# การจัดพอร์ตการลงทุนออพชันและการป้องกันความเสี่ยงของออพชันรูปแบบพิเศษที่เขียนบนดัชนีมินิเอสแอนด์พีห้าร้อยในตลาดที่มีสภาพไม่คล่องกับมูลค่าที่ความเสี่ยงแบบมีเงื่อนไข

นางสาวเบญญาณี โกษาพงศ์



วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์และวิทยาการคณนา
ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2565

# OPTIONS PORTFOLIO OPTIMIZATION AND HEDGING OF EXOTIC OPTIONS WRITTEN ON MINI S&P 500 INDEX IN AN ILLIQUID MARKET WITH CVAR

Miss Benyanee Kosapong



A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Applied Mathematics and Computational Science

Department of Mathematics and Computer Science

Faculty of Science

Chulalongkorn University

Academic Year 2022

| | |
|---|---|
| Thesis Title | OPTIONS PORTFOLIO OPTIMIZATION AND HEDGING OF EXOTIC OPTIONS WRITTEN ON MINI S&P 500 INDEX IN AN ILLIQUID MARKET WITH CVAR |
| By | Miss Benyanee Kosapong |
| Field of Study | Applied Mathematics and Computational Science |
| Thesis Advisor | Associate Professor Petarpa Boonserm, Ph.D. |
| Thesis Co-advisor | Assistant Professor Udomsak Rakwongwan, Ph.D. |

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

................................................ Dean of the Faculty of Science

(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

................................................ Chairman

(Assistant Professor Boonyarit Intiyot, Ph.D.)

................................................ Thesis Advisor

(Associate Professor Petarpa Boonserm, Ph.D.)

................................................ Thesis Co-advisor

(Assistant Professor Udomsak Rakwongwan, Ph.D.)

................................................ Examiner

(Raywat Tanadkithirun, Ph.D.)

................................................ External Examiner

(Associate Professor Sanae Rujivan, Ph.D.)

เบญญานี โกษาพงค์ : การจัดพอร์ตการลงทุนออพชันและการป้องกันความเสี่ยงของออพชันรูปแบบพิเศษที่เขียนบนดัชนีมินิเอสแอนด์พีห้าร้อยในตลาดที่มีสภาพไม่คล่องกับมูลค่าที่ความเสี่ยงแบบมีเงื่อนไข. (OPTIONS PORTFOLIO OPTIMIZATION AND HEDGING OF EXOTIC OPTIONS WRITTEN ON MINI S&P 500 INDEX IN AN ILLIQUID MARKET WITH CVAR) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : รศ.ดร. เพชรอาภา บุญเสริม, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : ผศ.ดร. อุดมศักดิ์ รักษ์วงวาน **??** หน้า.

งานวิจัยนี้ได้ทำการศึกษาวิธีการหาค่าพอร์ตการลงทุนที่เหมาะสมที่สุดเพื่อเลือกสัญญาซื้อขายล่วงหน้าที่มีผลตอบแทนสูงที่คาดหวังและมีความเสี่ยงต่ำ โดยใช้วิธีการวัดความเสี่ยงดังนี้ วิธีการวัดความแปรปรวน วิธีการวัดมูลค่าความเสี่ยง และวิธีการวัดมูลค่าความเสี่ยงแบบมีเงื่อนไข อย่างไรก็ตามงานวิจัยนี้เน้นการวิเคราะห์พอร์ตโฟลิโอโดยการใช้มูลค่าความเสี่ยงแบบมีเงื่อนไข เนื่องจากเป็นการวิธีการวัดความเสี่ยงที่มีคุณสมบัติความเกี่ยวพัน และอัตราการเปลี่ยนแปลงของความชันของกราฟความสัมพันธ์ระหว่างราคาและการขาดทุนที่จะเกิดขึ้นจากการลงทุนของสินทรัพย์นี้ ซึ่งใช้วิธีการของ Rockafellar และ Uryasev แต่นำมาปรับใช้กับออพชั่นที่เขียนไว้ในดัชนีดัชนีเอสแอนด์พี 500 และเรากำหนดให้ค่าดัชนีถูกจำลองโดยการใช้การแจกแจงความแปรปรวนและแกมมาหลังจากนั้นจะมูลค่าการขาดทุนที่น้อยที่สุดที่จะเกิดจากการลงทุนในออพชั่นนี้ ภายใต้ผลตอบแทนที่คาดหวังและเงื่อนไขของราคาเสนอมาพร้อมกับราคาเสนอซื้อและราคาเสนอขาย นอกจากนี้ยังศึกษาการเปลี่ยนแปลงในพอร์ตโฟลิโอที่ปรับให้เหมาะสมตามพารามิเตอร์ต่าง ๆ ซึ่งจากผลการทดลองพบว่า มูลค่าความเสี่ยงแบบมีเงื่อนไขขึ้นอยู่กับส่วนเบี่ยงเบนมาตรฐาน อัตราความแปรปรวน อัตราผลตอบแทนที่คาดหวังและระดับความเชื่อมั่น นอกจากนี้เรายังคำนวณราคาที่ไม่เห็นความแตกต่าง สำหรับมูลค่าการขายและการซื้อ มูลค่าบัญชี และกลยุทธ์การป้องกันความเสี่ยง จากการทดลองพบว่า ราคาขายจะมากกว่าราคาซื้อสำหรับทุก ๆ ค่าเบี่ยงเบนมาตรฐาน และเมื่อผลตอบแทนที่คาดหวังเท่ากับ 1,400% ราคาที่ไม่เห็นความแตกต่างจะอยู่ระหว่างราคาป้องกันความเสี่ยง

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์

สาขาวิชา คณิตศาสตร์ประยุกต์และวิทยาการคณนา

ปีการศึกษา 2565

ลายมือชื่อนิสิต ........................

ลายมือชื่อ อ.ที่ปรึกษาหลัก ..............

ลายมือชื่อ อ.ที่ปรึกษาร่วม ...............

## 6370011423 : MAJOR APPLIED MATHEMATICS AND COMPUTATIONAL SCIENCE

KEYWORDS : PORTFOLIO OPTIMIZATION / CONDITIONAL VALUE-AT-RISK (CVAR) / VARIANCE GAMMA DISTRIBUTION / INDIFFERENCE PRICE / ACCOUNTING VALUE / HEDGING PRICE

BENYANEE KOSAPONG : OPTIONS PORTFOLIO OPTIMIZATION AND HEDGING OF EXOTIC OPTIONS WRITTEN ON MINI S&P 500 INDEX IN AN ILLIQUID MARKET WITH CVAR. ADVISOR : ASSOC. PROF. PETARPA BOONSERM, Ph.D., CO-ADVISOR : ASST. PROF. UDOMSAK RAKWONGWAN, Ph.D., **??** pp.

This thesis investigates the derivatives for portfolio optimization. Risk measures such as Mean Variance (MV), Value-at-Risk (VaR), and Conditional Value-at-Risk (CVaR) are minimized. However, we focus primarily on CVaR because it is a coherent and convex risk measure. We adopt the method of Rockafellar and Uryasev (Journal of Risk 2, 3 (2000)), which minimizes CVaR for shares and convert this method to use with options written on the S&P500 Mini Index. The distribution is known and the index values are simulated by using the Variance Gamma (VG) distribution, over CVaR constraints. In particular, the approach can be used for minimizing the CVaR values under expected returns, and the conditions of the quotes come with the bid and ask prices as well as the sizes. We study the changes in optimized portfolios, subject to various modeling parameters. The values of CVaR depend on the standard deviation ($\sigma$), the variance rate ($\nu$), the required return ($Q$) and the confidence level ($\beta$). Moreover, we compute the indifference prices to obtain the selling and accounting values and the hedging strategy. As a result, for all sigma values, the selling prices are greater than the buying prices, and when the expected return equals 1,400%, the indifference prices are between the hedging prices.

| | | |
|---|---|---|
| Department : | Mathematics and Computer Science | Student's Signature ...................... |
| | | Advisor's Signature ...................... |
| Field of Study : | Applied Mathematics and Computational Science | Co-advisor's Signature .................. |
| Academic Year : | 2022 | |

# ACKNOWLEDGEMENTS

This thesis has become a reality with the kind support and help of many individuals. I would like to extend my sincere thanks to all of them.

Foremost, I would like to express my sincere gratitude to Assoc. Prof. Dr.Petarpa Boonserm and Asst. Prof. Dr.Udomsak Rakwongwan, my advisor and my co-advisor, for continuing to advocate of my research project and the constant encouragement throughout the course. They were always there to lift my spirits and provide me with motivation. Their research ideas are interesting. This thesis would not have been completed without the support from them. Moreover, I received a great deal of insight from our casual conversations, which cover a wide range of topics like how to deal with life and positive thinking.

I am very thankful to the Development and Promotion of Science and Technology Talents Project (DPST) scholarship for their financial support to this Master's degree student research project.



Lastly, I would like to thank my parents and friends for supporting me spiritually throughout the journey of writing this thesis and my life in general.

# CONTENTS

# LIST OF TABLES

# LIST OF FIGURES

# CHAPTER I

# INTRODUCTION

In every field, people face risks, be it in medicine, industry businesses, ecology, security or finance. It is impossible to avoid risks, however, through certain measures and strategies, we can reduce the occurrence of these risks. For instance, homeowners are capable of achieving more attractive risks for future cash flows by buying home insurance to curb the risk of investments declining in value due to economic developments or other events that affect the whole market when markets start to fluctuate.

In this thesis, we focus on financial risks, or the likelihood of losing money on an investment. For example, in daily life, currency risk affects investors who hold foreign currencies because of rate changes and monetary policy changes. For individual investments, if your goal is to pursue growth, you will plan to place the maximum amount, which is about 80% of your assets in stocks and as little as 20% in bonds. As far as we know, we can choose from a range of trading instruments and markets to diversify our investments. This is a method that reduces the risk of investments, which involves the uncertainty of earnings or unexpected outcomes from changes in market conditions such as asset prices, interest rates, volatility, and market liquidity. Moreover, portfolio diversification, the process of selecting various investments within each asset, can reduce investment risks. Therefore, in this thesis, we propose another method that can be used effectively to reduce the risk of investment, which is "portfolio optimization".



Portfolio optimization is concerned with the selection of investments, including financial assets such as shares, bonds, mutual funds, and derivatives (e.g., forwards, options, and futures), etc. Since there are many assets to select from, all investors try to manage their money and invest it in various securities to minimize investment risks while maximizing the return on their investment. In fact, the risk is subjective. This means that an individual perceives a possible unwanted event based on a person's opinion, emotions, gut feeling, or intuition. It may not necessarily be the same for everyone. It is an evaluation based on the individual's feelings at the time rather than a mathematical analysis of the circumstance. Since it is subjective, the risk acceptance of each investor is different. Therefore, we will seek the measures that can be preferred for the risk. The popular and widespread risk measures are variance, Value-at-Risk (VaR), and Conditional Value-at-Risk (CVaR). Then, we will inspect them for use as the risk measure for optimizing the portfolio selection. In the field of finance, portfolio optimization is one of the most prevalent and complicated problems [?].

In 1986, the primary method for solving the portfolio selection problem was devised by Markowitz [?]. In the so-called mean-variance (MV) portfolio optimization model, the expected return of the portfolio is evaluated, while the associated risk is estimated by the variance of the portfolio return. Therefore, one of the risk measures is variance. Moreover, Artzner, et al, [?, ?] has declared that the variance of a random variable lacks both sub-additive and positive homogeneity. Therefore, it is not a coherent risk measure. As a result, there are some drawbacks to variance. To replace the variance, alternative risk measures have been proposed.

The alternative risk measures are VaR and CVaR. Before explaining CVaR, we will recap about VaR. VaR is the chance of an undesirable event with a certain investment time horizon and a given confidence level (e.g., 90%, 95%, and 99%). From various literature reviews, we recognize that VaR is not a coherent risk measure because of the failure of some properties of the risk measure that is sub-additive. Besides, it is a non-convex function.



Another risk measure was recommended by Rockafellar and Uryasev [?] in 2000, which is CVaR. The mean of the loss values that exceed the VaR value at a specific significance level is referred to as CVaR. Then, obviously, CVaR value must be greater than or equal to the VaR value. Additionally, it can provide more information beyond VaR. As mentioned previously, this is the reason that CVaR is interesting and prevalent in the financial market. Furthermore, CVaR is a coherent and convex risk measure, so it has the minimum point. As a result, in this thesis, we will modify the model of CVaR minimization for the portfolio in our dataset. We would like to decrease the risk of the options portfolio.

Moreover, the financial market is important, so we will discuss the distinction between a liquid market and an illiquid market. The word "liquid market" refers to a market with numerous buyers and sellers, and low transaction costs. Liquid markets are usually found in financial assets like forex, futures, bonds, and stocks. Liquidity is the absolute opposite of illiquidity. In an illiquid market, it is difficult to sell assets due to their expense, lack of interested buyers, or other reasons. Examples of illiquid markets include some stocks with low trading volumes or collectibles. In illiquid markets, assets still have value, and in many circumstances, extremely high value, but they are simply difficult to sell.

We have interested in the options market, which is an illiquid market because it can make a sky-high return quicker than the stock market. However, many investors wrongly believe that options are always riskier investments than stocks because they may not completely understand what options are and how they work. Options can be used to hedge holdings and reduce risk and speculate on whether a stock will rise or fall, but with a lower risk than buying or shorting the underlying stock. As mentioned previously, these are the reasons why we are interested in studying options.

In this thesis, we adapted the mathematical model for minimization CVaR from Rockafellar and Uryasev [?]. They optimized the portfolio selection for shares but we would like to compute the model in option market. The data comes from an exotic option written on the Mini S&P 500 index in an illiquid market. The mainly objectives of this thesis are 3 points such as we minimize options portfolio by using CVaR to reduce the risk of investment, we would like to know what parameters that affect CVaR values and we consider the minimization portfolio with liability.

We have divided this thesis into five chapters. In chapter II, we study the risk measures, which are the measures that we use to optimize our portfolio. In chapter III, we study the portfolio optimization problem. In chapter IV, we study the method for obtaining the portfolio optimization problem and show all the results of the optimization problems with other constraints. Finally, in chapter V, we present the conclusion to this thesis and our future work.

# CHAPTER II

# RISK MEASURES

Mostly, a financial advisor suggests you take a risk assessment when you would like to invest in financial institutions. We will measure the probability of these terrible outcomes before making an investment. This is called a risk measure. Risk measures are used to measure a terrible outcome throughout the literature. The risk measure methods are used to determine the number of assets to be kept in reserve to cover unexpected losses. It is an excellent concept to reconsider what we expect of our risk measures.

In this chapter, we provide some information about a coherent and a convex risk measure, and then we will utilize the principle of the coherent risk measures developed by Artzner, et al. [?, ?] and the properties of the convex risk measure [?, ?] to review risk measures.

## 2.1 Coherent and convex risk measures

### 2.1.1 The coherent risk measure

**Definition 1** (Coherent risk measure, [?, ?]). The coherent risk measure is a function $\Gamma : L^\infty(\Omega, F, P) \to \mathbb{R}$ satisfying the followings for each $v_1, v_2 \in L^\infty(\Omega, F, P)$.



1. **Monotonicity:** If $v_1 \leq v_2$, then $\Gamma(v_1) \leq \Gamma(v_2)$.

   This means that if the loss $v_1$ is always less than or equal to the loss $v_2$, the risk associated with the loss $v_1$ will be no more than the risk associated with the loss $v_2$.

2. **Positive homogeneity:** $\Gamma(av_1) = a\Gamma(v_1)$, for any constant $a \geq 0$.

   This means that the risk of the loss $v_1$ is scaled by the positive value $a$, which is equal to $a$ times of the risk of the loss $v_1$.

3. **Sub-additivity:** $\Gamma(v_1 + v_2) \leq \Gamma(v_1) + \Gamma(v_2)$.

   This means that the risk occurring from investments in two portfolios is less than or equal to the risk of investment in each of them separately.

4. **Translation invariance:** If $a \in \mathbb{R}$, then $\Gamma(v_1 + a) = \Gamma(v_1) + a$.
   This means that the addition of a sure amount of capital reduces the risk by the same amount.

### 2.1.2 The convex risk measure

In this subsection, we would like to study the definition of the convex risk measure. Then, let's start with the definition of a convex set and convex function and then we will consider the convex risk measure.

**Definition 2.** (Convex set, [?, ?]). A set $C \subseteq \mathbb{R}^n$ is a convex set if for all $x_1, x_2 \in C$ and all $t \in [0, 1]$,

$$(1 - t)x_1 + tx_2 \in C.$$

This means that every point on a line connecting two points in the set is included in the set. The examples are shown in Figure **??**.

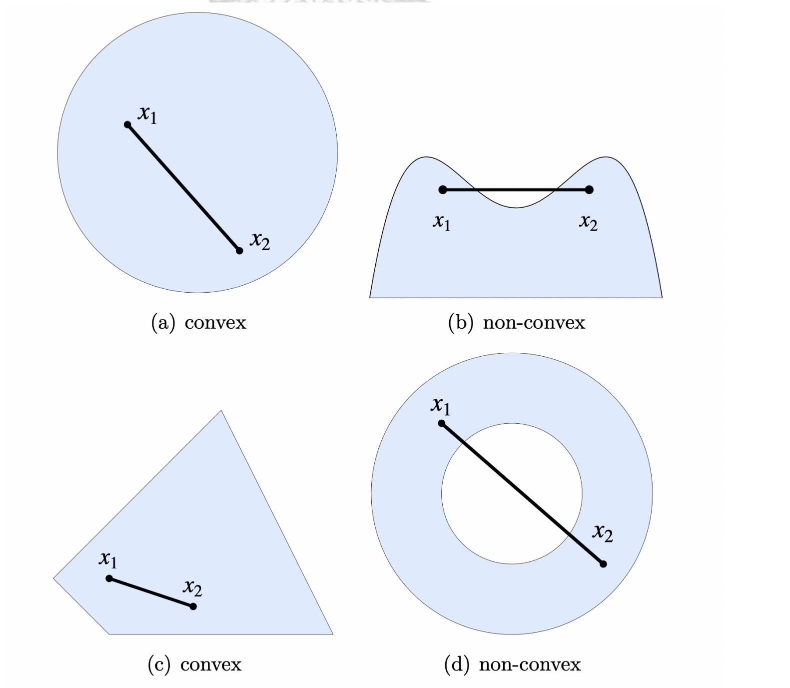



**Figure 2.1:** A graphic of convex sets and non-convex sets [?].
.

**Definition 3.** (Convex function, [?, ?, ?]). A real function $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function if its domain is a convex set and for all $x_1, x_2$ in its domain, and all $t \in [0, 1]$, we have

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2).$$

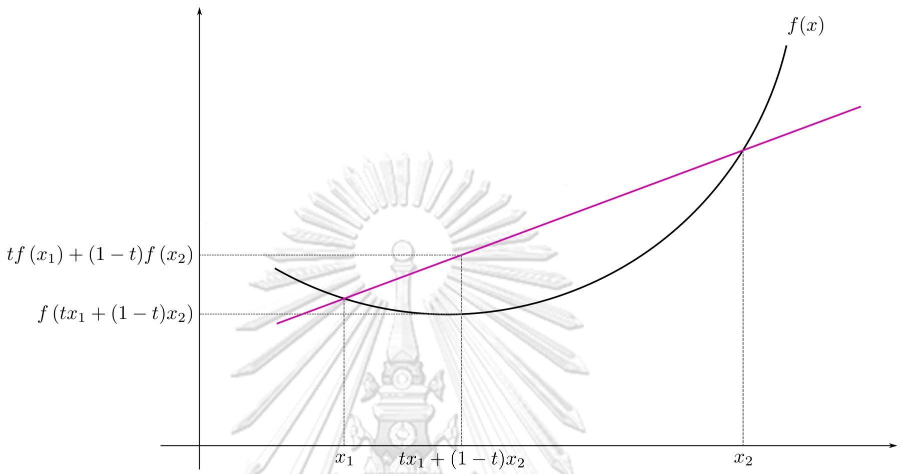



**Figure 2.2:** A graphic of a convex function [?].

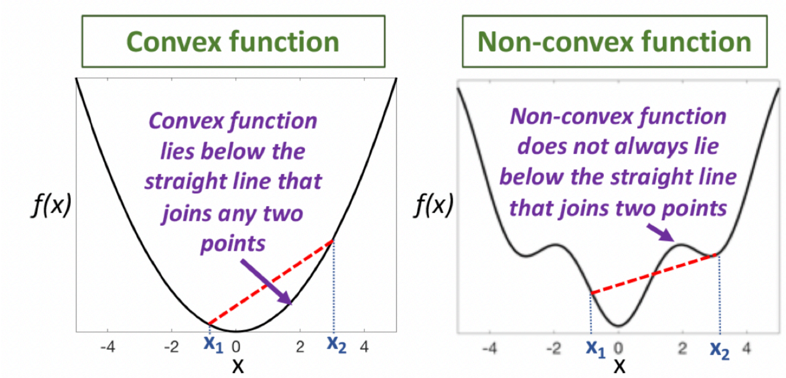



**Figure 2.3:** An example of a convex function and a non-convex function [?].

As can be seen in Definition **??**, if we take any two points $x_1$, $x_2$, then $f(x)$ evaluated by any convex combination of these two points is no greater than the same convex combination of $f(x_1)$ and $f(x_2)$ as shown in Figure **??**. The convex function is important because it can be used to indicate the only minimum point of this function. It is shown in Figure **??** and Figure **??**.

**Definition 4** (Convex risk measure, [?, ?]). A convex risk measure is a function $\gamma : L^\infty(\Omega, F, P) \to \mathbb{R}$ satisfying the followings for each $v_1, v_2 \in L^\infty(\Omega, F, P)$.

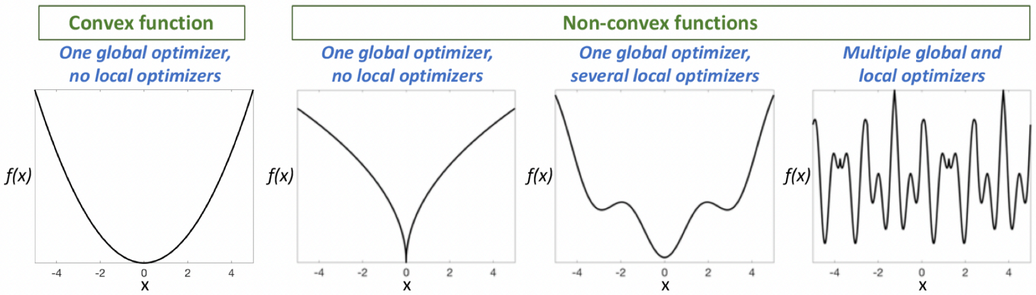



**Figure 2.4:** Examples of convex and non-convex functions with different global and local optimizers [?].

1. **Monotonicity:** If $v_1 \leq v_2$, then $\gamma(v_1) \leq \gamma(v_2)$.

2. **Translation invariance:** If $a \in \mathbb{R}$, then $\gamma(v_1 + a) = \gamma(v_1) + a$.

3. **Convexity:** $\gamma(\lambda v_1 + (1-\lambda)v_2) \leq \lambda\gamma(v_1) + (1-\lambda)\gamma(v_2)$, for $0 \leq \lambda \leq 1$.

As mentioned above, the convex risk measure has some same properties, which are monotonicity and translation invariance, as the coherent risk measure. It can be called the coherent risk measure if it satisfies positive homogeneity. Moreover, we do not need to consider sub-additivity because it is equivalent to sub-additivity directly when it is a convexity, according to the assumption of positive homogeneity.

In the next section, we will discuss the favored risk measures, e.g., variance, VaR, and CVaR, as to determine which one covers all the properties of the coherent and convex risk measures.

## 2.2 Variance

Variance is a familiar statistic used to measure variability. It is determined by averaging the squared deviations from the mean. We first assume that the expected value of the random variable $X$ is $\mathbb{E}(X)$ and then the variance of the random variable $X$ is determined as:

$$\sigma^2(X) = \mathbb{E}[(X - \mathbb{E}(X))^2]. \tag{2.1}$$

Additionally, it can be shown that the variance of a linear combination of two random variables is:

$$\sigma^2(aX + bY) = a^2\sigma^2(X) + b^2\sigma^2(Y) + 2abCov(X, Y), \tag{2.2}$$

where $X$ and $Y$ are random variables, $a$ and $b$ are real numbers, and $Cov(X, Y)$ is the covariance of two random variables, which is given as:

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))].$$

As previously stated, Markowitz [?] modified Equation (??) to optimize the portfolio. For instance, in a period of investment, Markowitz reduced the risk as variance of the portfolio, while providing the maximum expected return of this portfolio.

Let's start by determining all variables. We consider the number of the assets as $n$. The portfolio return of asset $i$ at the end of the period is $R_i$ which is a random variable due to the unknown nature of the future price. Let $w_i$ be the proportion of the asset $i$ invested in this portfolio. The space of portfolios is given as [?, ?, ?]:

$$W = \{(w_1, w_2 \ldots, w_n) \mid \sum_{i=1}^{n} w_i = 1, w_i \geq 0, \forall i \in \{1, 2, \ldots, n\}\}.$$

We use the asset's return $(R_i)$ with the distribution functions such as normal, lognormal, t and variance-gamma distributions to defined the portfolio return $(R_w)$ is given as:

$$R_w = w_1R_1 + w_2R_2 + \ldots + w_nR_n.$$

For one-period market with $n$ assets, the variance of the portfolio selection is determined by [?]:

$$\sigma^2(R_w) = \sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j Cov(R_i, R_j), \tag{2.3}$$

where $Cov(R_i, R_j)$ is the correlation between random variables $R_i$ and $R_j$.

The portfolio variance is written as a quadratic function of the required decisions $w_1, w_2 \ldots, w_n$. By definition, variance includes monotonicity and translation invariance. Therefore, we would like to confirm that the variance covers both monotonicity and translation invariance.

Firstly, we present that $\sigma^2(R_w)$ is monotonicity and translation invariance. The properties of the variance express that it is always monotonicity. For example, if $R_w^{(1)}$ is riskier than $R_w^{(2)}$, in terms of standard deviations when $\sigma(R_w) = \sqrt{\sigma^2(R_w)}$ for a random variable $R_w$, then $\sigma(R_w^{(1)}) \geq \sigma(R_w^{(2)})$.

Afterwards, we exhibit that the variance covers translation invariance. Let $R_w$ be a random variable and $a$ be a constant. Then,

$$
\begin{aligned}
\sigma^2(R_w + a) &= \mathbb{E}[(R_w + a - \mathbb{E}[R_w + a])^2], \\
&= \mathbb{E}[(R_w + a - \mathbb{E}[R_w] - a)^2], \\
&= \mathbb{E}[(R_w - \mathbb{E}[R_w]^2)], \\
&= \sigma^2(R_w).
\end{aligned}
$$

Therefore, variance does have monotonicity and translation invariance properties. However, it is not a coherent risk measure because it lacks positive homogeneity and subadditive. We will prove that $\sigma^2(R_w)$ is not a positive homogeneity by using the properties of variance of the random variable $R_w$. As we know, if we multiply its value by a scalar number, it is given as:

$$
\begin{aligned}
\sigma^2(aR_w) &= \mathbb{E}[(aR_w - \mathbb{E}[aR_w])^2], \\
&= \mathbb{E}[(aR_w - a\mathbb{E}[R_w])^2], \\
&= \mathbb{E}[a^2(R_w - \mathbb{E}[R_w])^2], \\
&= a^2\mathbb{E}[(R_w - \mathbb{E}[R_w])^2], \\
&= a^2\sigma^2(R_w).
\end{aligned}
$$

Since $a^2\sigma^2(R_w) \neq a\sigma^2(R_w)$, variance lacks a positive homogeneity.

Now we prove that $\sigma^2(R_w)$ is not a sub-additive. As we know, the correlation of two random variables $R_w^{(1)}$ and $R_w^{(2)}$ is defined as:

$$\rho(R_w^{(1)}, R_w^{(2)}) = \frac{Cov(R_w^{(1)}, R_w^{(2)})}{\sigma(R_w^{(1)})\sigma(R_w^{(2)})},$$

where $\sigma(R_w^{(1)})$ and $\sigma(R_w^{(2)})$ are the standard deviation of random variables $R_w^{(1)}$ and $R_w^{(2)}$, respectively. Through Equation (**??**) and the correlation definition, we have variance of two random variables, which can be written as:

$$\sigma^2(R_w^{(1)} + R_w^{(2)}) = \sigma^2(R_w^{(1)}) + \sigma^2(R_w^{(2)}) + 2\rho(R_w^{(1)}, R_w^{(2)})\sigma(R_w^{(1)})\sigma(R_w^{(2)}). \tag{2.4}$$

From Equation (**??**), the variance has a sub-additive property when $\rho(R_w^{(1)}, R_w^{(2)}) < 0$, but not for $\rho(R_w^{(1)}, R_w^{(2)}) \geq 0$ cases. If $\rho(R_w^{(1)}, R_w^{(2)}) = 0$ when $R_w^{(1)}$ and $R_w^{(2)}$ are not linearly independent or $\rho(R_w^{(1)}, R_w^{(2)}) > 0$ then $\sigma^2(R_w^{(1)} + R_w^{(2)}) \geq \sigma^2(R_w^{(1)}) + \sigma^2(R_w^{(2)})$. Hence, it lacks a sub-additive and it is not a coherent risk measure.

### 2.3 Value-at-Risk (VaR)

Jorion [**?**] provided an additional risk measure, VaR, which is the asymptotic risk measure. VaR has become a popular risk measure in the past decades. For an investment horizon time $T$ and a given confidence interval $\beta \in (0, 1)$, VaR is the greatest loss that can occur in the $100\beta\%$ scenarios. In mathematics, VaR describes a quantitative measure of loss as shown in Figure **??**.

**Definition 5.** (Jorion, [**?**]). At significant level $\beta \in (0, 1)$, VaR of a random variable $R_w$ is determined as the negative profits or positive losses. Then, the $\beta$-quantile of $R_w$ is defined as:

$$VaR_\beta(R_w) = \min\{\, r \in \mathbb{R} \mid P(R_w \leq r) \geq \beta \,\}.$$

Since VaR is used widely, there are many definitions of VaR. Rockafellar and Uryasev [?, ?] established VaR in different Jorion's definition, but they are equivalent. Firstly, we define all parameters and then explain the definition of VaR for Rockafellar and Uryasev. We assume that the loss function of portfolio $w = (w_1, w_2, \ldots, w_n)$ is denoted by $f(w, y)$ then $y \in \mathbb{R}^n$ exemplifies the uncertainly effect of loss, such as a return of share.

For each $w$, the loss function $f(w, y)$ is a random variable in $\mathbb{R}^n$ as $y$ is a random variable, and $p(y)$ indicates the probability density function (PDF) of $y$ in $\mathbb{R}$. The probability that $f(w, y)$ does not exceed $\alpha$ is defined as:

$$\Psi(w, \alpha) = \int_{f(w,y)\leq\alpha} p(y)\, dy. \tag{2.5}$$

**Definition 6.** (Value-at-Risk (VaR), [?]). The loss random variable associated with $w$ and probability level $\beta$ in $(0, 1)$ is called by $\beta$-VaR value, denoted by $\alpha_\beta(w)$.

$$\alpha_\beta(w) = \min\{\, \alpha \in \mathbb{R} \mid \Psi(w, \alpha) \geq \beta \,\}. \tag{2.6}$$

Since many authors argue that VaR is not a convex function [?] and lacks a subadditive property [?, ?], we would like to give an example to confirm that VaR is not a coherent risk measure. For any portfolios $R_w^{(1)}$ and $R_w^{(2)}$, the VaR of the combined portfolios $R_w^{(1)}$ and $R_w^{(2)}$ is not less than the sum of VaR of the portfolio $R_w^{(1)}$ and VaR of the portfolio $R_w^{(2)}$ as shown in the equation below. Therefore, VaR is not the coherent risk measure.

$$VaR(R_w^{(1)} + R_w^{(2)}) \geq VaR(R_w^{(1)}) + VaR(R_w^{(2)}). \tag{2.7}$$

## 2.4 Conditional Value-at-Risk (CVaR)

Since VaR is not a coherent risk measure, Rockafellar and Uryasev [?, ?, ?] suggested another risk measure, CVaR. CVaR is the average percentage of the worst case loss scenarios, which is also known as mean of loss exceeding the VaR cutoff point or the tail VaR. It is shown in Figure ??. By Definition ??, for a distribution function of random variable $R_w$, CVaR is defined as:

$$CVaR_\beta(R_w) = \mathbb{E}[R_w \mid R_w > VaR_\beta(R_w)].$$

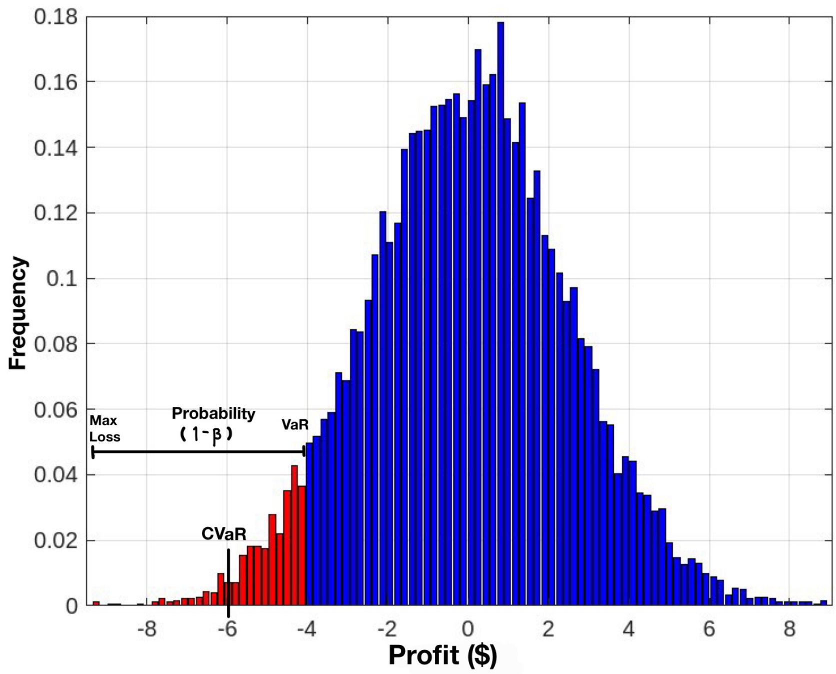



**Figure 2.5:** The graphical representation of VaR, CVaR and max loss (min profit).

This definition is also equivalent with the definition of CVaR that is defined in Rockafellar and Uryasev [?].

**Definition 7.** (Conditional Value-at-Risk (CVaR), [?]). The loss function which is a continuous random variable that is associated with $w$ and probability level $\beta$ in $(0,1)$ is called the $\beta$-CVaR value and it can be denoted by $\phi_\beta(w)$. The $\phi_\beta(w)$ is defined as follows:

$$\begin{aligned}\phi_\beta(w) &= \mathbb{E}[f(w,y) \mid f(w,y) \leq \alpha_\beta(x)],\\ &= \frac{\mathbb{E}[\mathbb{1}_{f(w,y)\leq\alpha_\beta(w)} f(w,y)]}{P(f(w,y) \leq \alpha_\beta(x))},\\ &= (1-\beta)^{-1} \int_{f(w,y)\geq\alpha_\beta(w)} f(w,y)p(y)\, dy, \qquad (2.8)\end{aligned}$$

where $f(w,y)$ is the loss function, $\alpha_\beta(w)$ is the VaR at a confident level $\beta$ and $p(y)$ is the probability density function of $y$.

Artzner et al. [?, ?], Rockafellar and Uryasev [?, ?] and Pflug [?] mention that CVaR is a coherent and convex risk measure.

In this chapter, we studied the properties of coherent and convex risk measures and then considered whether the risk measures such as Variance, VaR, and CVaR cover all the properties of a coherent risk measure. Afterwards, we found that CVaR is the only coherent risk measure from all risk measures as variance is neither positive homogeneity nor a sub-additivity and VaR lacks a sub-additivity,. Moreover, CVaR is convex, therefore, it can be mentioned that this continuous function has a minimum point. From this chapter, we knew that we should used CVaR for minimization because it is the risk measure that explained all information of distribution while VaR can tell the information at confident level and variance is not suitable for using in this thesis. In the next chapter, we will use CVaR to minimize the options portfolio.

# CHAPTER III

# PORTFOLIO OPTIMIZATION MODEL

In this chapter, we explain the illiquid market, options market and datasets. In addition, we consider the portfolio optimization model with CVaR and adopt the mathematical model for options market. We separate portfolio optimization into 2 main situations: portfolio without liability and portfolio with liability. We will start with the portfolio without liability. Then, we will examine the portfolio optimization with liability and will consider the portfolio for indifference pricing and accounting value. Lastly, we will study a hedging portfolio.

## 3.1 Illiquid markets

Liquid markets are markets that allow large amounts of assets to be bought and sold at any time, with minimal transaction costs. In contrast to the illiquid markets, it is hard to sell assets in the illiquid market because of the costs associated with conducting business and a shortage of potential buyers [**?**].

Illiquid asset types have become more popular in recent decades. According to Ang [**?**], pension fund holdings of illiquid asset types have increased from 5% in 1995 to 20% in 2011. However, if we add illiquid assets into the portfolio, the portfolio will then involve substantial risks. Examples of illiquid markets are some small cap stocks, real estate, and options [**?**]. In this thesis, we are interested in options, so we will describe the information associated with options in the next section.

## 3.2 Options

An option is a contract that gives its owner the right, but not the obligation, to buy or sell a security at a fixed price on or before a given date. Additionally, the value of an option is based on or derived from the value of the underlying securities or assets.

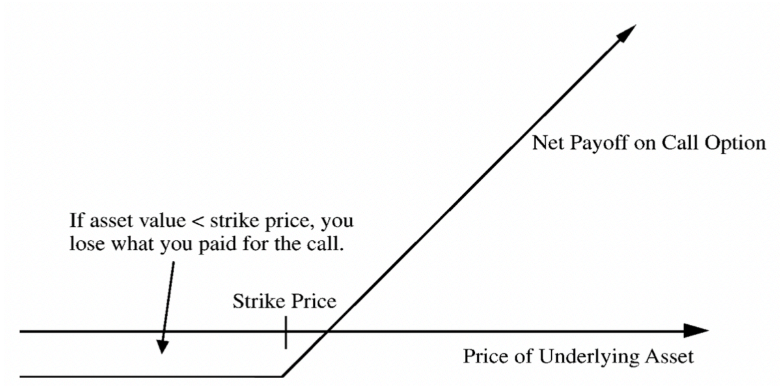
Various words are related to options. For example, exercising options, strike price or exercise price, expiration date and exercise style. Before explaining the different types of options, we will provide brief information about the words relating to options [**?**].

- Exercising options: This is an act of buying or selling the underlying asset.
- Strike price or exercise price: The fixed price specific to the options contract at which the holder is able to buy or sell the underlying asset.
- Expiration date: A limited life of an option because the option is said to expire at the end of its life. The expiration date is the last day that the option can be exercised.
- Exercise style: The exercise style of options governs the time at which the exercise can occur. For a research, a European option is recommended because it may be exercised only on the expiration date. Then, it is easy to calculate.

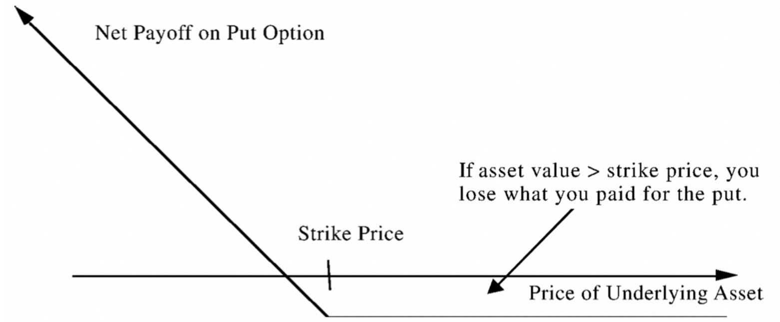
Options are separated into two types: call options and put options. A call option allows the owner to purchase an asset at a specified price for a given period of time. A put option allows the owner to sell an asset at a specified price for a certain period of time. Note that the popular options that are traded are American options because they can be bought or sold at any moment until the expiry date. However, in this thesis, we focus on European options because investors can exercise them only at the expiration date. Moreover, in the exchange-traded option market, investors must buy or sell in units of contract. One contract is 100 shares.

The cash payment upon option expiration is depicted in the payoff graph. In the case of call options, if the value of the underlying asset is lower than the strike price, the net payoff will be negative. The gross payment is the difference between the value of the underlying asset and the strike price, and the net payoff is the difference between the price of the underlying asset and the strike price. For instance, you should purchase a call option if you anticipate that the stock price will increase at the expiration date. This is illustrated in Figure **??**.

Additionally, if the value of the underlying asset exceeds the strike price, the net




**(a)** Call option.




**(b)** Put option.

**Figure 3.1:** The payoff of options for long position [**?**].

payout of put options will be negative. However, if the asset is below the strike price, the gross payoff will correspond to the difference between the underlying asset's value and the strike price. For instance, if you anticipate that the stock price will decline at expiration time, you should purchase a put option. This is demonstrated in Figure **??**.

**Table 3.1:** The value of options for each position.

| Position | Options | The value of this option |
|---|---|---|
| Long | Call | $\max\{(S_T - K), 0\}$ |
| | Put | $\max\{(K - S_T), 0\}$ |
| Short | Call | $-\max\{(S_T - K), 0\}$ or $\min\{(K - S_T, 0)\}$ |
| | Put | $-\max\{(K - S_T), 0\}$ or $\min\{(S_T - K, 0)\}$ |

Mathematically, the value of an option is represented by an option payoff function. Payoff of options evaluated as a function of the underlying stock price $S_T$ at a maturity time $T$. The value of the option for each position can consider both put and call options with a strike price $K$ [**?**, **?**]. The values of options for each position, which are long and short positions, are represented in Table **??**.

### 3.3 The dataset

We utilized a quotation for mini S&P500 index options. The payouts are determined by the underlying asset's value at maturity time $T$. Cash payouts are generally determined by the interest rate $(r)$, whereas option payouts are determined by the value of the underlying asset at maturity $(S_T)$ and the strike price $(K)$. The quotes were acquired from Bloomberg on December 26, 2020 at 2:55:00 PM. The value of mini S&P500 index was 295.42 and the maturity time $T$ was a month $(T = 0.0833)$. Therefore, the payoffs for holding units $w \in \mathbb{R}$ of an asset for a long position are shown in Table **??**.

**Table 3.2:** The payoffs as functions of the number of units $w$ held.

| Asset | Payoff as a function of the long position $w$ |
|---|---|
| Cash | $e^{rT}w$ |
| Call option | $\max\{(S_T - K), 0\}w$ |
| Put option | $\max\{(K - S_T), 0\}w$ |

Since the illiquid market is an incomplete market, in computation, we add the constraints such as the bid and ask prices representing the greatest possible prices for buyers and sellers in the market, with the bid and ask sizes. Therefore, for each strike price, options have a limited quantity to buy or sell. Examples of options are illustrated in Table **??** and Table **??**. Moreover, both tables present that when a call option's price goes down, the strike price goes up, whereas a put option's price increases as the strike price increases as well.



**Table 3.3:** Examples of call options in market quotes on December 26, 2020 at 2:55:00 PM for options.

| Strike $(K)$ | Bid price | Ask price | Bid size($\times 100$) | Ask size ($\times 100$) |
|---|---|---|---|---|
| 295 | 8.24 | 8.42 | 50 | 128 |
| 296 | 7.65 | 7.82 | 50 | 128 |
| 297 | 7.07 | 7.25 | 50 | 128 |
| 298 | 6.52 | 6.69 | 50 | 128 |
| 299 | 6.00 | 6.16 | 50 | 128 |
| 300 | 5.50 | 5.65 | 50 | 144 |

**Table 3.4:** Examples of put options in market quotes on December 26, 2020 at 2:55:00 PM for options.

| Strike ($K$) | Bid price | Ask price | Bid size($\times$100) | Ask size ($\times$100) |
|---|---|---|---|---|
| 295 | 8.37 | 8.61 | 50 | 50 |
| 296 | 8.77 | 9.02 | 50 | 50 |
| 297 | 9.18 | 9.44 | 142 | 13 |
| 298 | 9.62 | 9.90 | 13 | 13 |
| 299 | 10.09 | 10.37 | 125 | 13 |
| 300 | 10.59 | 10.87 | 112 | 13 |

### 3.4 Simulation of the stock price at time $t$

Most finance professionals believe that asset values are unpredictable and fluctuate. People have trouble comprehending exactly what this implies, yet it is crucial to have a strong understanding of it in order to deal with derivatives. We will construct a few models, such as the geometric Brownian motion (GBM) and the variance gamma distribution (VG) in this section to simulate stock prices in the future.

#### 3.4.1 Geometric Brownian motion (GBM)

We will examine a portfolio composed of only the initial stock, which follows the geometric brownian motion of a random variable $S(t)$, which is a normal distribution with the drift parameter $\mu$, the volatility $\sigma$ and the initial value $S(0)$. It can be written as:



$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \tag{3.1}$$

where $B(t)$ is a standard Brownian motion. Then, the solutions of this equation is an Itô process.

The stochastic differential equation implies

$$S(t) = \mu S(t)t + \sigma S(t)B(t),$$

where $S(t)$ is a stochastic process, $\mu$ is the percentage of drift and $\sigma$ is the percentage of volatility.

**Theorem 8.** (Itô's Lemma,[?]). Let $S(t)$ be a stochastic process satisfying Equation (**??**), and assume that we have $G(S,t), G : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$. The Itô's Lemma supposes that $G(S(t),t)$ follows the generalized Brownian motion as follows:

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2}\right) dt + \frac{\partial G}{\partial S} dB(t), \tag{3.2}$$

where $B(t)$ is a standard Brownian motion.

Rockafellar and Uryasev, [?] used the Monte Carlo method to simulate the number of prices ($n$ scenarios). Then, we solve Equation (**??**). We will change the form of the equation above to $G = \log(S)$ and apply Itô's Lemma to this equation. Then, we have

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0,$$

so, we get that

$$\begin{aligned} d(\log(S(t))) &= \mu dt + \sigma dB(t) - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma dB(t). \end{aligned}$$



We can rewrite the equation above to the equation below:

$$\log(S(t)) - \log(S(0)) = (\mu - \frac{1}{2}\sigma^2)t + \sigma B(t). \tag{3.3}$$

Afterwards, we take the exponential of the equation above. We have the stock price at maturity time $T$, which corresponds with the initial price $S_0$. It is shown in the equation below.

$$S(T) = S(0) \exp{(\mu - \frac{1}{2}\sigma^2)T + \sigma B(T)}. \tag{3.4}$$

### 3.4.2 Variance-Gamma (VG) distribution

These days, there are several distributions (e.g., normal distribution, $t$ distribution, and VG distribution) that may be used to simulate stock market returns. Daily stock market returns, as we all know, are not normally distributed because stock market return distributions appear to have tails that are significantly thicker than normal distributions, making it suited for employing other fat tail distributions. The interesting distributions which have fat tails are $t$ distribution and VG distribution. However, in this thesis, we choose VG distribution because VG is derived by the Brownian motion with a constant drift at the gamma distributed time change [?, ?].

Madan [?] introduced stock prices to the VG distribution, which is defined in the equation below, and then we will adopt it to the option prices.

$$S(t) = S(0)\exp(mt + X(t;\sigma_S,\nu_S,\theta_S) + \omega_S t), \tag{3.5}$$

where $\omega_S = \dfrac{1}{\nu_S}\ln(1-\theta_S\nu_S - \sigma_S^2\nu_S/2)$, $m$ is the average rate of return on the stock, and the subscript $S$ on the parameter VG indicates that these are the statistical parameters.

Next, we will change the average rate of return on the stock under this probability measure to the compound interest rate $r$. Let the risk neutral process be given by



$$S(t) = S(0)\exp(rt + X(t;\sigma_{RN},\nu_{RN},\theta_{RN}) + \omega_{RN} t), \tag{3.6}$$

where $\omega_{RN} = \dfrac{1}{\nu_{RN}}\ln(1-\theta_{RN}\nu_{RN} - \sigma_{RN}^2\nu_{RN}/2)$ and the subscript $RN$ on the parameter VG indicates that these are the risk neutral parameters.

**Theorem 9.** (Dilip B. Madan and Eugene Seneta, [?, ?]). The density function for price $z = S(t)$ at exercise time $(t)$ has a log-VG distribution dynamics of Equation (**??**), and is defined as:

$$f_{VG}(z) = \frac{2\exp(\theta x/\sigma^2)}{\nu^{(t/\nu)}\sqrt{2\pi}\sigma\Gamma(\frac{t}{\nu})}\left(\frac{x^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}} K_{\frac{t}{\nu}-\frac{1}{2}}\frac{1}{\sigma^2}\sqrt{x^2(2\sigma^2/\nu+\theta^2)}, \tag{3.7}$$

where $K$ is the modified Bessel function of the second kind and

$$x = \ln(z) - \ln(S(0)) - mt - \frac{t}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2).$$

Assume that $r$ is the interest rate and this expectation is taken under the risk neutral process of Equation (**??**). Then, in the long position, a call option price of strike $K$ and maturity time $T$ is called as $c(S(0);K,T)$. It is defined by

$$c(S(0);K,T) = e^{-rT}\mathbb{E}\left[\max(S(T) - K, 0)\right]. \tag{3.8}$$

## 3.5 Portfolio optimization model using CVaR

With all the above mentioned, in Chapter II, we point to minimizing the CVaR for a portfolio. Since Rockafellar and Uryasev considered the portfolio optimization in the stock market, we apply thie model in the options market. The datasets that area used to optimize the portfolio are the S&P500 Mini Index options.

First of all, we determine all the parameters. The number of assets is $n$ assets. The loss function associated with the portfolio or the decision vector $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$. The underlying price at an expiration date $T$ is a random variable and it is called $S_T$. Then, the return vector $R(S_T)$, which is the loss vector, is also a random variable [**?**].

**Table 3.5:** The return of portfolio ($R(S_T)$) for each asset.

| Assets | Position | Portfolio return of the position |
|---|---|---|
| Cash | - | $(e^{rT} - 1)/1$ |
| Call | Long | $\max\{(S_T - K), 0\}$/Ask price |
| | Short | $\max\{(S_T - K), 0\}$/Bid price |
| Put | Long | $\max\{(K - S_T), 0\}$/Ask price |
| | Short | $\max\{(K - S_T), 0\}$/Bid price |

Although options can be bought and sold in a similar way to stocks, the returns of the portfolio are completely different. We have displayed the return of portfolio $R(S_T)$

in Table **??**. Afterwards, we use the portfolio return from Table **??** to define the loss function or the negative of the portfolio return $f(w, S_T)$. It is determined as:

$$f(w, S_T) = -w^T R(S_T) = -\left[w_1 R_1(S_T^1) + \cdots + w_n R_n(S_T^n)\right]. \tag{3.9}$$

From Equation (**??**), we have the probability of loss function, not exceeding a threshold $\alpha$ is determined by the cumulative distribution function. It is defined in Equation (**??**).

$$\Psi(w, \alpha) = \int_{f(w,S_T)\leq\alpha} p(S_T)\, dS_T. \tag{3.10}$$

Moreover, from Definition **??** and Definition **??**, we can determine the $\beta$-VaR value ($\alpha_\beta(w)$) and the $\beta$-CVaR value ($\phi_\beta(w)$) as shown in Equation (**??**) and Equation (**??**), respectively.

$$\alpha_\beta(w) = \min\{\, \alpha \in \mathbb{R} \mid \Psi(w, \alpha) \geq \beta \,\}, \tag{3.11}$$

and

$$\phi_\beta(w) = (1-\beta)^{-1} \int_{f(w,S_T)\geq\alpha_\beta(w)} f(w, S_T) p(S_T)\, dS_T. \tag{3.12}$$

However, Equation (**??**) is extremely difficult to solve because it has the condition of the integrable term. So, we need to approximate CVaR by using the auxiliary function. Next, we will explain about this auxiliary function and minimization CVaR.

**Definition 10.** (Rockafellar and Uryasev, [**?**]). The practical method of approach is a description of $\phi_\beta(w)$ and $\alpha_\beta(w)$ in terms of the function $F_\beta$ on $w \times \mathbb{R}$ defined as follows:

$$F_\beta(w, \alpha) = \alpha + (1-\beta)^{-1} \int_{S_T\in\mathbb{R}^n} [f(w, S_T) - \alpha]^+ p(S_T)\, dS_T, \tag{3.13}$$

where

$$[f(w,S_T)-\alpha]^+ = \begin{cases} f(w,S_T)-\alpha, & f(w,S_T)-\alpha > 0, \\ 0, & f(w,S_T)-\alpha \le 0. \end{cases}$$

Due to a convex property, we will claim that the auxiliary function in Equation (**??**) is a convex function. A mathematical proof is shown in Lemma **??**.

*Assumption* 11 (Shapiro and Wardi, [**?**]). There is a random variable which is a positive value $C = C(\omega)$ where $\mathbb{E}[C]$ is finite and

$$\mid h(\theta_1,\omega) - h(\theta_2,\omega) \mid \le C(\omega)\|\theta_1-\theta_2\|,$$

for almost all $\omega \in \Omega$, for all $\theta_1, \theta_2 \in D$ where $D$ is an open subset in $\mathbb{R}^n$ and $\|\cdot\|$ is the Euclidean norm .

*Assumption* 12 (Shapiro and Wardi, [**?**]). The function $H : \mathbb{R}^n \to \mathbb{R}$ is a directionally differentiable at a point $\theta \in \mathbb{R}^n$ if the limit

$$H'(\theta,d) = \lim_{t\to 0^+} \frac{H(\theta+td)-H(\theta)}{t}$$

exists for all $d \in \mathbb{R}^n$.



*Assumption* 13 (Shapiro and Wardi, [**?**]). The function $h(\theta)$ with probability one is directionally differentiable at $\theta_0 \in D$.

**Proposition 14** (Shapiro and Wardi, [**?**])**.** Assume that either assumption **??** or **??** hold, or the function $h(\theta)$ is a convex with probability one. Then, the expected value of function $H(\theta)$ is directionally differentiable at $\theta_0 \in D$ and

$$H'(\theta_0,d) = \mathbb{E}h'(\theta_0,d). \tag{3.14}$$

**Lemma 15.** With $w$ fixed, let $G(\alpha) = \displaystyle\int_{S_T\in\mathbb{R}^n} g(\alpha,S_T)p(S_T)\,dS_T$, where $g(\alpha,S_T) = [f(w,S_T)-\alpha]^+$. Then, $G(\alpha)$ is a convex continuously differentiable function with derivative

$$G'(\alpha) = \Psi(w, \alpha) - 1. \tag{3.15}$$

*Proof.* Assume that $G(\alpha) = \int_{S_T \in \mathbb{R}^n} g(\alpha, S_T) p(S_T)\, dS_T$.
By Proposition **??**,

$$\begin{aligned}
G'(\alpha) &= \mathbb{E}[g'(\alpha, S_T)] \\
&= \mathbb{E}\left[\frac{\partial}{\partial \alpha}\left[f(w, S_T) - \alpha\right]^+\right] \\
&= \mathbb{E}\left[\frac{\partial}{\partial \alpha}\left[f(w, S_T) - \alpha\right] \mathbb{1}_{f(w,S_T) \geq \alpha}\right] \\
&= \mathbb{E}\left[\left[f(w, S_T) - \alpha\right] \frac{\partial}{\partial \alpha} \mathbb{1}_{f(w,S_T) \geq \alpha} + \mathbb{1}_{f(w,S_T) \geq \alpha}(-1)\right] \\
&= \mathbb{E}\left[-\mathbb{1}_{f(w,S_T) \geq \alpha}\right] \\
&= -\mathbb{E}\left[\mathbb{1}_{f(w,S_T) \geq \alpha}\right] \\
&= -\int_{f(w,S_T) \geq \alpha} p(S_T)\, dS_T \\
&= -\left(1 - \int_{f(w,S_T) \leq \alpha} p(S_T)\, dS_T\right) \\
&= \int_{f(w,S_T) \leq \alpha} p(S_T)\, dS_T - 1 \\
&= \Psi(w, \alpha) - 1.
\end{aligned}$$



As can be seen from the prove above, $G'(\alpha)$ is a increasing function in $\alpha$ and $G''(\alpha) > 0$ for all $\alpha \in \mathbb{R}$. Then, $G(\alpha)$ is a convex function. Thus, $F_\beta(w, \alpha)$ in Equation (**??**) is a convex function. Therefore, we use it to approximate CVaR or $\phi_\beta(w)$.

**Theorem 16** (Rockafellar and Uryasev, [**?**])**.** For a function of $\alpha$, if $F_\beta(w, \alpha)$ is convex and continuously differentiable, then we can determine the $\beta$-CVaR of the loss associated with any $w \in W$ as follows:

$$\phi_\beta(w) = \min_{\alpha \in \mathbb{R}} F_\beta(w, \alpha) \tag{3.16}$$

and furthermore,

$$\alpha_\beta(w) \in \operatorname{argmin}_{\alpha\in\mathbb{R}} F_\beta(w,\alpha) \text{ and } \phi_\beta(w) = F_\beta(w,\alpha_\beta(w)).$$

*Proof.* By Equation (**??**) and Lemma (**??**), $F_\beta(w,\alpha)$ is convex and continuously differentiable with a partial derivative,

$$\begin{aligned}
\frac{\partial}{\partial\alpha}F_\beta(w,\alpha) &= \frac{\partial}{\partial\alpha}\left[\alpha + (1-\beta)^{-1}\int_{S_T\in\mathbb{R}^n}[f(w,S_T)-\alpha]^+ p(S_T)\,dS_T\right]\\
&= \frac{\partial}{\partial\alpha}\left[\alpha + (1-\beta)^{-1}\int_{S_T\in\mathbb{R}^n}[g(\alpha,S_T)p(S_T)\,dS_T]\right]\\
&= \frac{\partial}{\partial\alpha}\left[\alpha + (1-\beta)^{-1}G(\alpha)\right]\\
&= 1 + (1-\beta)^{-1}G'(\alpha)\\
&= 1 + (1-\beta)^{-1}\left[\Psi(w,\alpha)-1\right] \qquad (\because \text{From lemma } \textbf{??})\\
&= 1 + (1-\beta)^{-1}\Psi(w,\alpha) - (1-\beta)^{-1}\\
&= (1-\beta)^{-1}\Psi(w,\alpha) + \frac{(1-\beta-1)}{1-\beta}\\
&= (1-\beta)^{-1}\Psi(w,\alpha) - (1-\beta)^{-1}\beta\\
&= (1-\beta)^{-1}[\Psi(w,\alpha)-\beta].
\end{aligned}$$

Therefore, the values of $\alpha$ providing the minimum of $F_\beta(w,\alpha)$ are accurately those for which $\Psi(w,\alpha)-\beta=0$. This further yields the validity of the $\beta$-VaR. In particular, we have



$$\begin{aligned}
\min_{\alpha\in\mathbb{R}} F_\beta(w,\alpha) &= F_\beta(w,\alpha_\beta(w))\\
&= \alpha_\beta(w) + (1-\beta)^{-1}\int_{S_T\in\mathbb{R}^n}[f(w,S_T)-\alpha_\beta]^+ p(S_T)\,dS_T.
\end{aligned}$$

The equation above is seen as difficulty due to an integrable term. So, we will identify the integral here as equal to

$$\begin{aligned}
&\int_{S_T\in\mathbb{R}^n}[f(w,S_T)-\alpha_\beta]^+ p(S_T)\,dS_T\\
&= \int_{f(w,S_T)\geq\alpha_\beta(w)}[f(w,S_T)-\alpha_\beta]p(S_T)\,dS_T\\
&= \int_{f(w,S_T)\geq\alpha_\beta(w)} f(w,S_T)p(S_T)\,dS_T - \alpha_\beta(w)\int_{f(w,S_T)\geq\alpha_\beta(w)} p(S_T)\,dS_T. \qquad (3.17)
\end{aligned}$$

By Equation (**??**),

$$\int_{f(w,S_T)\geq\alpha_\beta(w)} f(w,S_T)p(S_T)\,dS_T = \phi_\beta(w)(1-\beta).$$

And by Equation (**??**),

$$\beta = \Psi(w,\alpha_\beta(w)) = \int_{f(w,S_T)\leq\alpha_\beta(w)} p(S_T)\,dS_T = 1 - \int_{f(w,S_T)\geq\alpha_\beta(w)} p(S_T)\,dS_T.$$

Therefore, we get that

$$\int_{f(w,S_T)\geq\alpha_\beta(w)} p(S_T)\,dS_T = 1-\beta.$$

Afterwards, adding $\int_{S_T\in\mathbb{R}^n}[f(w,S_T)-\alpha_\beta]^+p(S_T)\,dS_T$ and $\int_{f(w,S_T)\geq\alpha_\beta(w)} p(S_T)\,dS_T$ into Equation (**??**), we get

$$\int_{S_T\in\mathbb{R}^n}[f(w,S_T)-\alpha_\beta]^+p(S_T)\,dS_T = \phi_\beta(w)(1-\beta)-\alpha_\beta(w)(1-\beta).$$

Finally, we can conclude that

$$\begin{aligned}\min_{\alpha\in\mathbb{R}} F_\beta(w,\alpha) &= \alpha_\beta(w) + (1-\beta)^{-1}\int_{S_T\in\mathbb{R}^n}[f(w,S_T)-\alpha_\beta]^+p(S_T)\,dS_T \\ &= \alpha_\beta(w) + (1-\beta)^{-1}\left[\phi_\beta(w)(1-\beta)-\alpha_\beta(w)(1-\beta)\right] \\ &= \alpha_\beta(w)+\phi_\beta(w)-\alpha_\beta(w) \\ &= \phi_\beta(w).\end{aligned}$$



**Theorem 17.** (Rockafellar and Uryasev, [**?**]) The minimization of $\beta$-CVaR of the loss associated with $w$ across all $w \in W$ is equivalent to minimizing $F_\beta(w,\alpha)$ across all $(w,\alpha)\in W\times\mathbb{R}$, in the sense that

$$\min_{w\in W}\phi_\beta(w) = \min_{(w,\alpha)\in W\times\mathbb{R}} F_\beta(w,\alpha), \tag{3.18}$$

where

$$F_\beta(w,\alpha) = \alpha + (1-\beta)^{-1} \int_{S_T \in \mathbb{R}^n} [f(w,S_T) - \alpha]^+ p(S_T)\, dS_T, \qquad (3.19)$$

and

$$[f(w,S_T) - \alpha]^+ = \begin{cases} f(w,S_T) - \alpha, & f(w,S_T) - \alpha > 0, \\ 0, & f(w,S_T) - \alpha \leq 0. \end{cases}$$

*Proof.* We will show that $\min_{w \in W} \min_{\alpha \in \mathbb{R}} F_\beta(w,\alpha) = \min_{(w,\alpha) \in W \times \mathbb{R}} F_\beta(w,\alpha)$. For each $w$,

$$\min_{\alpha \in \mathbb{R}} F_\beta(w,\alpha) \geq \min_{(w,\alpha) \in W \times \mathbb{R}} F_\beta(w,\alpha).$$

After taking minimum over $w$,

$$\min_{w \in W} \min_{\alpha \in \mathbb{R}} F_\beta(w,\alpha) \geq \min_{(w,\alpha) \in W \times \mathbb{R}} F_\beta(w,\alpha).$$

By Theorem **??**, $F_\beta(w,\alpha)$ is a convex function. We have $(w_0,\alpha_0)$ that makes $F_\beta(w,\alpha)$ minimum.



$$F_\beta(w_0,\alpha_0) = \min_{(w,\alpha) \in W \times \mathbb{R}} F_\beta(w,\alpha).$$

Then,

$$\min_{\alpha \in \mathbb{R}} F_\beta(w_0,\alpha) \leq F_\beta(w_0,\alpha_0) = \min_{(w,\alpha) \in W \times \mathbb{R}} F_\beta(w,\alpha).$$

Similarly,

$$\min_{\alpha \in \mathbb{R}} F_\beta(w,\alpha) \leq \min_{\alpha \in \mathbb{R}} F_\beta(w_0,\alpha).$$

Therefore,

$$\min_{w\in W}\min_{\alpha\in\mathbb{R}} F_\beta(w,\alpha) = \min_{(w,\alpha)\in W\times\mathbb{R}} F_\beta(w,\alpha).$$

From Theorem **??**,

$$\phi_\beta(w) = \min_{\alpha\in\mathbb{R}} F_\beta(w,\alpha).$$

Hence,

$$\min_{w\in W} \phi_\beta(w) = \min_{(w,\alpha)\in W\times\mathbb{R}} F_\beta(w,\alpha).$$

Rockafellar and Uryasev [?] claimed that we can use the auxiliary function $F_\beta(w,\alpha)$, which is shown in Equation (**??**), to minimize the CVaR in place of Equation (**??**). Since Equation (**??**) is a convex function and if $W$ is a convex set, then the CVaR minimization problem in Theorem **??** is a convex programming problem.

Moreover, the integral term is a multiple integral in Equation (**??**), since Theorem **??** specifies the auxiliary function used to estimate the CVaR value. This work, however, simply uses the underlying value ($S_T$). The current spot price is 295.42. The multiple integral is then converted to a one-dimensional integral. Then, the auxiliary function utilized to estimate the CVaR value is given as:



$$F_\beta(w,\alpha) = \alpha + (1-\beta)^{-1}\int_{S_T\in\mathbb{R}} [f(w,S_T)-\alpha]^+ p(S_T)\, dS_T, \tag{3.20}$$

where

$$[f(w,S_T)-\alpha]^+ = \begin{cases} f(w,S_T)-\alpha, & f(w,S_T)-\alpha > 0, \\ 0, & f(w,S_T)-\alpha \le 0. \end{cases}$$

### 3.5.1 The auxiliary functions of CVaR function for any mathematical techniques

The integral term in Equation (**??**) of $F_\beta(x,\alpha)$ can be approximated in various ways. For example, this can be done by sampling using the Monte Carlo integration technique, the Riemann sum and the Gaussian Legendre quadrature which we know the probability distribution function of a random variable $S_T$.

#### 3.5.1.1 Monte Carlo integration technique

Assume that $S_T^1, S_T^2, S_T^3, ..., S_T^q$ is a sample set. Then, we can adjust the auxiliary function (**??**) to the approximate function below and then we can get an approximate solution to the minimization of $F_\beta(w,\alpha)$ over $W \times \mathbb{R}$.

$$\tilde{F}_\beta(w,\alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} \left[ -w^T R(S_T^k) - \alpha \right]^+ . \tag{3.21}$$

The mathematical model for minimizing CVaR or $F_\beta(w,\alpha)$ without liability over $W \times \mathbb{R}$ [**?**] can be written by

$$\min_{(w,\alpha)\in W\times\mathbb{R}} \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} u_k, \tag{3.22}$$



where $u_k = \left[ -w^T R(S_T^k) - \alpha \right]^+$,
subject to

$$\begin{aligned} f(w,S_T) &= -\left[ w_1 R_1(S_T^1) + \cdots + w_n R_n(S_T^n) \right] = -w^T R(S_T), \\ u_k &\geq 0, \\ w^T R(S_T^k) + \alpha + u_k &\geq 0, \quad k = 1,2,...,q. \end{aligned}$$

We can simulate the stock prices for each asset using the normal distribution, variance-gamma (VG) distribution, geometric brownian motion (GBM), and lognormal distribution from the portfolio optimization above, and then plug them into Equation (**??**) to solve the linear programming problem. However, the Monte Carlo integration technique can take a long time because of the simulation of the stock prices. Furthermore, we would like to suggest other techniques such as the Riemann integral and the Gaussian Legendre quadrature.

#### 3.5.1.2 Riemann integration

The Riemann sum is a well-known technique which enables the integration of any continuous function. As a result, we would recommend the use of this technique so we can convert the integrable term in Equation (**??**) to another function that can approximate the CVaR of this portfolio.

**Definition 18.** (Riemann sum, [**?**]). Let $[a, b]$ be the closed and bounded interval. This interval is partitioned by points into $n$ subintervals where $a < x_1 < x_2 < \ldots < x_{n-1} < b$. Then, Riemann sum of function $f(x)$ over interval $[a, b]$ is equal to

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),$$

where $x_i^*$ is the point between $x_{i-1}$ and $x_i$.

By Definition **??**, we can create approximate function which is used to solve the integral function in Equation (**??**). Firstly, let $a$ and $b$ be the lower bound and the upper bound of the stock prices $S$ at time $T$, respectively. Then, the mathematical model for minimizing the CVaR value using the Riemann sum technique is equal to the model below under the same constraints that are used in the mathematical model **??**.

$$\min_{(w,\alpha)\in W\times\mathbb{R}} \alpha + \frac{1}{(1-\beta)} \sum_{k=1}^{n} u_k p(S_T^k)\Delta S_T, \tag{3.23}$$

$$\text{or} \min_{(w,\alpha)\in W\times\mathbb{R}} \alpha + \frac{p(S_T)\Delta S_T}{(1-\beta)} \sum_{k=1}^{n} u_k, \tag{3.24}$$

where

$$u_k = \left[w^T S_T^k - \alpha\right]^+ \text{ and } \Delta S_T = \frac{b-a}{n}.$$

Although Riemann integration takes less time than the Monte Carlo technique, the stock prices are not just sampled because they are the values between the lowest and largest stock prices. This means that the stock prices are really specific. Therefore, the Gaussian Legendre quadrature is advocated.

#### 3.5.1.3 Gaussian Legendre quadrature

The Gaussian Legendre quadrature is one of the many numerical integral techniques. It is defined by the closed interval $[-1, 1]$ and the corresponding weight $m_k$ for each point $x_k$. The weight is defined as [?]:

$$m_k = \frac{2}{(1-(x_k)^2)[P_q'(x_k)]^2}, \qquad \text{for } k = 1, \cdots, q, \tag{3.25}$$

where $q$ is the number of the Gaussian quadrature points, and the Legendre polynomial $P_q(x)$ is the related orthogonal polynomial. If we consider the $q^{th}$ polynomial normalized by the given $P_q(1) = 1$, the $k^{th}$ Gaussian node, $x_k$ is the $k^{th}$ root of $P_q$.

Since the assumption of the Gaussian Legendre quadrature is complicated, we will give examples of weights and points [?] in Table **??**.

**Table 3.6:** Example points and weights of the Gaussian Legendre quadrature [?].

| A number of points ($k$) | points ($x_k$) | weights ($m_k$) |
|---|---|---|
| 1 | $0$ | $2$ |
| 2 | $\pm\frac{1}{\sqrt{3}}$ | $1$ |
| 3 | $0$ | $\frac{8}{9}$ |
| | $\pm\sqrt{\frac{3}{5}}$ | $\frac{5}{9}$ |
| 4 | $\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$ | $\frac{18+\sqrt{30}}{36}$ |
| | $\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$ | $\frac{18-\sqrt{30}}{36}$ |

Thus, we minimize the CVaR or $F_\beta$ over $W \times \mathbb{R}$, which is the same as the model (**??**), but changing the simulation method to the Gaussian Legendre quadrature. It is defined as:

$$\min_{(w,\alpha)\in W\times\mathbb{R}} \alpha + \frac{1}{(1-\beta)}\sum_{k=1}^{q} u_k p(S_T^k) m_k, \tag{3.26}$$

where $S_T^k$ is the $k^{th}$ point corresponding to the weight $m_k$ and $u_k = \left[f(w, S_T^k) - \alpha\right]^+$. Moreover, we optimize this linear programming by using the same constraints as in the mathematical model of Equation (**??**) and Equation (**??**).

In 2000, Rockafellar and Uryasev [**?**] used the Monte Carlo method to simulate values of the underlying price at time $T$, however, this thesis employs the Gaussian quadrature method. As a result, the elapsed time spent to estimate expectation value, simulate option prices and solve an optimization problem is considerably reduced. As we know, many simulations are needed to accurately estimate the expected value. This is even more true for derivative portfolios that require more simulated paths. In addition to the long time required for simulation, the large number of simulated paths improves optimization. This is because, we should insert a dummy variable to convert Lemma **??** into a linear programming problem to solve the optimization problem.

Another issue in estimating the expected value of the optimization problem using a simulation approach is that the underlying simulation value is not far enough from the spot at maturity. This means that some derivatives, such as call options with very high strike prices, will not expire in cash. The best solution is to sell this option as short as possible. However, the probability of a call option expiring in money is not zero.

The expected value is an essential integral and its domain is a non-negative real number. However, such improper integrals can not be evaluated. Therefore, we only estimate in the domain where the underlying value ranges from zero to a sufficiently large number at maturity. We will derive the background of New theorem in Appendix A.

**Theorem 19** (New theorem)**.** The minimization of CVaR in the derivatives market where the derivatives are written only on a single underlying value, which can be written as:

$$\min_{(w,\alpha)\in W\times\mathbb{R}} \tilde{F}_\beta(w,\alpha) = \min_{(w,\alpha)\in W\times\mathbb{R}} \alpha + \frac{1}{(1-\beta)} \sum_{k=1}^{q} \left[ -\sum_{i=1}^{n} \left[ w_i R_i(S_T^k) \right] - \alpha \right]^+ p(S_T^k) m_k, \tag{3.27}$$

where $S_T^k$ is the solution of the Gaussian polynomial on the interval $[0, c]$, where $c$ is a large number, $p(S_T^k)$ is a probability density function of $S_T^k$ and $m_k$ is the corresponding weight of $S_T^k$ that is explained in Equation (**??**).

### 3.5.2 An application of portfolio optimization without liability

As can be seen, we have an indicator function in terms of $[f(w, S_T) - \alpha]^+$ on the auxiliary function $(\tilde{F}_\beta(w,\alpha))$ in Equation (**??**). Since the indicator function in this form is hard to solve, we then reformed it using a similar technique by Rockafellar and Uryasev, [**?**], which will make the function easier to use. Therefore, our minimization portfolio is now defined as follows:

$$\min_{(w,\alpha)\in W\times\mathbb{R}} \tilde{F}_\beta(w,\alpha) = \min_{(w,\alpha)\in W\times\mathbb{R}} \alpha + \frac{1}{(1-\beta)} \sum_{i=1}^{q} u(S_T^i) p(S_T^i) m_i, \tag{3.28}$$

where



$$u(S_T^i) = \left[ f(w, S_T^i) - \alpha \right]^+.$$

Subject to

$$\begin{aligned} u(S_T^i) &\geq 0, \\ u(S_T^i) + \left[ w^T R(S_T^i) + \alpha \right] &\geq 0, \quad i = 1, 2, ..., n, \\ w^T \bar{R}(S_T) &\geq Q, \\ \sum_{i=1}^{n} w_i &= 1, \\ w_i, \alpha &\in \mathbb{R}, \quad i = 1, 2, ..., n, \end{aligned}$$

where $\bar{R}(S_T)$ is the average return and $Q$ is the required return.

### 3.5.3 An application of portfolio optimization with liability

In the previous section, the portfolio optimization model was considered in case of no liability, but now we will minimize the CVaR of the portfolio while having a liability. Then, we will explain in more detail about indifference pricing and accounting value.

#### 3.5.3.1 Indifference pricing

Indifference pricing was established by Stewart Hodges and Anthony Neuberger in 1989 and has since become widely used, particularly in academic research. It is frequently used to price securities in incomplete markets where traditional risk-neutral valuation fails due to a lack of traded assets to create a replicating portfolio [?].

In the financial and insurance industries, indifference pricing is commonly used. The fundamental idea behind indifference pricing is that it is a natural way of pricing and hedging financial instruments with cash flows that cannot be replicated by financial market activities. Furthermore, the indifference price of traded cash flows is non-linear and is influenced by the agent's existing liabilities, risk preferences, and underlying probability measure. Now, we focus on the issue of how much risk must be calculated so that the risk measured in terms of subjective uselessness does not increase in comparison to the risk at the beginning position [?, ?].

Let $\varphi(W_0, C_0)$ be an objective value or the CVaR value from portfolio optimization. Assume that for a trader with an initial wealth $W_0$ and an initial liability $C_0$, the indifference price of the trader with future liability $\bar{C} \in L^\infty(\Omega, \mathcal{F}, \mathcal{P})$ can be defined as:

$$\pi_s\big(W_0, C_0, \bar{C}\big) := \inf\{\bar{W} | \varphi(W_0 + \bar{W}, C_0 + \bar{C}) \leq \varphi(W_0, C_0)\},$$

where $L^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is a set of bounded measurable functions from a sample space.

Then,

$$\pi_s(0,0,\bar{C}) := \inf\{\bar{W}|\varphi(\bar{W},\bar{C}) \leq \varphi(0,0)\},$$

in case of zero initial wealth and zero liability. Therefore, the minimization model of the indifference price for selling is

$$\min_{(w,\alpha)\in W\times\mathbb{R}} \alpha + \frac{1}{(1-\beta)}\sum_{i=1}^{q} u(S_T^i)p(S_T^i)m_i, \tag{3.29}$$

where $u_T^i = \left[f(w,S_T^i) - \alpha + C_T\right]^+$, $C_T = \dfrac{10^4}{W_0}[S_T - K]^+$, and $W_0 = 1.0000 \times 10^5$,

subject to

$$\begin{aligned} f(w,S_i) &= -[w_1 S_T^1 + ... + w_n S_T^n] = -w^T S_T, \\ u_i &\geq 0, \\ u_i + \left[w^T S_T + \alpha - C_T\right] &\geq 0, \quad i = 1,2,...,n. \end{aligned}$$

In addition, if we use the minimization model of the indifference price for buying, we will change the variable $C_T$ from the value of the options that is shown in Table **??** and multiply it by the scalar value $-\dfrac{10^4}{W_0}$.

#### 3.5.3.2 Accounting value

In addition, if the values of $\varphi(0,0) = 0$, then it cannot be true because the indifference prices for selling and the accounting values of the liabilities will be the same.

Moreover, the accounting value is defined as:

$$\pi_s^0(\bar{C}) := \inf\{\bar{W}|\varphi(\bar{W},\bar{C}) \leq 0\}, \tag{3.30}$$

where $\bar{C} \in L^\infty(\Omega,\mathcal{F},\mathcal{P})$.

Since a portfolio has a liability, then the CVaR value must be greater than the

CVaR value of a portfolio without liabilities. Thus, the purpose of indifference pricing and accounting value is to find the smallest amount of money that can make the CVaR value the same as that of a portfolio without liabilities for indifference pricing, however, for the accounting value, the CVaR value should approach zero. The bisection method then becomes an important technique in solving this problem.

#### 3.5.3.3 The bisection method

Although Newton's method is a popular way for solving nonlinear equations, there are other methods that may be useful in some cases. The bisection method is yet another way for solving the nonlinear problem $f(x) = 0$, and it may be utilized if the function f is continuous. For continuous functions, Bolzano's theorem provides the inspiration for this technique:

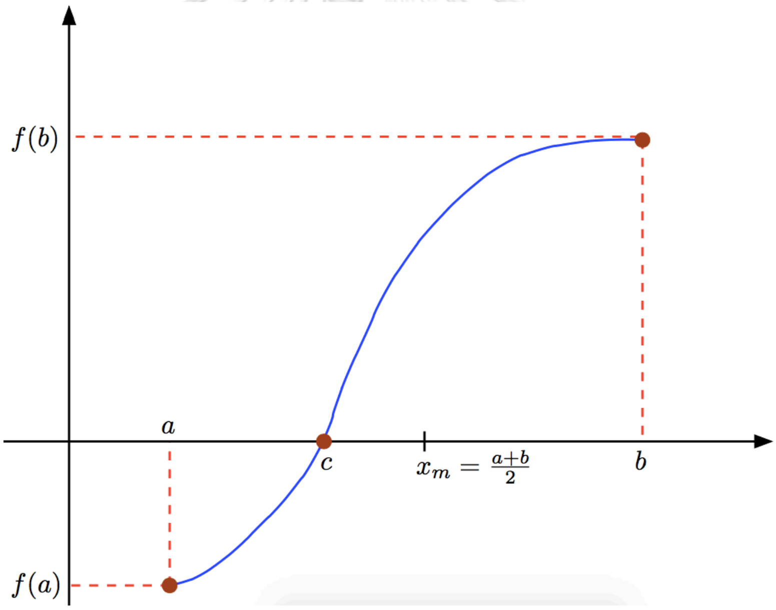



**Figure 3.2:** Illustration of the bisection method [**?**].

**Definition 20** (Bolzano's theorem [**?**, **?**]). If the continuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ and $f(a)f(b) < 0$, then there exist $c \in [a, b]$ such that $f(c) = 0$.

As you see in Figure **??**, by evaluating whether it belongs to either of the two sub-intervals $[a, x_m]$, $[x_m, b]$, where $x_m$ is the midpoint, the bisection method needs to find the value $c$ where the graph of $f(c)$ crosses over zero.

Then, the algorithm of the bisection method follows:

- We have our solution ($x_m$) if $f(x_m) = 0$, and the process ends.
- We will change the mid point if $f(x_m)$ has the opposite sign, this means there is either $f(a)f(x_m) < 0$, or $f(x_m)f(b) < 0$.

Through the above, we will modify this to find the initial wealth added into the portfolio and then project the same CVaR value before or make the CVaR value equal to zero in case of accouting value. So $f(x_m)$ is the continuous function which shows the CVaR value after solving for the solution, and makes $f(x_m)$ approach the CVaR value without liability or make it equal to zero for indifference pricing and accounting values.

#### 3.5.3.4 An application to hedging

Due to high investment risks, some investors want to reduce their risks by hedging. These investors are called hedgers. Hedging is used to insure the risk of investing by opening the buying or selling transactions in the same currency, simultaneously, while reducing the uncertainty and limiting losses. It, however, does not mean making a lot of profit. Although the goal of hedging is to decrease investment risks, there is no assurance that the outcome will be better than that without it [?]. In a complete financial market, every risky claim can be hedged perfectly, but in an incomplete market, it is possible to stay on the safe side by super-hedging [?].

Hedging is commonly used with investment instruments, such as options and futures. For instance, assume that we own shares of stock A and we are convinced in this corporation's long term performance, however, we may be concerned about the industry's losses in the short term. Then, we can buy put options on Stock A in order to hedge investment risks. To hedge investment risks from the downside of the stock A. This strategy is called married put.

In this section, we focus on the question of optimal hedging of a given risky position in an incomplete market. Then, we would introduce the super-hedging strategy. After that, we will derive the sub-hedging strategy by adapting the super-hedging strategy, and we will illustrate how hedgers can reduce their risks with options [?].

Super-hedging is a strategy that uses a self-financing trade plan to insure equities. In a complete market, it may be similar to the hedging price for the initial portfolio, but in an incomplete market, it employs the lowest price that can be paid for a hedged portfolio in order for its value to be greater or equal to that of the initial portfolio at some time in the future.

At maturity time $T$, if we have the option price $S_T$ with portfolio $w$, the liability $C_T$ and the initial wealth $W_0$, then super hedging can be described as the mathematical equation below.

$$\pi_{\sup} = \inf\{S_T \cdot w \geq C_T, P - a.s.\}. \tag{3.31}$$

We would like to minimize

$$\min S_0 \cdot w, \tag{3.32}$$

subject to

$$\begin{aligned} S_0 \cdot w &\leq W_0, \\ S_T \cdot w - C_T &\geq 0, \\ l \leq w_i &\leq u, \quad i = 1, 2, ..., n. \end{aligned}$$



On the other hand, the sub-hedging price is the highest amount that may be paid to insure that for any future situation. So the sub-hedging costs with a liability $C_T$ is given by

$$\pi_{\inf} = \sup\{S_T \cdot w \leq C_T, P - a.s.\}. \tag{3.33}$$

Since it is the opposite of super-hedging, then the minimization can be changed to a negative form of the super-hedging. It can be defined as:

$$\min -(S_0 \cdot w), \tag{3.34}$$

subject to

$$\begin{aligned} S_0 \cdot w &\leq W_0, \\ S_T \cdot w + C_T &\geq 0, \\ l \leq w_i &\leq u, \quad i = 1, 2, ..., n. \end{aligned}$$

As we know, the super-hedging price is the least amount of the necessary price that is used to insure the portfolio, while the sub-hedging price is the greatest earning one can get by entering the position of super-hedging with the negative liability $C_T$. In general, the expenses of super-hedging and sub-hedging are to be considered comparable in the claims quotations.

## 3.6 Numerical implementation

### 3.6.1 Linear programming to solve portfolio optimization using CVaR without liability



We have the auxiliary function ($\tilde{F}_\beta(w, \alpha)$) in Equation (**??**), which is used to approximate the CVaR value. In 2000, Rockafellar and Uryasev suggested the Monte Carlo technique to solve the integration function as you can see in Equation (**??**). Now, we adapt this technique to the Gaussian quadrature technique. Thus, the minimization of CVaR for this portfolio as shown in Equation (**??**) is solved by using Linprog that is a built-in function in Matlab as this problem is one of the linear problems.

Next, we will explain the Linprog's syntex to solve the linear programming problem.

To begin with, let $z^T = \left[w^1, w^2, \cdots, w^n, \alpha, u^1, u^2, \cdots, u^m\right]$, then the Linprog's syntex is as follows:

$$\min f^T z, \tag{3.35}$$

subject to

$$A \times z \leq b,$$
$$Aeq \times z = beq,$$
$$lb \leq z \leq ub,$$

where $f, b, beq, lb, ub$ are vectors and $A, Aeq$ are matrices.

The function called for solving linear problems is $linprog(f, A, b, Aeq, beq, lb, ub)$ and then the important outputs are the optimal portfolio $z$ and the optimal value of $f^T z$.

After we change the minimization portfolio from model (**??**) to the linear programming problem in model (**??**), we have the solutions to the optimal portfolio and optimal value, which are the selected portfolio and the CVaR value, respectively.

For the maturity time $T$, our problem can be written as,



$$\min f^T z, \tag{3.36}$$

subject to

$$A \times z \leq b,$$
$$Aeq \times z = beq,$$
$$lb \leq z \leq ub,$$

where $f^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & \frac{1}{1-\beta}p(S_1)w_1 & \frac{1}{1-\beta}p(S_2)w_2 & \cdots & \frac{1}{1-\beta}p(S_m)w_m \end{bmatrix}$,

$$A = -\begin{bmatrix} R(S_1^1) & R(S_1^2) & \cdots & R(S_1^n) & 1 & 1 & 0 & \cdots & 0 \\ R(S_2^1) & R(S_2^2) & \cdots & R(S_2^n) & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R(S_m^1) & R(S_m^2) & \cdots & R(S_m^n) & 1 & 0 & 0 & \cdots & 1 \\ \bar{R}(S^1) & \bar{R}(S^2) & \cdots & \bar{R}(S^m) & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Q \end{bmatrix},$$

$$Aeq = -\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, beq = \begin{bmatrix} 1 \end{bmatrix},$$

$R(S_i^j)$ is a return of the $j^{th}$ asset of the $i^{th}$ scenario and $\bar{R}(S^j)$ is a mean return of the $j^{th}$ asset.

### 3.6.2 Linear programming to solve portfolio optimization using CVaR with asset-liability management

With some liability, the loss value of the investment may go up. Then, we would like to know how much of the minimum value of the initial wealth we need to add to the portfolio to attain the same CVaR value as the portfolio with no liability. There are 2 strategies that involve selling and buying.

#### 3.6.2.1 Selling

We can change the minimization portfolio with liability $C_T = \frac{10^4}{W_0}[S_T - K]^+$, where $W_0$ is the initial wealth from the model (**??**) to the linear programming problem in the model (**??**). For the maturiy time $T$, our problem can be written as,

$$\min f^T z, \tag{3.37}$$

subject to

$$A \times z \leq b,$$

$$Aeq \times z = beq,$$

$$lb \leq z \leq ub,$$

where $f^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & \frac{1}{1-\beta}p(S_1)w_1 & \frac{1}{1-\beta}p(S_2)w_2 & \cdots & \frac{1}{1-\beta}p(S_m)w_m \end{bmatrix}$,

$$A = -\begin{bmatrix} R(S_1^1) & R(S_1^2) & \cdots & R(S_1^n) & 1 & 1 & 0 & \cdots & 0 \\ R(S_2^1) & R(S_2^2) & \cdots & R(S_2^n) & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R(S_m^1) & R(S_m^2) & \cdots & R(S_m^n) & 1 & 0 & 0 & \cdots & 1 \\ \bar{R}(S^1) & \bar{R}(S^2) & \cdots & \bar{R}(S^m) & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, b = \begin{bmatrix} -C_T \\ -C_T \\ \vdots \\ -C_T \\ Q \end{bmatrix},$$

$Aeq = -\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, $beq = \begin{bmatrix} 1 \end{bmatrix}$,

$R(S_i^j)$ is a return of the $j^{th}$ asset of the $i^{th}$ scenario and $\bar{R}(S^j)$ is a mean return of the $j^{th}$ asset.

#### 3.6.2.2 Buying

For buying, the liability is the negative of the liability for a selling situation, so we can change the minimization portfolio from model (**??**) to the linear programming problem in model (**??**). For the maturity time $T$, our problem can be written in the same way as in model (**??**), but by changing the value of liability $C_T$.

#### 3.6.2.3 Hedging

After we explain about the super-hedging strategy, then we will know the linear programming for the sub-hedging as the sub-hedging value is opposite to the super-hedging value. To begin with, let $z^T = \begin{bmatrix} w^1, w^2, \cdots, w^n, \alpha, u^1, u^2, \cdots, u^m \end{bmatrix}$, then the Linprog's syntex is as follows:

$$\min W_0, \tag{3.38}$$

subject to

$$A \times z \le b,$$
$$lb \le z \le ub,$$

where $A = -\begin{bmatrix} S_1^0 & -1 \\ S_2^0 & -1 \\ \vdots & \vdots \\ S_m^0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

In Chapter III, we studied illiquid markets, options, and datasets. Afterwards, we adopted the minimization model over portfolio $W$ using CVaR for the options portfolio. Additionally, portfolio optimization was separated into 2 main sections: portfolio optimization without liability and with liability. Moreover, we described the hedging strategy, which is an important technique to insure our portfolio. After that, we changed the mathematical model to linear programming and showed them in the numerical implementation for compliance in MATLAB. In the last section, we explained the distributions used in simulating the options prices. There are 2 common distributions, which are the normal distribution and the VG distribution. In the next chapter, we will present all the results from the portfolio minimization using the CVaR value.

# CHAPTER IV

# RESULTS AND DISCUSSIONS

This chapter is divided into 2 main sections. There are the results of the options portfolio optimization with minimizing CVaR, and then we consider the details about the optimal portfolios with liability; for example, indifference pricing, accounting value, and hedging situations.

Furthermore, we assume that the time to maturity $T = \frac{1}{12}$ year, the investment's beginning wealth is $\$100,000$ and the required return percentage is 1,000, which is a high amount since the required return of options must be more than the required return of shares for trading. We selected the portfolio with the lowest CVaR and the given expected return. A restricted quantity of buying and selling is permitted. This means the amount of bid sizes, ask sizes, bid prices and ask prices are used as the constraints.

## 4.1 Portfolio optimization without liability

We will compare the distributions used for simulated options prices in this section. Let's begin with the geometric Brownian motion (GBM) and then go on to the variance-gamma distribution (VG). We will demonstrate the results of minimizing the CVaR value by using the Monte Carlo technique to simulate the log returns of the mini S&P 500 index and the underlying prices, respectively. After that, we will compare the Monte Carlo method's results with the Gaussian Legendre quadrature's results.

### 4.1.1 Efficiency of portfolio minimization

#### 4.1.1.1 Results from the Monte Carlo technique

Firstly, we suppose that the options prices are simulated by GBM, then afterwards, we minimize this linear problem in Equation (**??**). The assumption of all parameters for this minimization are assumed in Table **??**, but for the VG distribution, the parameters used in computation are in Table **??**. After that, Equation (**??**) is used to minimize the

CVaR for options portfolio.

**Table 4.1:** Base-case parameters including the Black-Scholes model, the compounded interest rate($r$) and the required return ($Q$) for this minimization.

| $\mu$ | $\sigma$ | $r$ | $Q$ |
|---|---|---|---|
| 0.0100 | 0.3500 | 0 | 10 |

**Table 4.2:** Base-case parameters including VG parameters, the compounded interest rate($r$) and the required return ($Q$) for this minimization.

| $\mu$ | $\sigma$ | $\nu$ | $\theta$ | $r$ | $Q$ |
|---|---|---|---|---|---|
| 0 | 0.1206 | 0.0031 | 0 | 0 | 10 |

We generate the options prices for 100,000 iterations used to minimize the CVaR in Equation (**??**). All results are shown in Table **??**, which presents the standard deviation, 95%VaR and 95%CVaR of the CVaR minimization portfolio with any distribution. As we can see, the standard deviation values of the GBM portfolio and the VG portfolio are unlike. There is about a tenfold difference. The distributions of losses in functions from GBM and VG are illustrated in Figure **??** and Figure **??**, respectively. Then, we observe that VaR values are also different. They are a positive value and a negative value from the GBM and the VG distribution, respectively. However, the CVaR values are positive values, meaning that, we will lose money if we invest in this portfolio.

**Table 4.3:** The standard deviation, the VaR and CVaR of the CVaR minimization portfolio with any confidence levels.

| Distribution | Standard deviation ($) | VaR ($) | CVaR ($) | Cash ($) |
|---|---|---|---|---|
| Geometric brownian motion | 6,871,100.00 | 224,810.00 | 226,590.00 | 822,470.00 |
| Variance Gamma (VG) | 567,780.00 | -207,920.00 | 865,370.00 | 3,994,000.00 |

On the other hand, the portfolio selections in Figure **??** and Figure **??** are dissimilar. Further, our data is suitable for the VG distribution more than the GBM because VG distribution has fat tails with elapsed times of 357.5733 and 179.6685 seconds for GBM and VG, respectively. Thus, the elapsed time for the simulation of the VG distribution is less than for GBM by about twofolds. Therefore, we will use VG distribution in the next experiment.

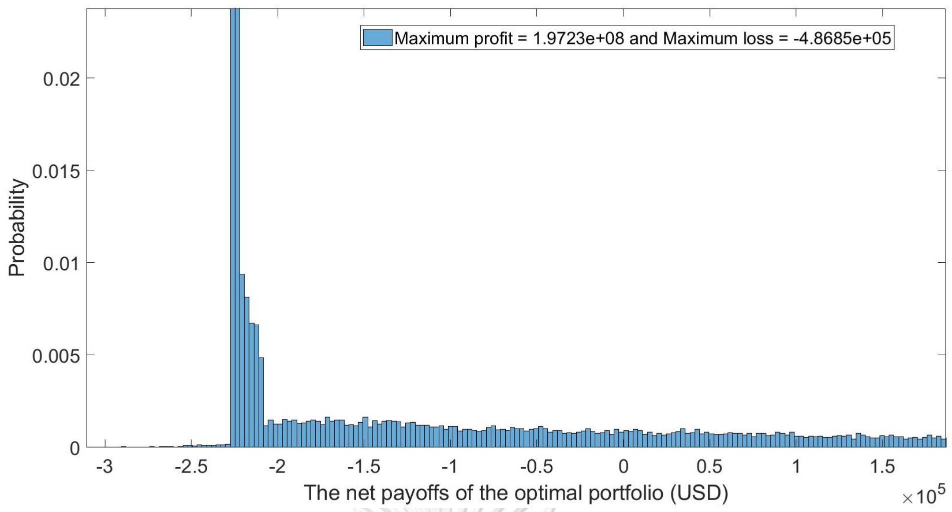



**Figure 4.1:** The net payoff of the portfolio selected by the minimization of CVaR after 30 days for 100,000 iterations with GBM distribution.

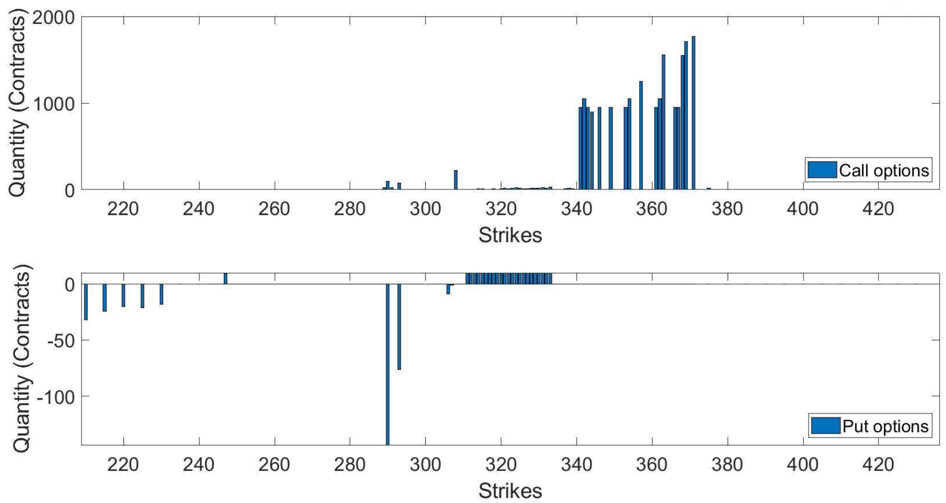



**Figure 4.2:** The portfolio selected by the minimization of CVaR after 30 days for 100,000 iterations with GBM distribution.

#### 4.1.1.2 Results from the Gaussian Legendre quadrature

Since the elapsed times of simulation of the Monte Carlo technique for the GBM and VG distribution takes longer, we would like to change the simulation method from the Monte Carlo to the Gaussian Legendre quadrature. For the Gaussian Legendre quadrature, we require the number of points used to simulate the options prices to be equal to 500 points. As a result, the elaped time for the minimization becomes 5.3631 seconds,

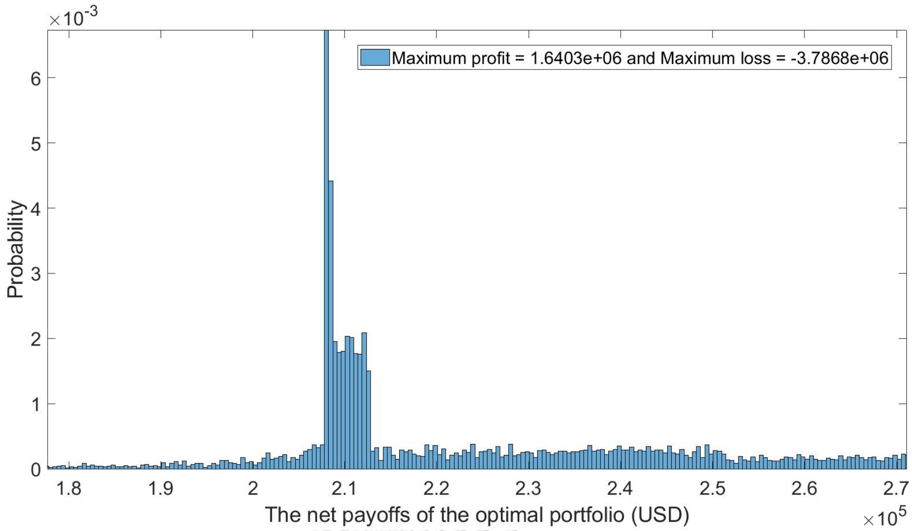



**Figure 4.3:** The net payoff of the portfolio selected by the minimization of CVaR after 30 days for 100,000 iterations with VG distribution.

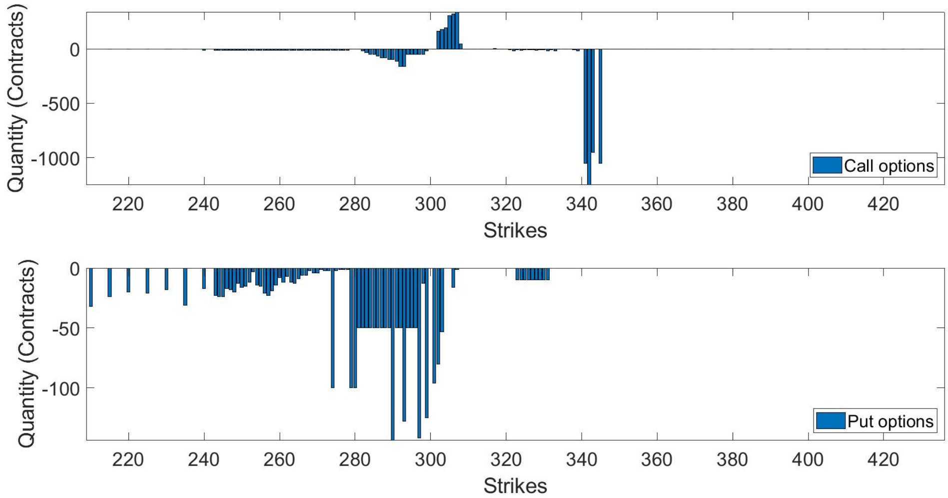



**Figure 4.4:** The portfolio selected by the minimization of CVaR after 30 days for 100,000 iterations with VG distribution.

which is less than the elaped time that is used for the Monte Carlo technique, which is approximately thirty-four folds. Furthermore, in Figure **??** and Figure **??**, the VaR values are nearly equal to the CVaR value in Figure **??** is $80,555.00.

### 4.1.2 The main model

From the previous subsection, we know that VG distribution and the Gaussian Legendre quadrature are appropriate for solving the integral term in Equation (**??**) because

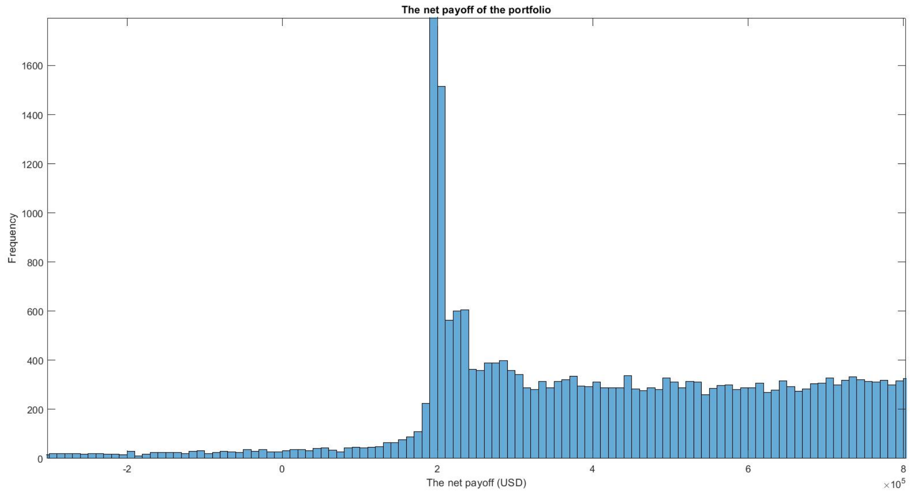



**Figure 4.5:** The net payoff of the portfolio selected by the minimization of CVaR after 30 days for 500 points with VG distribution from the Gaussian Legendre quadrature.

it can simulate many options prices in a short time. Then, these become the reasons to support the use of VG distribution and the Gaussian Legendre quadrature with random options prices and in approximating the auxiliary function that is used to minimize the CVaR value. The aim of this experiment is to find the optimization portfolio by using CVaR and then we get the optimal portfolio from this. Afterwards, we change the parameters (i.e., the confident level, $\sigma$, $\nu$ and the required return $Q$) to consider the behaviour of CVaR values.

**Table 4.4:** The expectation, the standard deviation, the VaR and CVaR of the CVaR minimization portfolio with any confidence levels.

| Confidence levels | Expectation ($) | Standard deviation ($) | VaR ($) | CVaR ($) |
|---|---|---|---|---|
| 90 | 1,098,754.8910 | 496,263.4816 | -500,439.2953 | -134,825.5103 |
| 95 | 1,098,684.8730 | 571,959.8916 | -197,953.5713 | 80,554.9436 |
| 99 | 1,098,400.9050 | 712,572.5382 | 292,746.9923 | 513,764.3919 |

Table **??** provides information about the expectation, the standard deviation, the VaR and CVaR of the CVaR minimization for portfolio with any confidence levels. Overall, if we compare the expectation and the standard deviation of the payoff for 90%, 95% and 99% confidence levels, we can see that the expected values decrease while the standard deviation values and the confidence levels are increasing.

In addition, the 90% VaR, 95% VaR and 99% VaR are -500,439.2953, -197,953.5713 and 292,746.9923, respectively. The 95% VaR is more than 90% VaR and they are negative values. This means we will make a profit if we invest in assets as follows in Figure **??**. In contrast, the 99% VaR is a positive value, therefore, we can make a loss from this portfolio at 99% confidence level. Similarly, we consider the 90% CVaR, 95% CVaR and 99% CVaR which are one negative value at 90% confidence level and two positive values at 95% and 99% confidence levels. The CVaR value at 99% confidence level is the highest and equal to 513,764.3919. From Figure **??**, the expected return and the CVaR values will also increase if the confidence levels increase. This figure seems like an exponential function.

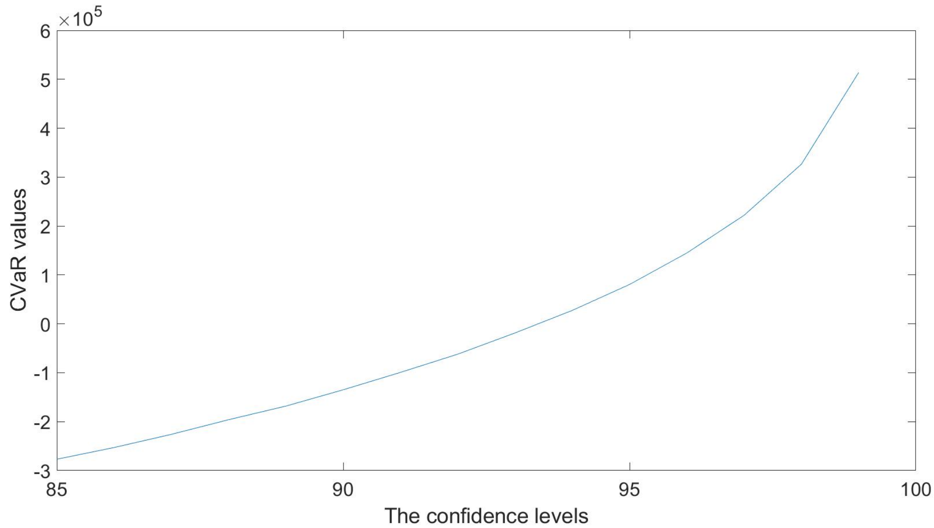



**Figure 4.6:** The graph of the CVaR value with any confidence level.



Moreover, Figure **??** and Figure **??** demonstrate the distribution of the net payoff of the portfolio after 30 days with 95% and 99% confidence levels and present the points of the VaR value with 95% and 99% confidence levels as different from one another because the 95% VaR value is a negative value but 99% VaR value is a positive value. And then, we can approximate the 95% CVaR and the 99% CVaR by finding the mean of the left tails that is beyond the 95% VaR and the 99% VaR cut points, respectively.

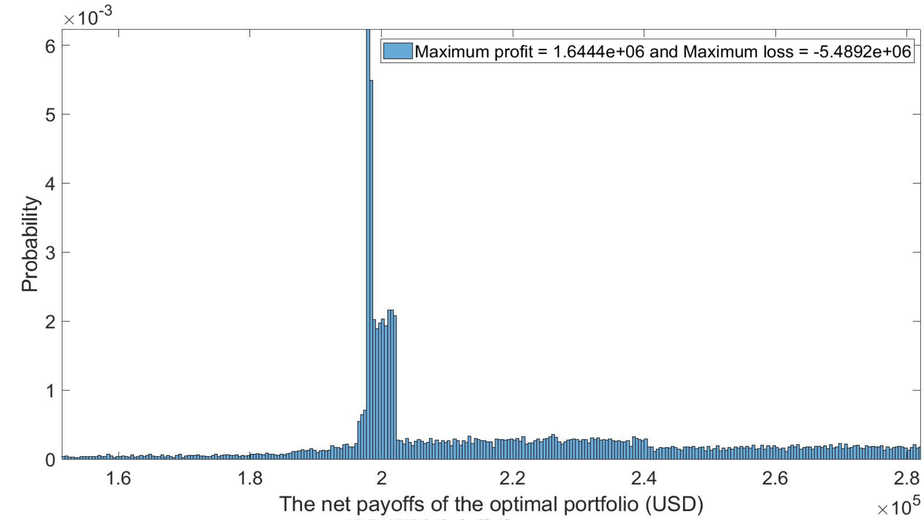



**Figure 4.7:** The net payoff of the portfolio selected by the minimization of CVaR after 30 days with 95% confidence level.

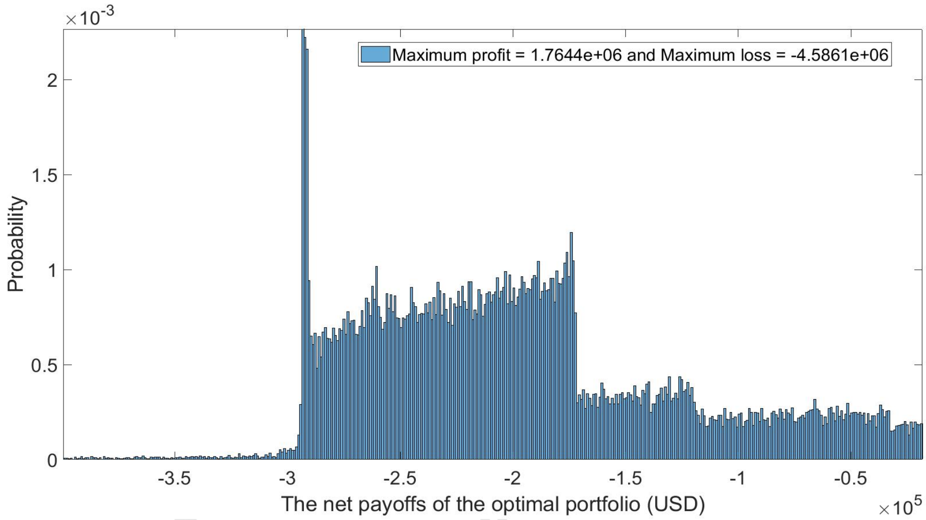





**Figure 4.8:** The net payoff of the portfolio selected by the minimization of CVaR after 30 days with 99% confidence level.

**Table 4.5:** The expectation, the standard deviation, the 95%VaR and 95%CVaR of the CVaR minimization portfolio after changing $\sigma$.

| $\sigma$ | Expectation (\$) | Standard deviation (\$) | VaR (\$) | CVaR (\$) |
|---|---|---|---|---|
| 0.0100 | 2,447,307.2550 | 6,023.9430 | -2,343,309.3110 | -2,332,935.6250 |
| 0.0600 | 2,266,687.0430 | 268,691.2197 | -1,633,867.5360 | -1,313,287.6290 |
| 0.1206 | 1,098,684.8730 | 571,959.8916 | -197,953.5713 | 80,554.9436 |
| 0.1600 | 1,096,318.8480 | 983,849.2708 | 814,298.3118 | 1,572,524.8320 |

Afterwards, we repeat the optimization and alter the settings for simulating the index values using VG distribution such as $\sigma$ and $\nu$ to examine the impact of the optimal

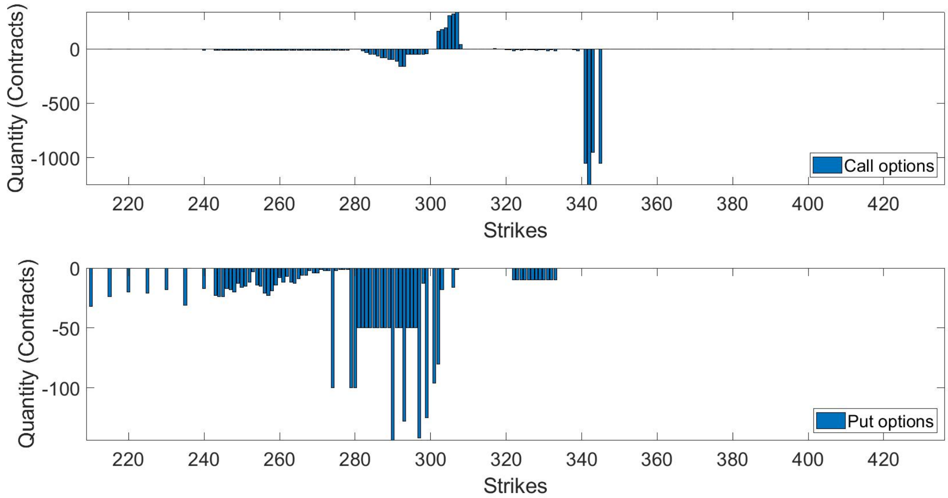



**Figure 4.9:** The proportion of cash and asset $j$ from optimizing the portfolio selected after 30 days with 95% confidence level.

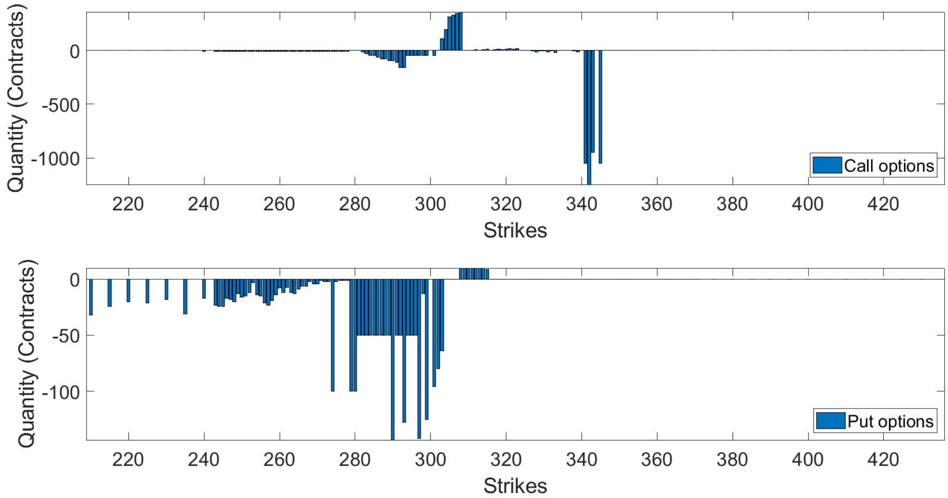



**Figure 4.10:** The proportion of cash and asset $j$ from optimizing the portfolio selected after 30 days with 99% confidence level.

**Table 4.6:** The expectation, the standard deviation, the 95%VaR and 95%CVaR of the CVaR minimization portfolio.

| $\nu$ | Expectation ($) | Standard deviation ($) | VaR ($) | CVaR ($) |
|---|---|---|---|---|
| 0.0031 | 1,098,684.8730 | 571,959.8916 | -197,953.5713 | 80,554.9436 |
| 0.0050 | 1,098,603.7248 | 572,760.2695 | -197,478.7360 | 93,702.8259 |
| 0.0100 | 1,098,435.3860 | 575,311.0251 | -196,886.3640 | 126,445.5404 |
| 0.0500 | 1,098,154.4600 | 615,367.5069 | -105,229.7714 | 302,184.3148 |

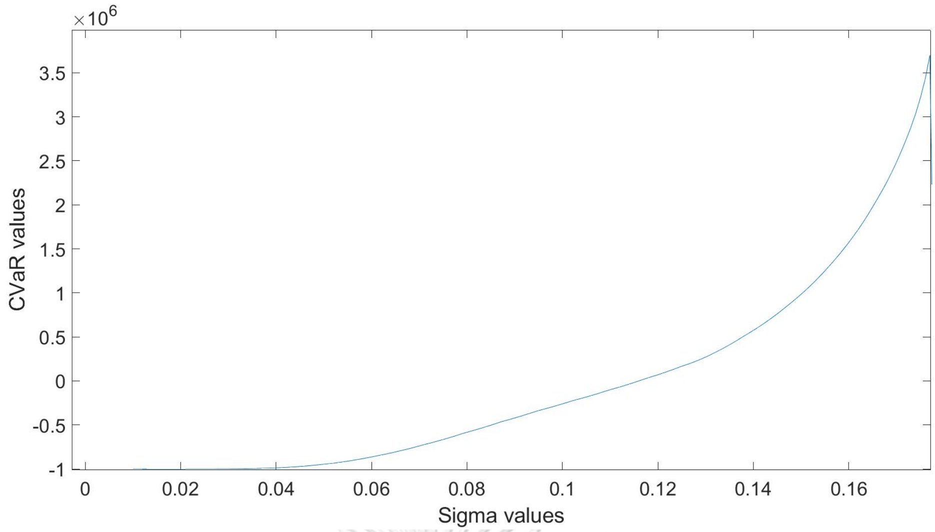



**Figure 4.11:** The graph of the CVaR value with a different $\sigma$.

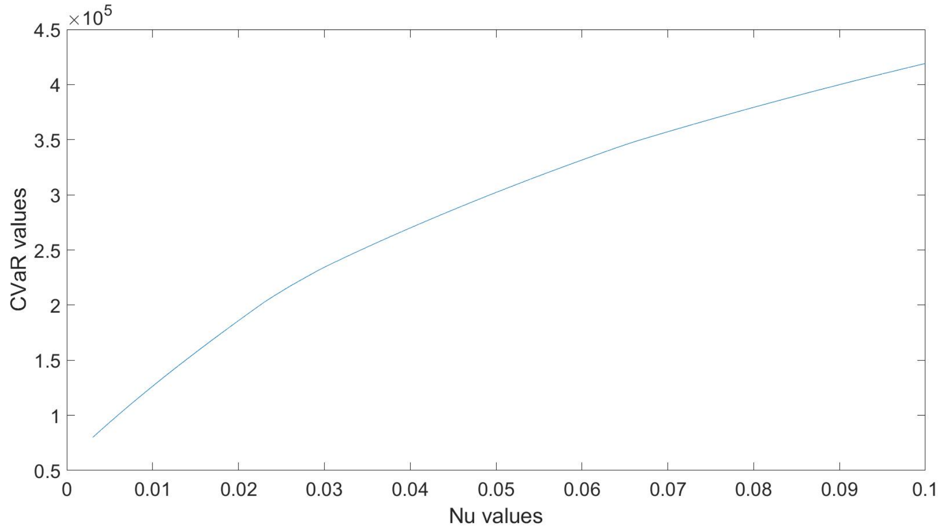





**Figure 4.12:** The graph of the CVaR value with a different $\nu$.

portfolio. Table **??** and Table **??** illustrate that when $\sigma$ and $\nu$ increase, the CVaR value increases as well.

Table **??** also shows the effect of adjusting the standard deviation. It is also interesting to note that the CVaR value increases as the standard deviation goes up, and we can see that if $\sigma = 0.1206$, the portfolio can lose money because CVaR is a positive number. As can be seen, with $\sigma = 0.16$, VaR and CVaR are positive values, implying that the portfolio has a 95% probability of generating more than \$814, 298.3118 and the average amount of the loss function that exceeds the VaR value is \$1, 572, 524.8320. Additionally,

**Table 4.7:** The expectation, the standard deviation, the 95%VaR and 95%CVaR of the CVaR minimization portfolio.

| Required return (×100%) | Expectation ($) | Standard deviation ($) | VaR ($) | CVaR ($) |
|---|---|---|---|---|
| 4 | 500,025.9123 | 173,296.0617 | -277,025.8312 | -182,627.4672 |
| 5 | 599,853.2901 | 253,386.0246 | -279,989.0965 | -165,062.9223 |
| 6 | 699,768.4104 | 298,425.7639 | -317,810.8022 | -135,336.4001 |
| 7 | 799,545.7056 | 372,815.1428 | -304,771.7842 | -95,465.9161 |
| 8 | 899,212.1172 | 440,351.6925 | -281,516.9284 | -48,169.4726 |
| 9 | 998,947.5817 | 508,836.1415 | -247,517.7584 | 8,816.5003 |
| 10 | 1,098,684.8730 | 571,959.8916 | -197,953.5713 | 80,554.9436 |
| 11 | 1,198,304.1340 | 599,717.5156 | -236,811.8675 | 187,709.4893 |
| 12 | 1,297,901.3660 | 634,393.7524 | -164,292.9537 | 338,779.0353 |
| 13 | 1,397,502.2780 | 679,316.5099 | -87,777.0852 | 499,900.0343 |
| 14 | 1,497,063.0400 | 733,950.0379 | -8,537.3502 | 669,934.6554 |
| 15 | 1,596,580.3840 | 796,464.7948 | 69,961.1988 | 847,590.6633 |

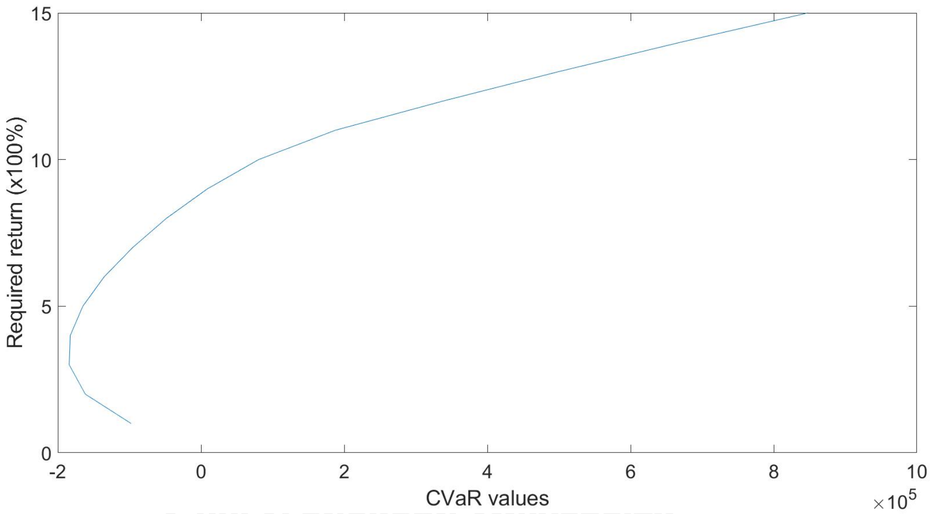





**Figure 4.13:** The graph of the CVaR value with other required returns.

Figure ?? presents that the CVaR values slightly go up in $\sigma \in [0.01, 0.08)$ and increase rapidly in $\sigma \in [0.08, 0.18)$. Similarly, the results after changing $\nu$ are shown in Table ??. When $\nu$ rises up, our CVaR value increases. It implies that $\nu$ has a minor impact on the CVaR value as shown in Figure ??.

Meanwhile, the expected rate of return has an influence on the expectation, standard deviation, 95%VaR, and 95%CVaR, as shown in Table ??. This has an impact on the profit of the selected portfolio. Furthermore, Figure ?? displays that if the required return is increasing, we may make more losses since the trend of the graph of the CVaR value with required returns is a log curve that is increasing. To summarize, high-risk investments may offer higher returns than other investments, but they also expose your money to more danger.

As a result of the growth of the CVaR values, our portfolios will be more sensitive than the base-case if the values of the three factors (e.g., $\sigma$, $\nu$ and $Q$) increase. If we compare the risky optimization portfolios with the base-case, we can see that they grow.

## 4.2 Asset-liability management for insurance companies

### 4.2.1 Indifference pricing

For indifference pricing, we solve the initial wealth that we should add in the portfolio optimization with liability, and then we will attain the same CVaR value when we consider the portfolio without liability. The mathematical model that are used to solve these problems are defined in Equation (**??**). As we can see, the results of the indifference pricing for buying and selling are shown in Table **??**. All selling prices are greater than all buying prices for each required return. Moreover, the buying prices and the selling prices decrease, while the required returns and CVaR values rise up.

Furthermore, if we change $\sigma$ that affects the fluctuations of the options prices, the indifference prices for selling become more than the indifference prices for buying. We can see this in Figure **??**.

**Table 4.8:** The 95%CVaR, buying prices and selling prices after minimizing the CVaR of the liability portfolio (Indifference pricing).

| Required return ($\times$100%) | CVaR ($) | Buying prices ($) | Selling prices ($) |
|---|---|---|---|
| 10 | 80,555.00 | 9.10 | 11.12 |
| 11 | 171,040.00 | 6.47 | 7.10 |
| 12 | 162,970.00 | 6.09 | 6.30 |
| 13 | 159,150.00 | 5.77 | 5.92 |
| 14 | 156,030.00 | 5.60 | 5.64 |
| 15 | 153,580.00 | 5.30 | 5.34 |

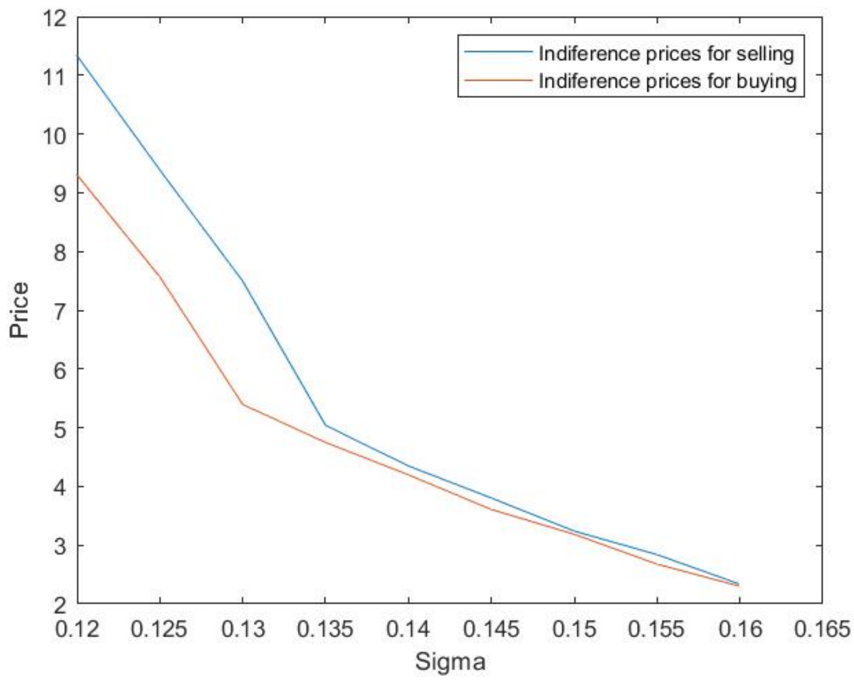



**Figure 4.14:** The graph of the indifference prices with any sigma value.

### 4.2.2 Accounting value

For the accounting value, we compute the initial wealth that we should include in the portfolio optimization with liability, with the CVaR values equal to zero. A similar equation with indifference pricing is considered, but the liability is changed. The solution of the accounting value is displayed in Table **??**. For each required return, all selling prices are higher than all buying prices. Additionally, the selling and buying prices rise, while the expected returns go up.

**Table 4.9:** The buying prices and selling prices after minimizing the CVaR of the liability portfolio (Accounting Values).

| Required return (×100%) | Buying Price ($) | Selling prices ($) |
|---|---|---|
| 10 | 0.13 | 25.38 |
| 11 | 9.87 | 35.38 |
| 12 | 19.87 | 45.38 |
| 13 | 29.87 | 55.38 |
| 14 | 39.87 | 65.38 |
| 15 | 49.87 | 75.38 |

### 4.2.3 Hedging strategies

A hedge is a risk minimization strategy that is independent of the expected return. As a result, if we consider super and sub-hedges, the prices of a hedging portfolio are the lowest price and the highest price that can be paid in any scenario at a specific time in the future.

Table **??** illustrates the prices of call options when we hedge a portfolio. In case of super-hedging, we get that the unit price of the buying price is lower than the unit price of selling. The prices of super-hedging are different from the prices of sub-hedging. Moreover, Figure **??** presents the relation between hedging portfolio, liability, super-hedge, and sub-hedge. This is when the hedging portfolio is the optimal portfolio with liability minus the optimal portfolio without liability. The graph of the hedging portfolio is a little bit shifted from strike 300 to strike 305. However, it is just between super-hedge and sub-hedge and it is greater than the liabilities for all the strike prices.

**Table 4.10:** The hedge and the amount of money ($W$) added after minimizing the liability portfolio for call options in selling strategy.

| Hedge | Initial wealth ($) | Unit price ($) |
|---|---|---|
| Super | $5.6600 \times 10^4$ | 5.66 |
| Sub | $5.3753 \times 10^4$ | 5.38 |

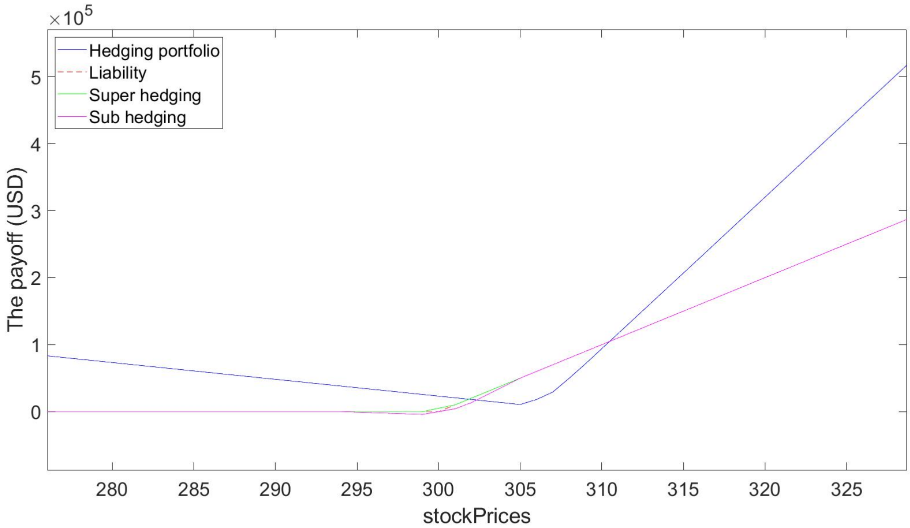



**Figure 4.15:** The graph of the hedging strategy for selling price.

In Chapter IV, we began with the optimization portfolio without any liabilities and then collected the CVaR value for this strategy. Subsequently, we tried to find the optimal value of the initial wealth that is added to this portfolio to achieve the same CVaR value as the optimization portfolio without any liabilities or a CVaR value equals to zero, which are the concepts of indifference pricing and accounting value, respectively. Lastly, we considered the hedging strategy for call options. From all the results, we have determined that the CVaR values depend on 3 parameters, which are $\sigma, \nu$ and $Q$. When the required return equals to 1,400%, the indifference prices for selling and buying are between the hedging prices. This means the prices after minimizing the portfolio with liability are related to the definition. Next, we will conclude all results in Chapter V.

# CHAPTER V

# CONCLUSIONS AND FUTURE WORK

In this chapter, we discuss the conclusion of this thesis and provide some possibilities about future work.

## 5.1 Conclusions

In Chapter I, we studied the background of portfolio optimization, risk measures i.e., variance, VaR and CVaR and the illiquid market that we are interested in. Moreover, this chapter also explains the history of why we use CVaR as a risk measure for portfolio optimization. In 1986, Markowitz [?] used mean-variance (MV) to measure the risk of a portfolio. Many years later, the authors suggested other risk measures because it is a symmetric distribution, so we can acquire the profit and loss with the same probability. Rockafellar and Uryasev [?] proposed that CVaR is an alternative measure of risk because CVaR has better properties than VaR, and a lot of the information on the loss can be acquired from the distribution of loss function.



In Chapter II, we studied the properties of the coherent risk measures i.e., monotonicity, positive homogeneity, sub-additivity, translation invariance and risk measures such as the Mean-Variance (MV), the Value-at-Risk (VaR), and the Conditional Value-at-Risk (CVaR) after that, we considered what is the appropriate risk measure that is a coherent risk measure. We came to learn that CVaR is the only coherent risk measure from all risk measures.

In Chapter III, we studied the value of options and the market, which are the datasets that were used for consideration in this thesis. Moreover, we considered the portfolio optimization by using CVaR, which allowed the portfolio optimization to be separated into 2 parts. We defined the optimization model without liability. In addition, we adopted the simulation from the Monte Carlo method to the Gaussian Legendre quadrature. Next, we considered the portfolio that has a liability, then we studied more information about indifference pricing, accounting value and the strategy of hedging.

In Chapter IV, we demonstrated all the results that were used with the minimization model in Chapter III. Firstly, we discussed the technique used to approximate the CVaR value and then compared them. As the results show, the efficiency for solving this problem of the Gaussian Legendre quadrature is better than the Monte Carlo technique because it takes less time to compute and can be used to consider big data. Therefore, the Gaussian Legendre quadrature method is the main tool. Additionally, we simulated the option values by using the variance-gamma approximations since the daily stock index has fat tails, then it becomes more suitable for the VG distribution than the normal distribution. Moreover, we knew that the CVaR value depends on many parameters, such as the standard deviation the confidence levels, $\sigma$, the variance rate $\nu$ and the expected return $Q$. Lastly, this chapter also considered the portfolio having liabilities in case of indifference prices and hedging strategies. The indifference prices for selling are higher than for buying for the standard deviations and when we choose the expected return of 1,400%, the indifference prices become the interval of hedging prices.

In this thesis, CVaR is chosen to measure the risk of options portfolio. We minimize CVaR of the options portfolio for reducing an investment risk. After optimizing portfolio, we get the portfolio selection that covers two constraints such as a minimum risk and a given expected return. From the experiments in Chapter IV, we knew that if we change parameters (i.e., the standard deviation the confidence levels, $\sigma$, the variance rate $\nu$ and the expected return $Q$), CVaR values change in the same direction. Furthermore, we compute the new portfolio that include the liability. We consider in 3 parts. For an indifference pricing and an accounting value, we get that the initial wealth which is added into the portfolio is different when the required returns change because the required return is one of effect parameters for CVaR values. However, the selling prices are greater than the buying prices for all experiments. For hedging strategy, the super-hedge price and sub-hedge price are upper bound and lower bound of indifference pricing at required return 1,400%.

## 5.2 Future work

We have made suggestions about potential future work that can be developed from this thesis as follows.

- We are interested in the optimization of portfolio in other markets, i.e., S&P500 Index, Dow jones and Nikkei 225 because they are the large datasets, which draw global interested in investment.
- We will minimize the CVaR value for a larger period of times and multiple time points, since in this thesis, we only considered the options in one period.
- We would like to minimize the CVaR value and standard deviation at the same time because the trends of both are similar.

# APPENDIX

**APPENDIX A :** We present the background of Theorem **??**. Since we would like to minimize CVaR function that is represented in Definition **??**. For this thesis, we consider the one-dimension for simulating stock prices ($S_T$) at maturity time $T$ so, we can change the dimension of $S_T$ from $\mathbb{R}^n$ to $\mathbb{R}$. Then, the auxiliary function for approximate CVaR value is defined as:

$$F_\beta(w,\alpha) = \alpha + \frac{1}{(1-\beta)} \int_{S_T \in \mathbb{R}} [f(w,S_T) - \alpha]^+ p(S_T)\, dS_T,$$

where

$$[f(w,S_T) - \alpha]^+ = \begin{cases} f(w,S_T) - \alpha, & f(w,S_T) - \alpha > 0, \\ 0, & f(w,S_T) - \alpha \le 0, \end{cases}$$

$p(S_T)$ is the probability density function of $S_T$.

- For Monte Carlo method, we suppose that $S_T^1, S_T^2, S_T^3, \ldots, S_T^q$ is a sample set and a number of simulated values is $q$ values. Then, the auxiliary function can be written as:

$$F_\beta(w,\alpha) = \alpha + \frac{1}{(1-\beta)} \left(\frac{1}{q}\right) \sum_{k=1}^{q} [f(w,S_T) - \alpha]^+,$$



- For Riemann summation, we assume that the number of simulated values $S_T^i$ is n values where the interval of simulation values is $[a,b]$. Then, the auxiliary function can be defined as:

$$\begin{aligned} F_\beta(w,\alpha) &= \alpha + \frac{1}{(1-\beta)} \sum_{i=1}^{n} [f(w,S_T) - \alpha]^+ \, p(S_T^i)\Delta S_T, \\ &= \alpha + \frac{1}{(1-\beta)} p(S_T)\Delta S_T \sum_{i=1}^{n} \left[f(w,S_T^i) - \alpha\right]^+, \end{aligned}$$

where $S_T^i = a + i\Delta S_T$, $\Delta S_T = \dfrac{b-a}{n}$ and $p(S_T)$ is the probability density function of $S_T$.

- For Gaussian Legendre quadrature, we use the stock prices from the solution of Gaussian polynomial. We assume that the loss function $f(w, S_T) = -\sum_{i=1}^{n} \left[w_i R_i(S_T^k)\right]$ and the asset's return $(R_i(S_T^k))$ is determined in Table ??. Then, we can define the auxiliary function. It can be written as:

$$
\begin{aligned}
F_\beta(w, \alpha) &= \alpha + \frac{1}{(1-\beta)} \sum_{k=1}^{q} \left[f(w, S_T) - \alpha\right]^+ p(S_T^k) m_k, \\
&= \alpha + \frac{1}{(1-\beta)} \sum_{k=1}^{q} \left[-\sum_{i=1}^{n} \left[w_i R_i(S_T^k)\right] - \alpha\right]^+ p(S_T^k) m_k,
\end{aligned}
$$

where $S_T^k$ is the point from the Gaussian polynomial on the interval $[0, c]$, where $c$ is a large number, $p(S_T^k)$ is a probability density function of $S_T^k$ and $m_k$ is the corresponding weight of $S_T^k$.

In Appendix B, we show the Mathlab code to optimize the CVaR value of the portfolio and compare the VaR value and the CVar value for each portfolio after changing the parameters that effect these values. Next, we will calculate the indifference pricing for selling and buying. Finally, we will solve the hedging price that is suitable for reducing the risk of adverse price movements in options.

**APPENDIX B1 :** The coding of the Monte Carlo technique for the simulation of the underlying prices using the Black Scholes model.

```
% Code : The Monte Carlo simulation stock Prices following the Geometric Brownian motion (GBM)
function [prices]=MonteCarloBSPrices(mu,sigma,timeToMaturity,spotPrice,numberOfScenarios)
%purpose :  Simulate numberOfScenarios paths of prices following the GBM
%input   :    mu = drift (1/year)
%           sigma = volatility (1/sqrt(year))
%           timeToMaturity = time interval of simulation (year)
%           spotPrice = current price (USD)
%           numberOfScenarios = the number of simulated prices
%output  :   an array of prices (USD)

rng('default');
epsilon=randn(1,numberOfScenarios);
logPrices=log(spotPrice)+(mu-(1/2)*sigma^2)*timeToMaturity+sigma*sqrt(timeToMaturity)*epsilon;
prices=exp(logPrices);
%assert(mean(log(prices/spotPrice))-(mu-(1/2)*sigma^2)*(timeToMaturity)<0.01);
%assert(std(log(prices/spotPrice))-sigma*sqrt(timeToMaturity)<0.01);
end
```

**APPENDIX B2 :** The coding of the computational matrix of all return values.

```
%Code :  The computation of the return matrix when we take bid and ask spread into an account
function[returnMatrix,ST]=computeReturnMatrixBidAsk(stockPrices,S0,T)
%purpose :  compute return matrix and prices at timeToMaturity by considering bid and ask prices
%input   :  stockPrices=the prices simulated by Monte Carlo at timeToMaturity
%output  :  returnMatrix = the matrix of return
%           ST = prices of all assets at timeToMaturity

[numberOfScenarios,~]=size(stockPrices);
% a = a number of assert (bank + stockPrice + call&put option)
% m = a number of assets (cash + bid + ask)
m=size(S0);
a=((m(2)-1)/2)+1;
ST=zeros(numberOfScenarios,a);
returnMatrix=zeros(numberOfScenarios,m(2));
cash=exp(0*(1/12));

%assign values of each asset at time T
ST(:,1)=cash;
for (i=2:a) %for (i=3:a)
    if T.Isput(i-1)==0
        ST(:,i)=max(0,stockPrices-T.Strike(i-1));        % call option
    else
        ST(:,i)=max(0,T.Strike(i-1)-stockPrices);        % put option
    end
end
ST=[ST ST(:,2:a)];
%compute return=(ST-S0)/S0
for(j=1:m(2))
    for (i=1:numberOfScenarios)
      returnMatrix(i,j)=(ST(i,j)-S0(j))/S0(j);
    end
end
end
```

**APPENDIX B3 :** The coding of the simulated underlying prices using the variance gamma distribution.

```
function [prices] = simulatePricePaths( mu, theta, sigma, nu, timeToMaturity ,spotPrice, numberOfScenarios )
    nSteps=1;
    dt = timeToMaturity/nSteps;
    rng(1)      % Fix random number
    shape = dt/nu;
    scale = nu;
    gam=gamrnd(shape,scale,numberOfScenarios,nSteps);
    rng(1)
    dW = randn(numberOfScenarios,nSteps);
    ds=sigma*sqrt(gam).*dW;
    s = log( spotPrice ) + cumsum(ds,2);
    S = horzcat(spotPrice*ones(numberOfScenarios,1),exp(s));
    times_ = linspace(0,timeToMaturity,nSteps+1);
    prices = S(:,end);
end
```

**APPENDIX B4 :** The coding of the probability density function for each simulated underlying prices, which is simulated using the variance gamma distribution.

```
function ret=pdfVG2(z,m,theta,sigma,v,t,S0)
%x=log(z) -log(S0)- m*t - (t/v)*log(1-theta*v-sigma^2*v/2);
x=log(z) -log(S0)- m*t -0*t;
     a=(2* exp((theta*x)/sigma^2)) /  (v^(t/v)*sqrt(2*pi)*sigma*gamma(t/v));
     b=( (x.^2) ./ ((2*sigma^2)/v+theta^2) ) .^(t/(2*v)-1/4);
     c=besselk( t/v-1/2 , (1/(sigma^2)) .*sqrt( (x.^2)* ( (2*sigma^2)/v+theta^2 ) ) );
     ret=a.*b.*c;
     ret=ret./z;
end
```

**APPENDIX B5 :** The coding of the cvarMinimization function, which is mainly the function used to find the optimal portfolio with the CVaR risk measure.

```
%Code :  The minimization of the conditional value-at-risk with bid and ask spread
function[portMean,portSd,Var,Cvar,var_cal,Cvar_cal]
  =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, W,K,buy,percent)
%purpose :  to find the portfolio with a desire mean but has the minimum
%           Cvar Bid and ask will be taken into an account.
%input :    mu = drift (1/year)
%           sigma = volatility (1/sqrt(year))
%           timeToMaturity = time interval of simulation (year)
%           spotPrice = current price (USD)
%           numberOfScenarios = the number of simulated prices
%           r = return
%           m = a number of assets (cash + bid + ask)
%           a = a number of options
%           buy = 0 ---> selling , 1---> buying , 2---> no liability
%output :   x = vector of the invested unit of each asset
%           portMean = mean of the portfolio at timeToMaturity
%           portSd = standard deviation of the portfolio at timeToMaturity
%           Var = Value-at-Risk
%           CVaR = COnditional Value-at-Risk

%% Input data
T=readtable("Edit01_26122020.xlsx","Range","A3:F301");  % all option
T.Properties.VariableNames = {'Strike' 'Bid' 'Ask' 'Isput' 'Bid_size' 'Ask_size'};
size(T);

% Cleaning data the strike is equal to K
toDelete = find(T.Strike==K);
T(toDelete,:) = [];
%size(T)

S0=vertcat(1,T.Bid,T.Ask);
S0=S0';

%% Simulation stockprices for histogram ad stockPrice 3 for finding PDF of S_i
m=size(S0);
a=(m(2)-1)/2;
%stockPrices=MonteCarloBSPrices(mu,sigma,timeToMaturity,spotPrice,numberOfScenarios);          % Normal distribution
stockPrices=simulatePricePaths(mu,theta,sigma,nu,timeToMaturity,spotPrice,numberOfScenarios ); % Variance-gamma

% -------------- Gaussian legendre quadrature ---------------------
n_Gauss=500;
[stockPrice3,W_stockPrice3]=lgwt(n_Gauss,100,500);

%% Return matrix

[returnMatrix, ST]=computeReturnMatrixBidAsk(stockPrices,S0,T);

%% Varaince-gamma distribution for finding PDF of each option
Ps=pdfVG2(stockPrice3',mu,theta,sigma,nu,timeToMaturity,spotPrice);

%% Indifference pricing C_T for simulation so use stockprice
if buy==0
    C_T_sim=10^4*max(stockPrices-K,0);
    C_T_opt=10^4*max(stockPrice3-K,0)/W;
elseif buy==1
    C_T_sim=-10^4*max(stockPrices-K,0);
    C_T_opt=-10^4*max(stockPrice3-K,0)/W;
else
    C_T_sim=0*max(stockPrices-K,0);
    C_T_opt=0*max(stockPrice3-K,0)/W;
end

%% set down all parameters for linprog (Gaussian quadrature)
num=size(stockPrice3,1);
mReturn(1)= mean(returnMatrix3(:,1));
for i=2:m(2)
    mReturn(i)=W_stockPrice3'.*Ps*returnMatrix3(:,i);
    %mReturn(i) = returnMatrix3(:,i)'*W_stockPrice3;
end
```

```
A=[-sparse(returnMatrix3) -ones(num,1) -speye(num)];
b=-C_T_opt.*ones(num,1);
Aeq=[ones(1,m(2)) 0 zeros(1,num); mReturn 0 zeros(1,num)];
beq=[1;(((1+r)*10^5)/W)-1];
%% Lower bound and upper bound
% Adding bid-ask size
Bid_constraint=(T.Bid_size.*T.Bid*100)/(W);
Ask_constraint=(T.Ask_size.*T.Ask*100)/(W);
l=[-Inf -Bid_constraint' zeros(1,a) -Inf zeros(1,num)]';   % general case
u=[Inf zeros(1,a) Ask_constraint' Inf Inf*ones(1,num)]';    % general case

% ------------------------ For Gaussian ------------------------------
ui=(1/(1-(percent/100)))*W_stockPrice3'.*Ps;
f=[zeros(1,m(2)) 1 ui];
f=f';

%use linprog to solve the linear programming
addpath('C:\Program Files\Mosek\9.3\toolbox\R2015a');
[w,v,exitflag]=linprog(f,A,b,Aeq,beq,l,u,[],[]);
rmpath('C:\Program Files\Mosek\9.3\toolbox\R2015a');
%W=100000;
Cvar_cal=v*W;                    % CvaR by using linprog
var_cal = w(m(2)+1)*W;           % VaR by using linprog
w=w(1:m(2));
%------------------------------------------------------------------------------
%% compute money invested in each asset
investedMoney=w'*W;

%compute quantities of each asset (x)
x=investedMoney./S0;
portMean=mean(ST*x');
portMean_ST3=mean(ST3*x');
portSd=std(ST*x');
portSd_ST3=std(ST3*x');

%compute VaR
initialPortValue=S0*x';
initialPortValueAfter30Days=initialPortValue*exp(0*(1/12)); % no interate rate
PayoffsOfPort3=ST3*x';
PayoffsOfPort=ST*x';
netPayoffsOfPort=(ST*x'-C_T_sim-initialPortValueAfter30Days); % Adding liability C_T
Var=-prctile(netPayoffsOfPort,100-percent);
%compute CvaR
summ=0;
count=0;
for (i=1:numberOfScenarios)
    if(netPayoffsOfPort(i)<=-Var)
        summ=summ+netPayoffsOfPort(i);
        count=count+1;
    end
end
Cvar=-summ/count;
%assert(exitflag==1);
%assert(abs(mean(returnMatrix)*w-0.10588405)<10^(-3));
portMean
portSd
Var
Cvar
max(netPayoffsOfPort)
min(netPayoffsOfPort)

end
```

**APPENDIX B6 :** The coding of the bisection function for finding the indifference price and the accounting value.

```
function [xc,root,portMean_c,portSd_c,Var_c,Cvar_c,var_cal_c,fc]
  = bisection_M(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, a, b, K, TOL, imax,buy,percent)
        % Input
        % a = initial wealth Wa
        % b = initial wea;th Wb
        % Tol = tolerance error
        % imax = maximum of the loops that finding c (root)
        % K = strike
        % bisection method

        % ---------------------- For selling ---------------------------
        if (buy==0)
        i = 1;
        [xa,portMean_a,portSd_a,Var_a,Cvar_a,var_cal_a,fa]
          =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, a, K, buy,percent);
        [xb,portMean_b,portSd_b,Var_b,Cvar_b,var_cal_b,fb]
          =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, b, K, buy,percent);
        % --------- fix W=10^5 -------------------------
        W=10^5;
        [xW,portMean_W,portSd_W,Var_W,Cvar_W,var_cal_W,fW]
          =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, W, K, 2,percent);

        ga=fa-fW;   %use Cvar opt
        %ga=fa-fW;  % use Cvar sim
        gb=fb-fW;
        %gb=fb-fW;
        if ga*gb>0
            disp('Given initial values do not bracket the root.');
        end
        % sets initial guess
        c = (a+b)/2;
        [xc,portMean_c,portSd_c,Var_c,Cvar_c,var_cal_c,fc]
          =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, c, K, buy,percent);

        while (abs(fc-fW) > TOL) && (i < imax)
            %gc=fc-fW;
            %gc=fc-fW;
            % updates interval
            if abs(fc-fW)==0   %abs(fc-fW)==0
                  break;
            elseif (fb-fW)*(fc-fW)>0
                b = c;
            else
                a = c;
            end
            % updates root estimate
            c = (a+b)/2;
            % increments loop index
            i = i+1;
            [xc,portMean_c,portSd_c,Var_c,Cvar_c,var_cal_c,fc]
              =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, c, K, buy,percent);
        end
        root = c;
        end

        % ---------------------- For buying ---------------------------
        if (buy==1)
        i = 1;
        [xa,portMean_a,portSd_a,Var_a,Cvar_a,var_cal_a,fa]
          =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, a, K, buy,percent);
        [xb,portMean_b,portSd_b,Var_b,Cvar_b,var_cal_b,fb]
          =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, b, K, buy,percent);
        % --------- fix W=10^5 -------------------------
        W=10^5;
        [xW,portMean_W,portSd_W,Var_W,Cvar_W,var_cal_W,fW]
          =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, W, K, 2,percent);
        ga=fa-fW; %use Cvar opt
        %ga=fa-fW; % use Cvar sim
        gb=fb-fW;
        %gb=fb-fW;
        if ga*gb>0
            disp('Given initial values do not bracket the root.');
        end
        % sets initial guess
        c = (a+b)/2;
        [xc,portMean_c,portSd_c,Var_c,Cvar_c,var_cal_c,fc]
          =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, c, K, buy,percent);
```

```
        while (abs(fc-fW) > TOL) && (i < imax)
            % updates interval
            if abs(fc-fW)==0   %abs(fc-fW)==0
                  break;
            elseif (fa-fW)*(fc-fW)>0
                a = c;
            else
                b = c;
            end
            % updates root estimate
            c = (a+b)/2;
            % increments loop index
            i = i+1;
            [xc,portMean_c,portSd_c,Var_c,Cvar_c,var_cal_c,fc]
              =cvarMinimization(mu, theta, sigma, nu, timeToMaturity, spotPrice, numberOfScenarios, r, c, K, buy,percent);
        end
        root = c;
        end
end
```

# BIOGRAPHY

| | |
|---|---|
| **NAME** | Miss Benyanee Kosapong |
| **DATE OF BIRTH** | 06 February 1997 |
| **PLACE OF BIRTH** | Nakhon Si Thammarat, Thailand |
| **EDUCATION** | B.Sc. (Computational Science), Walailak University, 2020 |
| **AWARD RECEIVED** | Fullscholarships from Development and Promotion of Science and Technology Talents Project (DPST) |