha}}\right) + \exp\left(-B_5 n^{\frac{\alpha}{2+\alpha}}\right)$$

where $B_j$'s are constants which depend on $\alpha$ and are given by

$$B_1 = 0.0020(2^{\frac{(2+\alpha)(1+\alpha)}{2}})(6^{\frac{1+\alpha}{2}}),$$

$$B_2 = 10706.8184(3^{\frac{1}{\alpha}})(2^{\frac{\alpha^2-\alpha+2}{2}})\left(\frac{2^{2+\alpha}}{8} + \frac{2^{1+2\alpha}}{48(2^{1-\alpha}-1)}\right)^{\frac{3-\alpha}{2}},$$

$$B_3 = 30.5664(3^{\frac{1}{\alpha}})\left(\frac{2^{2+\alpha}}{8} + \frac{2^{1+2\alpha}}{48(2^{1-\alpha}-1)}\right)^{\frac{1}{2+\alpha}},$$

$$B_4 = \frac{1}{96(3^{\frac{2}{\alpha}})\left(\frac{2^{2+\alpha}}{8} + \frac{2^{1+2\alpha}}{48(2^{1-\alpha}-1)}\right)^{\frac{2}{2+\alpha}}},$$

and $$B_5 = \frac{\upsilon}{4\pi^2(3^{\frac{2}{\alpha}})\left(\frac{2^{2+\alpha}}{8} + \frac{2^{1+2\alpha}}{48(2^{1-\alpha}-1)}\right)^{\frac{2}{2+\alpha}}}.$$

Observe that Example 2.10 cannot be applied by Theorem 2.1, since $\beta_j = 0$ for $j \geq 2$. □

# CHAPTER III

# Local Limit Theorems for Lattice Random Variables with Finite Third Moment

In chapter 2, we obtain the local limit theorems for sums of independent integer-valued lattice random variables without assuming finite third moment. However, we directly obtain the results in case of finite third moment which are Corollary 2.3 and Corollary 2.4. The main results of this chapter is the improvement of constants in Corollary 2.3 and Corollary 2.4.

Throughout this chapter, let $X_1, X_2, \dots, X_n$ be independent integer-valued lattice random variables with parameter $(a, d)$, mean $\mu_j$ and variance $\sigma_j^2$ such that $E|X_j|^3 < \infty$ for all $j = 1, 2, \dots, n$ and let

$$S_n = \sum_{j=1}^{n} X_j, \qquad \mu = \sum_{j=1}^{n} \mu_j, \quad \text{and} \quad \sigma^2 = \sum_{j=1}^{n} \sigma_j^2.$$

Recall that

$$\begin{aligned} \Delta_n^{(a,d)} &= \sup_{k \in \mathbb{Z}} \left| P(S_n = na + kd) - \frac{d}{\sigma\sqrt{2\pi}} e^{-\frac{(na+kd-\mu)^2}{2\sigma^2}} \right|, \\ \beta &= \sum_{j=1}^{n} \beta_j, \\ \text{and} \quad \upsilon &= \min_{1 \le j \le n} \upsilon_j \end{aligned}$$

are defined in Chapter 2 where

$$\begin{aligned} \beta_j &= 2 \sum_{m=-\infty}^{\infty} p_{jm} p_{j(m+1)}, \\ \upsilon_j &= 2 \sum_{m=-\infty}^{\infty} p_{jm} p_{j(m+j)}, \end{aligned}$$

$$\text{and} \quad p_{jm} = P(X_j = a + md).$$

The following statements are our main results.

**Theorem 3.1.** *If* $\sigma^2 > d^2$ *and* $\beta > 0$, *then*

$$\Delta_n^{(a,d)} \leq \frac{0.6607d}{\sigma^4}\sum_{j=1}^{n} E|X_j - a|^3 + \frac{0.3184d^2}{\sigma^2\kappa}e^{-\frac{\sigma^2\kappa^2}{2d^2}} + \frac{1.5708}{\kappa\beta}e^{-\frac{\kappa^2\beta}{\pi^2}}$$

*where* $\kappa = \frac{d}{3\left(\sum\limits_{j=1}^{n} E|X_j-a|^3\right)^{\frac{1}{3}}}$.
*Furthermore, if* $X_1, X_2, \ldots, X_n$ *are identically distributed such that* $\sigma^2 > d^2$ *and* $\beta_1 > 0$, *then*

$$\Delta_n^{(a,d)} \leq \frac{C_5}{n}$$

*where*

$$C_5 = \frac{17.8543dE|X_1 - a|^3}{\sigma_1^4} + \frac{418.5858E|X_1 - a|^3}{d^3\beta_1^2}.$$

**Theorem 3.2.** *If* $\sigma^2 > d^2$ *and* $\upsilon_j > 0$ *for all* $j = 1, 2, \ldots, n$, *then*

$$\Delta_n^{(a,d)} \leq \frac{0.6607d}{\sigma^4}\sum_{j=1}^{n} E|X_j - a|^3 + \frac{0.3184d^2}{\sigma^2\kappa}e^{-\frac{\sigma^2\kappa^2}{2d^2}} + e^{-\frac{n\upsilon}{4}\min\left(1,\left(\frac{n\kappa}{2\pi}\right)^2\right)}$$

*where* $\kappa = \frac{d}{3\left(\sum\limits_{j=1}^{n} E|X_j-a|^3\right)^{\frac{1}{3}}}$.
*Furthermore, if* $X_1, X_2, \ldots, X_n$ *are identically distributed such that* $\sigma^2 > d^2$ *and* $\upsilon_j > 0$ *for all* $j = 1, 2, \ldots, n$, *then for* $n \geq \left(\frac{6\pi(E|X_1-a|^3)^{\frac{1}{3}}}{d}\right)^{\frac{3}{2}}$,

$$\Delta_n^{(a,d)} \leq \frac{C_6}{n} + e^{-\frac{n\upsilon}{6}}$$

*where*

$$C_6 = \frac{17.8543 d E|X_1 - a|^3}{\sigma_1^4}.$$

This chapter is organized as follows. First, we give the auxiliary result in Section 3.1 which will be used to prove the main theorems in Section 3.2. Finally, we give some examples in Section 3.3.

## 3.1 Auxiliary Result

Let $\psi_1, \psi_2, \ldots, \psi_n$ and $\psi$ be the characteristic functions of $X_1, X_2, \ldots, X_n$ and $S_n$, respectively. In Lemma 2.7, we gave the bound of $\psi$ as follows:

$$\left|\psi(t) - e^{it\mu - \frac{1}{2}\sigma^2 t^2}\right| \leq 12.5606 \sum_{j=1}^{n} E|X_j|^3 |t|^3 e^{-\frac{1}{2}\sigma^2 t^2}$$

for all $|t| \leq \kappa_0$ and $\kappa_0 = \frac{1}{3\left(\sum\limits_{j=1}^{n} E|X_j|^3\right)^{\frac{1}{3}}}$. In the following lemma, we improve the constant of this result.

**Lemma 3.3.**

$$\left|\psi(t) - e^{it\mu - \frac{1}{2}\sigma^2 t^2}\right| \leq 1.0378 \sum_{j=1}^{n} E|X_j|^3 |t|^3 e^{-\frac{1}{2}\sigma^2 t^2}$$

*for all* $|t| \leq \kappa_0$.

**Proof.** From (2.2), (2.3) and (2.9), we know that for $0 < \alpha \leq 1$ and $j = 1, 2, \ldots, n$,

$$|Ee^{itX_j}| \geq 1 - 2^{1-\alpha} E|X_j|^\alpha |t|^\alpha$$

and there exists a function $g_j$ such that

$$\psi_j(t) = \exp\left\{ i\mu_j t - \frac{1}{2}\sigma_j^2 t^2 + \int_0^t \frac{g_j(s)}{\psi_j(s)}\, \mathrm{d}s \right\} \tag{3.1}$$

for $|t| \leq \left(\frac{1}{3E|X|^{\alpha}}\right)^{\frac{1}{\alpha}}$ and

$$|g_j(t)| \leq \left(2^{1-\alpha} + \frac{2^{2-\alpha}}{1+\alpha}\right)E|X|^{2+\alpha}|t|^{1+\alpha}.$$

So, we have

$$|Ee^{itX_j}| \geq 1 - E|X_j||t|$$

and

$$|g_j(t)| \leq 2E|X_j|^3|t|^2.$$

From these facts, we obtain that for all $|t| \leq \frac{1}{3E|X_j|}$,

$$|\psi_j(t)| = |Ee^{itX_j}| \geq \frac{2}{3}$$

and then

$$\int_0^t \left|\frac{g_j(s)}{\psi_j(s)}\right| \text{d}s \leq 3\int_0^t E|X_j|^3|s|^2 \text{ d}s = E|X_j|^3|t|^3. \tag{3.2}$$

From Lyapunov's inequality, we have that for all $l = 1, 2, \ldots, n$,

$$E|X_l| \leq (E|X_l|^3)^{\frac{1}{3}}$$

and then we can show that

$$\left(\frac{1}{\sum\limits_{j=1}^{n} E|X_j|^3}\right)^{\frac{1}{3}} \leq \left(\frac{1}{E|X_l|^3}\right)^{\frac{1}{3}} \leq \frac{1}{E|X_l|}.$$

From this fact, (3.1), (3.2) and the inequality $|e^z - 1| \leq |z|e^{|z|}$ for a complex number

$z$, we have that for $t \le \kappa_0$,

$$\begin{aligned}
\left|\psi(t) - e^{i\mu t - \frac{1}{2}\sigma^2 t^2}\right| &= \left|\exp\left\{i\mu t - \frac{1}{2}\sigma^2 t^2 + \sum_{j=1}^{n} \int_0^t \left|\frac{g_j(s)}{\psi_j(s)}\right| \mathrm{d}s\right\} - \exp\left\{i\mu t - \frac{1}{2}\sigma^2 t^2\right\}\right| \\
&= \left|\exp\left\{i\mu t - \frac{1}{2}\sigma^2 t^2\right\}\right| \left|\exp\left\{\sum_{j=1}^{n} \int_0^t \left|\frac{g_j(s)}{\psi_j(s)}\right| \mathrm{d}s\right\} - 1\right| \\
&\le \sum_{j=1}^{n} \left|\int_0^t \frac{g_{X_j}(s)}{\psi_{X_j}(s)} \mathrm{d}s\right| \times \exp\left\{ -\frac{1}{2}\sigma^2 t^2 + \sum_{j=1}^{n} \left|\int_0^t \frac{g_{X_j}(s)}{\psi_{X_j}(s)} \mathrm{d}s\right|\right\} \\
&\le \sum_{j=1}^{n} E|X_j|^3 |t|^3 \times \exp\left\{ -\frac{1}{2}\sigma^2 t^2 + \sum_{j=1}^{n} E|X_j|^3 |t|^3\right\} \\
&\le \sum_{j=1}^{n} E|X_j|^3 |t|^3 \times \exp\left\{ -\frac{1}{2}\sigma^2 t^2 + \frac{1}{27}\right\} \\
&\le 1.0378 \sum_{j=1}^{n} E|X_j|^3 |t|^3 e^{-\frac{1}{2}\sigma^2 t^2}.
\end{aligned}$$

□

## 3.2 Proof of Main Results

### 3.2.1 Proof of Theorem 3.1

**Proof.** Let $Y_1, Y_2, \ldots, Y_n$ be independent integer-valued random variables and let

$$W_n = Y_1 + Y_2 + \cdots + Y_n$$

with $E(W_n) = \mu_{\scriptscriptstyle W}$, $Var(W_n) = \sigma_{\scriptscriptstyle W}^2 > 1$ and $\psi_{\scriptscriptstyle W}$ be the characteristic function of $W_n$. Suppose that $\beta_{Y_j} = 2 \sum_{m=-\infty}^{\infty} P(Y_j = m)P(Y_j = m+1) > 0$ for all $j = 1, 2, \ldots, n$. From (2.12) and (2.14) - (2.15), we have

$$\left|P(W_n = k) - \frac{1}{\sigma_{\scriptscriptstyle W}\sqrt{2\pi}} e^{-\frac{(k-\mu_{\scriptscriptstyle W})^2}{2\sigma_{\scriptscriptstyle W}^2}}\right| \le A + B + C \tag{3.3}$$

where

$$A \le \frac{1}{2\pi} \int_{|t|<\kappa_0} |\psi_{\scriptscriptstyle W}(t) - e^{it\mu_{\scriptscriptstyle W} - \frac{1}{2}\sigma_{\scriptscriptstyle W}^2 t^2}| \, \mathrm{d}t,$$

$$B \le \frac{0.3184}{\sigma_{_W}^2 \kappa_0} e^{\frac{-\sigma_{_W}^2 \kappa_0^2}{2}},$$
$$C \le \frac{1.5708}{\kappa_0 \beta_{_W}} e^{-\frac{\kappa_0^2 \beta_{_W}}{\pi^2}},$$
$$\kappa_0 = \frac{1}{3\big(\sum_{j=1}^{n} E|X_j|^3\big)^{\frac{1}{3}}},$$
$$\text{and} \quad \beta_{_W} = \sum_{j=1}^{n} \beta_{_{Y_j}}.$$

By Lemma 3.3, we obtain that

$$\begin{aligned} A &\le \frac{1.0378}{\pi} \sum_{j=1}^{n} E|Y_j|^3 \int_0^{\kappa_0} |t|^3 e^{-\frac{1}{2}\sigma_{_W}^2 t^2} \, \mathrm{d}t \\ &\le \frac{1.0378}{\pi} \sum_{j=1}^{n} E|Y_j|^3 \left(\frac{2}{\sigma_{_W}^4}\right) \\ &\le \frac{0.6607}{\sigma_{_W}^4} \sum_{j=1}^{n} E|Y_j|^3. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), these provide that

$$\begin{aligned} &\left| P(W_n = k) - \frac{1}{\sigma_{_W}\sqrt{2\pi}} e^{-\frac{(k-\mu_{_W})^2}{2\sigma_{_W}^2}} \right| \\ &\le \frac{0.6607}{\sigma_{_W}^4} \sum_{j=1}^{n} E|Y_j|^3 + \frac{0.3184}{\sigma_{_W}^2 \kappa_0} e^{\frac{-\sigma_{_W}^2 \kappa_0^2}{2}} + \frac{1.5708}{\kappa_0 \beta_{_W}} e^{-\frac{\kappa_0^2 \beta_{_W}}{\pi^2}} \end{aligned} \tag{3.5}$$

Let $Y_j = \frac{X_j - a}{d}$ and $W_n = Y_1 + Y_2 + \cdots + Y_n$. Observe that $Y_1, Y_2, \ldots, Y_n$ are independent integer-valued random variables and

$$\mu_{_W} = \frac{\mu - na}{d}, \qquad \sigma_{_W}^2 = \frac{\sigma^2}{d^2}, \qquad P(Y_j = m) = P(X_j = a + dm), \tag{3.6}$$

$$E|Y_j|^3 = \frac{E|X_j - a|^3}{d^3}, \qquad \kappa_0 = \kappa = \frac{d}{3\left(\sum_{j=1}^{n} E|X_j - a|^3\right)^{\frac{1}{3}}} \quad \text{and} \quad \beta_{_W} = \beta. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\left|P(S_n = na + kd) - \frac{d}{\sigma\sqrt{2\pi}}e^{-\frac{(na+kd-\mu)^2}{2\sigma^2}}\right| = \left|P(W_n = k) - \frac{1}{\sigma_{W}\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right|$$

and then we have the conclusion of the theorem by applying this fact with (3.5). Furthermore, if $X_1, X_2, \ldots, X_n$ are identical, then

$$\mu = n\mu_1, \qquad \sigma = \sigma_1\sqrt{n}, \qquad \sum_{j=1}^{n} E|X_j - a|^3 = nE|X_1 - a|^3 \text{ and } \beta = n\beta_1$$

which imply that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \\
&\le \frac{0.6607dE|X_1 - a|^3}{\sigma_1^4 n} + \frac{0.9552d(E|X_1 - a|^3)^{\frac{1}{3}}}{\sigma_1^2 n^{\frac{2}{3}}}\exp\left(\frac{-\sigma_1^2 n^{\frac{1}{3}}}{18(E|X_1 - a|^3)^{\frac{2}{3}}}\right) \\
&\quad + \frac{4.7124(E|X_1 - a|^3)^{\frac{1}{3}}}{d\beta_1 n^{\frac{2}{3}}}\exp\left(\frac{-d^2\beta_1 n^{\frac{1}{3}}}{9\pi^2(E|X_1 - a|^3)^{\frac{2}{3}}}\right). \qquad (3.8)
\end{aligned}$$

Since $e^{-x} \le \frac{1}{x}$ for a real number $x > 0$, we obtain that

$$\begin{aligned}
&\frac{0.9552d(E|X_1 - a|^3)^{\frac{1}{3}}}{\sigma_1^2 n^{\frac{2}{3}}}\exp\left(\frac{-\sigma_1^2 n^{\frac{1}{3}}}{18(E|X_1 - a|^3)^{\frac{2}{3}}}\right) \\
&\le \frac{0.9552d(E|X_1 - a|^3)^{\frac{1}{3}}}{\sigma_1^2 n^{\frac{2}{3}}}\left(\frac{18(E|X_1 - a|^3)^{\frac{2}{3}}}{\sigma_1^2 n^{\frac{1}{3}}}\right) \\
&= \frac{17.1936dE|X_1 - a|^3}{\sigma_1^4 n}. \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{4.7124(E|X_1 - a|^3)^{\frac{1}{3}}}{d\beta_1 n^{\frac{2}{3}}}\exp\left(\frac{-d^2\beta_1 n^{\frac{1}{3}}}{9\pi^2(E|X_1 - a|^3)^{\frac{2}{3}}}\right) \\
&\le \frac{4.7124(E|X_1 - a|^3)^{\frac{1}{3}}}{d\beta_1 n^{\frac{2}{3}}}\left(\frac{9\pi^2(E|X_1 - a|^3)^{\frac{2}{3}}}{d^2\beta_1 n^{\frac{1}{3}}}\right) \\
&\le \frac{418.5858E|X_1 - a|^3}{d^3\beta_1^2 n}. \qquad (3.10)
\end{aligned}$$

From (3.8) - (3.10), we have

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = na+kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \le \frac{C_5}{n}$$

where

$$C_5 = \frac{17.8543dE|X_1-a|^3}{\sigma_1^4} + \frac{418.5858E|X_1-a|^3}{d^3\beta_1^2}. \qquad \square$$

### 3.2.2 Proof of Theorem 3.2

**Proof.** By the same trace of Theorem 3.1, we can only provide to the case of integer-valued random variables. Let $Y_1, Y_2, \dots, Y_n$ be independent integer-valued random variables with the characteristic functions $\psi_{Y_j}$ and let

$$W_n = Y_1 + Y_2 + \cdots + Y_n$$

with $E(W_n) = \mu_{W}$, $Var(W_n) = \sigma_{W}^2 > 1$ and $\psi_{W}$ be the characteristic function of $W_n$. Suppose that $\upsilon_{Y_j} = 2\sum_{m=-\infty}^{\infty} P(Y_j = m)P(Y_j = m+j) > 0$ for all $j = 1, 2\dots, n$. From (2.12), (2.14) and (2.23), we have

$$\left|P(W_n = k) - \frac{1}{\sigma_{W}\sqrt{2\pi}}e^{-\frac{(k-\mu_{W})^2}{2\sigma_{W}^2}}\right| \le A + B + C$$

where

$$A \le \frac{1}{2\pi}\int_{|t|<\kappa_0} |\psi_{W}(t) - e^{it\mu_{W} - \frac{1}{2}\sigma_{W}^2t^2}|\, \mathrm{d}t,$$

$$B \le \frac{0.3184}{\sigma_{W}^2\kappa_0}e^{\frac{-\sigma_{W}^2\kappa_0^2}{2}},$$

$$C \le e^{-\frac{n\upsilon_W}{4}\min\left(1, \left(\frac{n\kappa_0}{2\pi}\right)^2\right)},$$

$$\kappa_0 = \frac{1}{3\left(\sum_{j=1}^{n} E|X_j|^3\right)^{\frac{1}{3}}},$$

and $\upsilon_{W} = \min_{1\le j\le n} \upsilon_{Y_j}$.

From this fact and (3.4), we have

$$\left|P(W_n = k) - \frac{1}{\sigma_{W}\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right|$$
$$\leq \frac{0.6607}{\sigma_W^4}\sum_{j=1}^{n} E|Y_j|^3 + \frac{0.3184}{\sigma_W^2\kappa_0}e^{\frac{-\sigma_W^2\kappa_0^2}{2}} + e^{-\frac{n\upsilon_W}{4}\min\left(1,\left(\frac{n\kappa_0}{2\pi}\right)^2\right)}. \tag{3.11}$$

Hence, we can conclude from (3.6), (3.7) and (3.11) that

$$\left|P(W_n = k) - \frac{1}{\sigma_{W}\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right|$$
$$\leq \frac{0.6607d}{\sigma^4}\sum_{j=1}^{n} E|X_j - a|^3 + \frac{0.3184d^2}{\sigma^2\kappa}e^{-\frac{\sigma^2\kappa^2}{2d^2}} + e^{-\frac{n\upsilon}{4}\min\left(1,\left(\frac{n\kappa}{2\pi}\right)^2\right)}.$$

Furthermore, if $X_1, X_2, \ldots, X_n$ are identical and $\upsilon_j > 0$ for all $j = 1, 2, \ldots, n$, then

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right|$$
$$\leq \frac{0.6607dE|X_1 - a|^3}{\sigma_1^4 n} + \frac{0.9552d(E|X_1 - a|^3)^{\frac{1}{3}}}{\sigma_1^2 n^{\frac{2}{3}}}\exp\left(\frac{-\sigma_1^2 n^{\frac{1}{3}}}{18(E|X_1 - a|^3)^{\frac{2}{3}}}\right)$$
$$+ \exp\left(-\frac{n\upsilon}{4}\min\left(1,\left(\frac{dn^{\frac{2}{3}}}{6\pi(E|X_j - a|^3)^{\frac{1}{3}}}\right)^2\right)\right).$$

From this fact and (3.9), we obtain that for $n \geq \left(\frac{6\pi(E|X_1-a|^3)^{\frac{1}{3}}}{d}\right)^{\frac{3}{2}}$,

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \leq \frac{C_6}{n} + e^{-\frac{n\upsilon}{4}}$$

where

$$C_6 = \frac{17.8543dE|X_1 - a|^3}{\sigma_1^4}.$$

□

## 3.3 Examples

The following examples are to apply Theorem 3.1 and Theorem 3.2 and show that the error bounds are smaller than those in Corollary 2.3 and Corollary 2.4, respectively.

**Example 3.4.** Let $X_j$'s be independent random variables defined by $P(X_j = 0) = P(X_j = 3) = P(X_j = 6) = \frac{1}{3}$. We see that $X_j$'s are lattice random variables with common parameter $(0, 3)$ and

$$\mu_j = 3,\ \sigma_j^2 = 6,\ E|X_j|^3 = 81 \text{ and } \beta_j = \frac{4}{9}.$$

Denote

$$\Delta_n^{(0,3)} = \sup_{k\in\mathbb{Z}} \left| P(S_n = 3k) - \frac{3}{2\sqrt{3n\pi}} e^{-\frac{3(k-n)^2}{4n}} \right|.$$

Applying Theorem 3.1, we obtain that

$$\Delta_n^{(0,3)} \leq \frac{4.4594}{n} + \frac{2.0665}{n^{\frac{2}{3}}} e^{-0.0178n^{\frac{1}{3}}} + \frac{15.2921}{n^{\frac{2}{3}}} e^{-0.0024n^{\frac{1}{3}}}.$$

Notice that this bound is smaller than the bound from Corollary 2.3 which is given by

$$\Delta_n^{(0,3)} \leq \frac{93.4970}{n} + \frac{2.0665}{n^{\frac{2}{3}}} e^{-0.0178n^{\frac{1}{3}}} + \frac{15.2921}{n^{\frac{2}{3}}} e^{-0.0024n^{\frac{1}{3}}}.$$ □

**Example 3.5.** Let $X_j$'s be independent random variables defined by $P(X_j = 0) = \frac{2}{3}$ and $P(X_j = 3j) = \frac{1}{3}$. We see that $X_j$'s are lattice random variables with common parameter $(0, 3)$ and

$$\mu_j = j,\ \sigma_j^2 = 2j^2,\ E|X_j|^3 = 9j^3 \text{ and } \upsilon_j = \frac{4}{9}$$

which imply that

$$\mu = \frac{n(n+1)}{2},\ \sigma^2 = \frac{n(n+1)(2n+1)}{3} \quad \text{and} \quad \sum_{j=1}^{n} E|X_j|^3 = \frac{9n^2(n+1)^2}{4}.$$

Then,

$$\frac{2n^3}{3} \leq \sigma^2 \ \text{ and } \ \frac{9n^4}{4} \leq \sum_{j=1}^{n} E|X_j|^3 \leq 9n^4. \tag{3.12}$$

From (3.12) and Theorem 3.2, we have

$$\Delta_n^{(0,3)} \leq \frac{40.1340}{n^2} + \frac{8.9411}{n^{\frac{5}{3}}} e^{-0.0085 n^{\frac{1}{3}}} + e^{-0.0040 n^{\frac{1}{3}}}. \tag{3.13}$$

If we use Corollary 2.4, we have the bound as follows:

$$\Delta_n^{(0,3)} \leq \frac{841.4665}{n^2} + \frac{0.0241}{n^4} + \frac{8.9411}{n^{\frac{5}{3}}} e^{-0.0085^{\frac{1}{3}}} + e^{-0.0040 n^{\frac{1}{3}}},$$

which is larger than the bound in (3.13). □

# CHAPTER IV

# Local Limit Theorems for General Weighted Sums of Bernoulli Random Variables

Recall that De Moivre and Laplace [13] gave the first local limit theorem for Binomial random variable which is sums of $X_j$'s being Bernoulli random variables with parameter $p$, i.e., $P(X_j = 1) = 1 - P(X_j = 0) = p$ for all $j = 1, 2, \ldots, n$. From the past until today, there are many works which investigate the local limit theorem for a generality of Bernoulli sums such as our results in Chapter 2 and Chapter 3 which show the local limit theorem for sums of independent lattice random variables taking values in $\{a + md : m \in \mathbb{Z}\}$ for integers $a$ and $d \geq 1$ in the case that

$$\sum_{m=-\infty}^{\infty} P(X_j = a + md)P(X_j = a + (m+1)d) > 0 \tag{4.1}$$

or

$$\sum_{m=-\infty}^{\infty} P(X_j = a + md)P(X_j = a + (m+j)d) > 0 \tag{4.2}$$

for all $j = 1, 2, \ldots, n$. The weighted sums of Bernoulli random variables are other general forms of Bernoulli sums and they are defined by

$$\tilde{S}_n = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

where $X_j$'s are Bernoulli random variables and $a_j$'s are any integers. On the other hand, the weighted sums of Bernoulli random variables are sums of random variables $X_j$'s satisfying $P(X_j = a_j) = 1 - P(X_j = 0) > 0$ for some integers $a_j$. Recall that Giuliano and Weber [21] gave the error bound of the local limit theorem for the weighted sums of Bernoulli random variables when all weights are distinct positive integer, i.e., $S_n = \sum_{j=1}^{n} X_j$ where each $X_j$ satisfies $P(X_j = a_j) = 1 - P(X_j = 0) = p_j$ for some positive integer $a_j$ and

$$a_i \neq a_j \tag{4.3}$$

for any $i, j = 1, 2, \ldots, n$ such that $i \neq j$. The following inequality is their result:

$$\sup_{k\in\mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{C}{\sum_{j=1}^{n} p_j} \tag{4.4}$$

for some an unknown constant $C$.

In this chapter, let $X_1, X_2, \ldots, X_n$ be independent integer-valued random variables with mean $\mu_j$ and variance $\sigma_j^2$ such that $E|X_j|^{2+\alpha} < \infty$, where $0 < \alpha \leq 1$, for all $j = 1, 2, \ldots, n$ and let

$$S_n = \sum_{j=1}^{n} X_j, \qquad \mu = \sum_{j=1}^{n} \mu_j, \qquad \sigma^2 = \sum_{j=1}^{n} \sigma_j^2.$$

We generalize the condition (4.3) to the following condition

$$|j_m - j_l| \neq |k_r - k_s| \tag{4.5}$$

for any $j_m \neq j_l$ in $\operatorname{Im} X_j$, $k_r \neq k_s$ in $\operatorname{Im} X_k$ and $j \neq k$ and give the local limit theorem for this general weighted sums in the following theorems.

**Theorem 4.1.** *If $\sigma^2 > 1$ and the condition* (4.5) *holds, then*

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \\
&\quad \leq \frac{0.0020}{\left(\sum_{j=1}^{n} E|X_j|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171(3^{\frac{1}{\alpha}})}{\sigma^4} \left(\sum_{j=1}^{n} E|X_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}} \\
&\qquad + \frac{0.3184}{\sigma^2\tau_\alpha} e^{-\frac{\sigma^2\tau_\alpha^2}{2}} + \frac{8}{n}
\end{aligned}$$

*where* $\tau_\alpha = \frac{1}{3^{\frac{1}{\alpha}}\left(\sum\limits_{j=1}^{n} E|X_j|^{2+\alpha}\right)^{\frac{1}{2+\alpha}}}$.

**Theorem 4.2.** *If* $\sigma^2 > 1$*, the condition* (4.5) *holds and* $E|X_j|^3 < \infty$ *for all* $j = 1, 2, \ldots, n$*, then*

$$\sup_{k\in\mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{0.6607}{\sigma^4} \sum_{j=1}^{n} E|X_j|^3 + \frac{0.3184}{\sigma^2\tau_1} e^{-\frac{\sigma^2\tau_1^2}{2}} + \frac{8}{n}$$

*where* $\tau_1 = \frac{1}{3\left(\sum\limits_{j=1}^{n} E|X_j|^3\right)^{\frac{1}{3}}}$.

## 4.1 Auxiliary Result

We also use the characteristic function methods in order to prove Theorem 4.1 and Theorem 4.2. Bounding the characteristic function is the main concept for this method. Giuliano and Weber [21] investigated $\tilde{S_n}$ which is a sum of independent random variables $X_j$ defined by $P(X_j = a_j) = 1 - P(X_j = 0) = p_j$. They showed that if the condition (4.3) holds, then

$$\int_{-\pi}^{\pi} |\tilde{\psi}(t)| \, \mathrm{d}t \leq \frac{C}{\sum\limits_{j=1}^{n} p_j(1-p_j)}$$

for some constant $C$, where $\tilde{\psi}$ is the characteristic function of $\tilde{S_n}$. In this work, we use the idea of Giuliano and Weber in order to obtain our result in the case that $X_j$'s are any integer-valued random variables.

**Theorem 4.3.** *Let* $\psi$ *be the characteristic function of* $S_n$*. If the condition* (4.5) *holds, then*

$$\int_{0}^{\pi} |\psi(t)| \, dt \leq \frac{8\pi}{n}.$$

**Proof.** Let $\operatorname{Im} X_j = \{j_m \mid m \in \mathbb{Z}\}$ and $P(X_j = j_m) = p_{jm}$. For $c \in (0, 1]$, let

$$E_c = \left\{ t \in [0, \pi] : \left| \sum_{j=1}^{n} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} p_{jm} p_{jl} \cos\big((j_m - j_l)t\big) \right| > cn \right\}.$$

Since $|\psi(t)| \leq 1$ for any real number $t$, we obtain that

$$\int_0^\pi |\psi(t)| \, \mathrm{d}t = \int_{E_c} |\psi(t)| \, \mathrm{d}t + \int_{[0,\pi]\setminus E_c} |\psi(t)| \, \mathrm{d}t$$
$$\leq \lambda(E_c) + \int_{[0,\pi]\setminus E_c} |\psi(t)| \, \mathrm{d}t, \tag{4.6}$$

where $\lambda$ is the Lebesgue measure on $\mathbb{R}$. Since

$$1 \leq \frac{1}{cn}\Big|\sum_{j=l}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\cos\big((j_m - j_l)t\big)\Big|$$

for all $t \in E_c$, we have that

$$\lambda(E_c) = \int_{E_c} 1 \, \mathrm{d}\lambda$$
$$\leq \frac{1}{(cn)^2}\int_{E_c}\Big|\sum_{j=l}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\cos\big((j_m - j_l)t\big)\Big|^2 \mathrm{d}\lambda$$
$$\leq \frac{1}{(cn)^2}\int_{0}^{\pi}\Big|\sum_{j=l}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\cos\big((j_m - j_l)t\big)\Big|^2 \mathrm{d}t. \tag{4.7}$$

Note that

$$\Big|\sum_{j=1}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\cos\big((j_m - j_l)t\big)\Big|^2$$
$$= \sum_{j=1}^{n}\Big|\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\cos\big((j_m - j_l)t\big)\Big|^2$$
$$+ \sum_{j\neq k}\Big|\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} p_{jm}p_{jl}p_{kr}p_{ks}\cos\big((j_m - j_l)t\big)\cos\big((k_r - k_s)t\big)\Big|$$

and

$$\int_0^\pi \cos(at)\cos(bt) \, \mathrm{d}t = \left[\frac{\sin\big((b+a)t\big)}{2(b+a)} + \frac{\sin\big((b-a)t\big)}{2(b-a)}\right]_0^\pi = 0$$

for any integer numbers $a, b$ such that $|a| \neq |b|$. Then, we obtain

$$\begin{aligned}
\int_0^\pi \Big|\sum_{j=1}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\cos\big((j_m - j_l)t\big)\Big|^2 \,\mathrm{d}t
&\le \sum_{j=1}^{n}\int_0^\pi \Big|\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\cos\big((j_m - j_l)t\big)\Big|^2 \,\mathrm{d}t \\
&\le \sum_{j=1}^{n}\int_0^\pi \Big|\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\Big|^2 \,\mathrm{d}t \\
&= \sum_{j=1}^{n}\int_0^\pi 1 \,\mathrm{d}t \\
&= n\pi.
\end{aligned}$$

From this fact and (4.7), we have

$$\lambda(E_c) \le \frac{\pi}{c^2 n}. \tag{4.8}$$

Let $\psi_1, \psi_2, \ldots, \psi_n$ be the characteristic functions of $X_1, X_2, \ldots, X_n$, respectively. From the fact that

$$ln(|\psi_j(t)|) \le -\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\sin^2\left((j_m - j_l)\frac{t}{2}\right)$$

([9], p.6), we have that for $t \in [0, \pi] \setminus E_c$,

$$\begin{aligned}
ln(|\psi(t)|) &= \sum_{j=1}^{n} ln(|\psi_j(t)|) \\
&\le -\sum_{j=1}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\left(\frac{1}{2} - \frac{\cos((j_m - j_l)t}{2}\right) \\
&= -\frac{n}{2} + \frac{1}{2}\sum_{j=1}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_{jm}p_{jl}\Big(\cos((j_m - j_l)t\Big) \\
&\le -\frac{(1-c)n}{2}.
\end{aligned}$$

So,

$$|\psi(t)| \le e^{-\frac{(1-c)n}{2}}$$

and then

$$\begin{aligned}\int\limits_{[0,\pi]\setminus E_c} |\psi(t)|\,\mathrm{d}t &\le \int\limits_{[0,\pi]\setminus E_c} e^{-\frac{(1-c)n}{2}}\,\mathrm{d}t\\ &\le e^{-\frac{(1-c)n}{2}}\lambda[0,\pi]\\ &= \pi e^{-\frac{(1-c)n}{2}}. \end{aligned} \tag{4.9}$$

From (4.6), (4.8) and (4.9), we get that

$$\int\limits_0^\pi |\psi(t)|\,\mathrm{d}t \le \frac{\pi}{c^2 n} + \pi e^{-\frac{(1-c)n}{2}}.$$

From the fact that $e^{-x} \le \frac{1}{x}$ for all positive real number $x$, we obtain that for $c = \frac{1}{2}$,

$$\int\limits_0^\pi |\psi(t)|\,\mathrm{d}t \le \frac{4\pi}{n} + \frac{4\pi}{n} = \frac{8\pi}{n}.$$ □

## 4.2 Proof of Main Results

### 4.2.1 Proof of Theorem 4.1

**Proof.** From (2.12) - (2.14), we have that

$$\left|P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}\right| \le A + B + C \tag{4.10}$$

where

$$\begin{aligned} A &\le \frac{1}{2\pi}\int\limits_{|t|<\tau_\alpha} |\psi(t) - e^{it\mu - \frac{1}{2}\sigma^2 t^2}|\,\mathrm{d}t\\ B &\le \frac{0.3184}{\sigma^2\tau_\alpha}e^{-\frac{\sigma^2\tau_\alpha^2}{2}} \text{ and} \end{aligned}$$

$$C \leq \frac{1}{\pi} \int_{\tau_\alpha}^{\pi} |\psi(t)| \, \mathrm{d}t,$$

and

$$A \leq \frac{0.0020}{\left(\sum_{j=1}^{n} E|X_j|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}} + \frac{4.6171 \cdot 3^{\frac{1}{\alpha}}}{\sigma^4} \left(\sum_{j=1}^{n} E|X_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}}. \tag{4.11}$$

By Theorem 4.3, we have that

$$C \leq \frac{1}{\pi} \int_{0}^{\pi} |\psi(t)| \, \mathrm{d}t \leq \frac{8}{n}. \tag{4.12}$$

From (4.10) - (4.12), we obtain the theorem. □

## 4.2.2 Proof of Theorem 4.2

**Proof.** From (4.10) and (4.12) in the case of $\alpha = 1$, we have that

$$\left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq A + B + \frac{8}{n} \tag{4.13}$$

where

$$A \leq \frac{1}{2\pi} \int_{|t|<\tau_1} |\psi(t) - e^{it\mu - \frac{1}{2}\sigma^2 t^2}| \, \mathrm{d}t \text{ and}$$
$$B \leq \frac{0.3184}{\sigma^2 \tau_1} e^{-\frac{\sigma^2 \tau_1^2}{2}}.$$

From (3.4), we know that

$$A \leq \frac{0.6607}{\sigma^4} \sum_{j=1}^{n} E|X_j|^3. \tag{4.14}$$

From (4.13) and (4.14), we have the conclusion. □

## 4.3 Examples

We specify an unknown constant in (4.4) in Example 4.4.

**Example 4.4.** Let $X_1, X_2, \dots, X_n$ be Bernoulli random variables defined by

$$P(X_j = 0) = 1 - p_j \text{ and } P(X_j = 1) = p_j.$$

for all $j = 1, 2, \dots, n$ and let $a_1, a_2, \dots, a_n$ be distinct positive integers. Then, we define

$$\tilde{S_n} = a_1X_1 + a_2X_2 + \cdots + a_nX_n$$

with mean $\mu$ and variance $\sigma^2$. So, we obtain that

$$\mu = \sum_{j=1}^{n} a_jp_j, \quad \sigma^2 = \sum_{j=1}^{n} a_j^2p_j(1-p_j) \text{ and } \sum_{j=1}^{n} E|a_jX_j|^3 = \sum_{j=1}^{n} a_j^3p_j.$$

Let

$$\Delta_n = \sup_{k\in\mathbb{Z}} \left| P(\tilde{S_n} = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right|.$$

By Theorem 4.2, we obtain that

$$\Delta_n \leq \frac{0.6607}{\sigma^4} \sum_{j=1}^{n} a_j^3p_j + \frac{0.9552}{\sigma^2} \left( \sum_{j=1}^{n} a_j^3p_j \right)^{\frac{1}{3}} \exp\left( - \frac{0.0555\sigma^2}{\left( \sum\limits_{j=1}^{n} a_j^3p_j \right)^{\frac{2}{3}}} \right) + \frac{8}{n}.$$

Furthermore, in the case that $X_j$'s are i.i.d., i.e., $p_1 = p_2 = \cdots = p_n = p$, we have that

$$\begin{aligned} \Delta_n \leq & \frac{0.6607 \sum\limits_{j=1}^{n} a_j^3}{p(1-p)^2 \left( \sum\limits_{j=1}^{n} a_j^2 \right)^2} \\ & + \frac{0.9552 \left( \sum\limits_{j=1}^{n} a_j^3 \right)^{\frac{1}{3}}}{p^{\frac{2}{3}}(1-p) \sum\limits_{j=1}^{n} a_j^2} \exp\left( - \frac{0.0555 p^{\frac{1}{3}}(1-p) \sum\limits_{j=1}^{n} a_j^2}{\left( \sum\limits_{j=1}^{n} a_j^3 \right)^{\frac{2}{3}}} \right) + \frac{8}{n}. \end{aligned}$$ □

Example 4.5 is appropriate for Theorem 4.1. It cannot be applied by (4.4).

**Example 4.5.** Let $X_1, X_2, \ldots, X_n$ be independent random variables defined by

$$\begin{aligned} P(X_j = 0) &= 1 - \frac{1}{2^{j+1}-1} - \frac{1}{2(2^{j+1}-1)}, \\ P(X_j = 2^{j+1}-1) &= \frac{1}{2^{j+1}-1} \text{ and} \\ P(X_j = 2(2^{j+1}-1)) &= \frac{1}{2(2^{j+1}-1)} \end{aligned}$$

for each $j = 1, 2, \ldots, n$. Then,

$$\begin{aligned} &E|X_j|^{\frac{5}{2}} = (1 + 2^{\frac{3}{2}})(2^{j+1}-1)^{\frac{3}{2}}, \\ &\mu_j = 2 \text{ and} \\ &\sigma_j^2 = 6(2^j) - 7 \end{aligned}$$

which imply that $\mu = 2n$, $\sigma^2 = 12(2^n) - 7n - 12$ and

$$\begin{aligned} &2(2^n) \leq \sigma^2 \leq 12(2^n), \\ &2(2^{\frac{3n}{2}}) \leq \sum_{j=1}^{n} E|X_j|^{\frac{5}{2}} \leq 17(2^{\frac{3n}{2}}). \end{aligned}$$

Note that $X_1, X_2, \ldots, X_n$ satisfy the condition (4.5) since $2(2^x - 1) < 2^{x+1} - 1$ for all natural number $x$. By Theorem 4.1, we have

$$\Delta_n \leq \frac{0.0012}{2^{\frac{9n}{8}}} + \frac{358.6023}{2^{\frac{n}{8}}} + \frac{4.4501}{2^{\frac{2n}{5}}} e^{-\frac{0.0012}{2^{\frac{n}{5}}}} + \frac{8}{n}.$$

Note that $2^{\frac{9x}{8}} > x$ and $2^{\frac{2x}{5}} > 2^{\frac{x}{8}} > 400x$ for all integer $x \geq 125$. These provide that for $n \geq 125$,

$$\Delta_n \leq \frac{0.0012}{n} + \frac{0.8966}{n} + \frac{0.0112}{n} + \frac{8}{n} = \frac{8.9090}{n}. \qquad \square$$

**Remark 4.6.** *(i) From Example 4.5, we have* $\sum_{j=1}^{n} E|X_j|^3 \geq 6(4^n)$ *and* $\sigma^2 \leq$

$12(2^n)$. *Then,*

$$\frac{1}{\sigma^4}\sum_{j=1}^{n} E|X_j|^3 \geq \frac{1}{24}.$$

*So, we should not apply Theorem 4.2 with Example 4.5, since* $\Delta_n$ *does not converge to* $0$.

$(ii)$ *Observe that any random variables* $X_j$ *in Example 4.4 satisfy the following equations:* $\sum_{m=-\infty}^{\infty} P(X_j = m)P(X_j = m+1) = 0$ *and* $\sum_{m=-\infty}^{\infty} P(X_j = m)P(X_j = m+j) = 0$. *This instance cannot be applied by the theorems in Chapter 2 and Chapter 3, since it does not satisfy the conditions* (4.1) *and* (4.2).

# Appendices

Sunklodas and et al. ([24]) refer that the following equations are well-known expansions of the function $e^{ix}$ for any real number $x$ without proof. So, we give the proof of these expansions as follows.

**1.** For any real number $x$ and $\alpha \in (0,1]$, there exists a complex number $\Theta_1$ which $|\Theta_1| \leq 1$ such that

$$e^{ix} = 1 + 2^{1-\alpha}|x|^{\alpha}\Theta_1.$$

**Proof.** Recalling the Taylor's expansion of the function $e^{ix}$,

$$e^{ix} = \sum_{k=0}^{r-1}\left[\frac{(ix)^k}{k!}\right] + \frac{(ix)^r}{(r-1)!}\int_0^1 (1-u)^r e^{iux}\,\mathrm{d}u, \tag{A.1}$$

for $r \in \mathbb{N}$. Hence,

$$\begin{aligned} e^{ix} &= 1 + ix\int_0^1 (1-u)e^{iux}\,\mathrm{d}u \\ &= 1 + 2^{1-\alpha}|x|^{\alpha}\left[i2^{\alpha-1}|x|^{-\alpha}x\int_0^1 (1-u)e^{iux}\,\mathrm{d}u\right] \\ &= 1 + 2^{1-\alpha}|x|^{\alpha}\Theta_1 \end{aligned}$$

where $\Theta_1 = i2^{\alpha-1}|x|^{-\alpha}x\int_0^1 (1-u)e^{iux}\,\mathrm{d}u$. We need to show that $|\Theta_1| \leq 1$. If $|x| \leq 2$, then

$$\begin{aligned} |\Theta_1| &\leq 2^{\alpha-1}|x|^{1-\alpha}\int_0^1 |(1-u)e^{iux}|\,\mathrm{d}u \\ &= \left(\frac{|x|}{2}\right)^{1-\alpha}\int_0^1 (1-u)\,\mathrm{d}u \\ &\leq \left(u - \frac{u^2}{2}\right)\Big|_0^1 \\ &= \frac{1}{2}. \end{aligned}$$

If $|x| > 2$, then

$$
\begin{aligned}
|\Theta_1| &= 2^{\alpha-1}|x|^{1-\alpha}\bigg|\int_0^1 (1-u)e^{iux}\ \mathrm{d}u\bigg| \\
&= 2^{\alpha-1}|x|^{1-\alpha}\bigg|\frac{(1-\cos x)+i(x-\sin x)}{x^2}\bigg| \\
&= 2^{\alpha-1}|x|^{-1-\alpha}\sqrt{2-2\cos x-2x\sin x+x^2} \\
&\leq 2^{\alpha-1}|x|^{-1-\alpha}\sqrt{4+2|x|+x^2} \\
&= 2^{\alpha-1}|x|^{-1-\alpha}(|x|+2) \\
&= \frac{1}{2^{1-\alpha}|x|^{\alpha}} + \frac{2^{\alpha}}{|x|^{1+\alpha}} \\
&\leq \frac{1}{2^{1-\alpha}2^{\alpha}} + \frac{2^{\alpha}}{2^{1+\alpha}} \\
&= 1.
\end{aligned}
$$

□

**2.** For any real number $x$ and $\alpha \in (0,1]$, there exists a complex number $\Theta_2$ which $|\Theta_2| \leq 1$ such that

$$e^{ix} = 1 + ix + \frac{2^{1-\alpha}}{1+\alpha}|x|^{1+\alpha}\Theta_2.$$

**Proof.** From (A.1), we obtain that

$$
\begin{aligned}
e^{ix} &= 1 + ix + (ix)^2\int_0^1 (1-u)^2 e^{iux}\ \mathrm{d}u \\
&= 1 + ix + \frac{2^{1-\alpha}}{1+\alpha}|x|^{1+\alpha}\bigg[-\frac{(1+\alpha)x^2}{2^{1-\alpha}|x|^{1+\alpha}}\int_0^1 (1-u)^2 e^{iux}\ \mathrm{d}u\bigg] \\
&= 1 + ix + \frac{2^{1-\alpha}}{1+\alpha}|x|^{1+\alpha}\Theta_2
\end{aligned}
$$

where $\Theta_2 = -\frac{(1+\alpha)x^2}{2^{1-\alpha}|x|^{1+\alpha}}\int_0^1 (1-u)^2 e^{iux}\ \mathrm{d}u$. We remain to show that $|\Theta_2| \leq 1$. If $|x| \leq 3$, then

$$
\begin{aligned}
|\Theta_2| &\leq (1+\alpha)\bigg(\frac{|x|}{2}\bigg)^{1-\alpha}\int_0^1 |(1-u)^2 e^{iux}|\ \mathrm{d}u \\
&= (1+\alpha)\bigg(\frac{|x|}{2}\bigg)^{1-\alpha}\int_0^1 (1-u)^2\ \mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&= (1+\alpha)\left(\frac{|x|}{2}\right)^{1-\alpha}\left(u - u^2 + \frac{u^3}{3}\right)\Big|_0^1 \\
&= \frac{1+\alpha}{3}\left(\frac{|x|}{2}\right)^{1-\alpha} \\
&\leq \frac{1+\alpha}{3}\left(\frac{3}{2}\right)^{1-\alpha} \\
&\leq \frac{2}{3}\left(\frac{3}{2}\right) \\
&= 1.
\end{aligned}
$$

If $|x| > 3$, then

$$
\begin{aligned}
|\Theta_2| &= \frac{1+\alpha}{2^{1-\alpha}|x|^{\alpha-1}}\left|\int_0^1 (1-u)^2 e^{iux}\,\mathrm{d}u\right| \\
&= \frac{1+\alpha}{2^{1-\alpha}|x|^{\alpha-1}}\left|\frac{(2\sin x - 2x) + i(2 - 2\cos x - x^2)}{x^3}\right| \\
&= \frac{1+\alpha}{2^{1-\alpha}|x|^{2+\alpha}}\sqrt{x^4 + (4x^2-8)\cos x - 8x\sin x + 8} \\
&\leq \frac{1+\alpha}{2^{1-\alpha}|x|^{2+\alpha}}\sqrt{x^4 + 4x^2 + 8|x|} \\
&= \frac{1+\alpha}{2^{1-\alpha}|x|^{\alpha}}\sqrt{1 + \frac{4}{x^2} + \frac{8}{|x|^3}} \\
&\leq \frac{1+\alpha}{2}\left(\frac{2}{3}\right)^{\alpha}\sqrt{1 + \frac{4}{9} + \frac{8}{27}} \\
&\leq \frac{2}{3}\sqrt{\frac{47}{27}} \\
&\leq 1. \qquad \square
\end{aligned}
$$

# REFERENCES


[1] Doob, J.L.: Stochastic Processes. Wiley, Chapman and Hall, New York (1953)

[2] Prokhorov, Y.V., Rozanov, Y.A.: Probability Theory[in Russian]. Nauka, Moscow (1973)

[3] Statulyavichus, V.A.: Limit theorems for densities and asymptotic decompositions for distributions of sums of independent random variables. Theory of Probability and Its Applications **10**(4), 582–595 (1965)

[4] Ushakov, N.: Lower an upper bounds for characteristic functions. Journal of Mathematical Sciences **84**, 1179–1189 (1997)

[5] Ushakov, N.G.: Selected Topics in Characteristic Functions. VSP, Utrecht (1999)

[6] Benedicks, M.: An estimate of the modulus of the characteristic function of a lattice distribution with application to remainder term estimates in local limit theorems. Annals of Probability **3**, 162–165 (1975)

[7] Zhang, Z.: An upper bound for characteristic functions of lattice distributions with applications to survival probabilities of quantum states. Journal of Physics A: Mathematical and Theoretical **40**(1), 131–137 (2007)

[8] Zhang, Z.: Bound for characteristic functions and laplace tranforms of probability distributions. Theory of Probability and Its Applications **56**(2), 350–358 (2012)

[9] Siripraparat, T., Neammanee, K.: An improvement of convergence rate in the local limit theorem for integral-valued random variables. Journal of Inequalities and Applications **57**(2021), 1–18 (2021)

[10] McDonald, D.R.: On local limit theorems for integer valued random variables. Theory of Probability & Its Applications **24**(3), 613–619 (1980)

[11] Kolmogorov, A.: Sur les propriétés des fonctions de concentrations de m. p. lévy. Annales de l'institut Henri Poincaré **16**(1), 27–34 (1958)

[12] Aizenman, M., Germinet, F., Klein, A., Warzel, S.: On bernoulli decompositions for random variables, concentration bounds, and spectral localization. Probability Theory and Related Fields **143**, 219–238 (2009)

[13] McDnald, D.R.: The local limit theorem: a historical perspective. Journal of the Iranian Statistical Society **4**(2), 73–86 (2005)

[14] Vladimirovich, G.B.: On a local limit theorem of the theory of probability. Uspekhi Matematicheskikh Nauk **3**(25), 187–194 (1948)

[15] Giuliano, R., Weber, M.J.G.: Approximate local limit theorems with effective rate and application to random walks in random scenery. Bernoulli **23**(4B), 3268–3310 (2017)

[16] Siripraparat, T., Neammanee, K.: A local limit theorem for poisson binomial random variables. Scienceasia **47**, 111–116 (2021)

[17] Ibragimov, I.A., Linnik, I.V., Kingman, J.F.C.: Independent and Stationary Sequences of Random Variables. Wolters-Noordhoff Publishing, Groningen (1971)

[18] Petrov, V.V.: Sums of Independent Random Variables. Springer, Berlin Heidelberg (1975)

[19] Korolev, V., Zhukov, Y.V.: Convergence rate estimates in local limit theorems for poisson random sums. Annales de l'institut Henri Poincaré **99**(4), 1439–1444 (2000)

[20] Davis, B., McDonald, D.R.: An elementary proof of the local central limit theorem. Journal of Theoretical Probability **8**, 693–701 (1995)

[21] Giuliano, R., Weber, M.J.G.: Local limit theorems in some random models from number theory. Stochastic Analysis and Applications **34**(6), 941–960 (2016)

[22] Weber, M.J.G.: A sharp correlation inequality with an application to almost sure local limit theorem. Probability and Mathematical Statistics **31**(1), 79–98 (2011)

[23] Zolotukhin, A.Y., Nagaev, S., Chebotarev, V.: On a bound of absolute constant in the berry-esseen inequality for i.i.d. bernoulli random variables. Modern Stochastics: Theory and Applications **5**(3), 385–410 (2018)

[24] Sunklodas, J.K.: On the approximation of a binomial random sum. Lithuanian Mathematical Journal **54**(3), 356–365 (2014)

[25] Freiman, G.A., Pitman, J.: Partitions into distinct large parts. Journal of the Australian Mathematical Society **57**(3), 386–416 (1994)

[26] Feller, W.: An Introduction to Probability Theory and Its Applications, Vol.2. Wiley, New York (1971)

# VITA

| | |
|---|---|
| **Name** | Mr. Punyapat Kammoo |
| **Date of Birth** | 9 September 1997 |
| **Place of Birth** | Phitsanulok, Thailand |
| **Education** | B.Sc. (Mathematics),<br>Naresuan University, 2019 |
| **Scholarship** | Development and Promotion of Science and Technology<br>Talents Project (DPST) |
| **Conference** | **Presentation**<br>● *On Some Algebraic Structures of AG*-Groupoids*<br>at Annual Meeting in Mathematics<br>Conference : AMM 2019, 15-17 May 2019 at Burapha University |