จำนวนขีดแบ่งของกราฟหลายส่วนบริบูรณ์บางกราฟและส่วนเติมเต็มของกราฟเหล่านั้น

นางสาวธนพร สุมาลย์โรจน์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2565

THRESHOLD NUMBERS OF SOME COMPLETE MULTIPARTITE GRAPHS

AND THEIR COMPLEMENTS

Miss Thanaporn Sumalroj

A Dissertation Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy Program in Mathematics

Department of Mathematics and Computer Science

Faculty of Science

Chulalongkorn University

Academic Year 2022

Thesis Title	THRESHOLD NUMBERS OF SOME COMPLETE MULTIPARTITE GRAPHS AND THEIR COMPLEMENTS
By	Miss Thanaporn Sumalroj
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Teeradej Kittipassorn, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

. . . . Dean of the Faculty of Science

(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

. . . . Chairman

(Associate Professor Ratinan Boonklurb, Ph.D.)

. . . . Thesis Advisor

(Assistant Professor Teeradej Kittipassorn, Ph.D.)

. . . . Examiner

(Assistant Professor Teeraphong Phongpattanacharoen, Ph.D.)

. . . . Examiner

(Wutichai Chongchitmate, Ph.D.)

. . . . External Examiner

(Assistant Professor Chalermpong Worawannotai, Ph.D.)

ธนพร สุมาลย์โรจน์ : จำนวนขีดแบ่งของกราฟหลายส่วนบริบูรณ์บางกราฟและส่วนเติมเต็มของกราฟเหล่านั้น. (THRESHOLD NUMBERS OF SOME COMPLETE MULTIPARTITE GRAPHS AND THEIR COMPLEMENTS) อ.ที่ปรึกษาวิทยานิพนธ์ : ผศ.ดร.ธีระเดช กิตติภัสสร, 53 หน้า.

กราฟ ๆ หนึ่งเป็นกราฟ k-ขีดแบ่ง พร้อมด้วยขีดแบ่ง $\theta_1, \theta_2, \theta_3, \ldots, \theta_k$ ถ้าเราสามารถกำหนดจำนวนจริง r_v ให้กับแต่ละจุดยอด v โดยที่จุดยอด u และ v ใด ๆ ที่ต่างกัน ประชิดกันก็ต่อเมื่อขีดแบ่งที่มีค่าไม่เกิน $r_u + r_v$ มีอยู่เป็นจำนวนคี่ จำนวนขีดแบ่งของกราฟ ๆ หนึ่ง คือจำนวนเต็มบวก k ที่น้อยที่สุดที่ทำให้กราฟนั้นเป็นกราฟ k-ขีดแบ่ง กราฟหลายขีดแบ่งถูกนิยามโดยเจมิสันและสปราค ในฐานะที่เป็นนัยทั่วไปของกราฟขีดแบ่ง พวกเขาได้ตั้งคำถามหาจำนวนขีดแบ่งของกราฟหลายส่วนบริบูรณ์ โดยไม่นานมานี้ เชนและเฮาได้ตอบคำถามนี้แล้วบางส่วน นั่นคือ พวกเขาได้หาจำนวนขีดแบ่งของกราฟหลายส่วนบริบูรณ์ที่แต่ละส่วนมีขนาดไม่เล็กจนเกินไป นอกจากนั้นพวกเขาได้ตั้งคำถามหาจำนวนขีดแบ่งของกราฟหลายส่วนบริบูรณ์ที่แต่ละส่วนมีขนาดเท่ากับสาม งานวิจัยนี้ศึกษาและหาจำนวนขีดแบ่งของกราฟ $K_{3,3,3,\ldots,3}$ และกราฟ $K_{4,4,4,\ldots,4}$ พร้อมทั้งหาจำนวนขีดแบ่งของส่วนเติมเต็มของกราฟทั้งสอง ซึ่งเป็นการพัฒนาผลลัพธ์หนึ่งของพูลีโอ

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษา ....

ปีการศึกษา 2565

6072851223 : MAJOR MATHEMATICS

KEYWORDS : THRESHOLD GRAPHS / MULTITHRESHOLD GRAPHS / THRESHOLD NUMBERS / GRAPH REPRESENTATION / COMPLETE MULTIPARTITE GRAPHS

THANAPORN SUMALROJ : THRESHOLD NUMBERS OF SOME COMPLETE MULTIPARTITE GRAPHS AND THEIR COMPLEMENTS ADVISOR : ASST. PROF. TEERADEJ KITTIPASSORN, Ph.D.,53 pp.

A graph is a k-threshold graph with thresholds $\theta_1, \theta_2, \theta_3, \ldots, \theta_k$ if we can assign a real number r_v to each vertex v such that for any two distinct vertices u and v, uv is an edge if and only if the number of thresholds not exceeding $r_u + r_v$ is odd. The threshold number of a graph is the smallest k for which it is a k-threshold graph. Multithreshold graphs were introduced by Jamison and Sprague as a generalization of classical threshold graphs. They asked for the exact threshold numbers of complete multipartite graphs. Recently, Chen and Hao solved the problem for complete multipartite graphs where each part is not too small, and they asked for the case when each part has size 3. We determine the exact threshold numbers of $K_{3,3,3,\ldots,3}$, $K_{4,4,4,\ldots,4}$ and their complements, nK_3 and nK_4. This improves a result of Puleo.

Department : Mathematics and Computer Science Student's Signature ....

Field of Study : Mathematics Advisor's Signature ....

Academic Year : 2022

ACKNOWLEDGEMENTS

I would like to express my sincerest gratitude to my advisor, Assistant Professor Teeradej Kittipassorn, Ph.D., for his invaluable guidance, advice and consistent encouragement throughout the course of this dissertation. His knowledge and experience in teaching and advising have been invaluable to me. I would not have been able to complete this dissertation without his support.

I would also like to thank the members of my dissertation committee: Associate Professor Ratinan Boonklurb, Ph.D., Chairman; Assistant Professor Teeraphong Phongpattanacharoen, Ph.D.; Wutichai Chongchitmate, Ph.D.; and Assistant Professor Chalermpong Worawannotai, Ph.D., for their thoughtful recommendations and suggestions.

I am deeply grateful to the Development and Promotion of Science and Technology Talents Project (DPST) for their financial support and opportunities. Lastly, I would like to extend my sincerest thanks to my family, friends and all those who have supported and assisted me in this research.

CONTENTS

CHAPTER I

INTRODUCTION

In 1977, Chvátal and Hammer [3] introduced threshold graphs and studied these graphs for their application in integer linear programming problems. They defined these graphs as follows. A graph G is said to be a *threshold graph* if we can assign a real number r_v to each vertex v and there is a real number θ such that for any vertex subset U of G, $\sum_{v \in U} r_v \leq \theta$ if and only if U is independent in G.

As one of the fundamental classes of graphs, properties of threshold graphs have been extensively studied (see [5, 6, 7, 8, 10, 12, 14] and [18]), and since then many applications of these graphs have been found in various areas, such as scheduling theory, resource allocation and parallel processes (see [1, 4, 11, 13, 15] and [16]).

Threshold graphs can be characterized in a number of equivalent ways. For example, G is a threshold graph if and only if G has no induced subgraph isomorphic to $2K_2, P_4$ or C_4 (see [3] and [12]). Equivalently, a threshold graph is a graph that can be obtained from the single-vertex graph by repeatedly adding an isolated vertex or a universal vertex (see [3] and [12]). Moreover, G is a threshold graph if and only if we can assign a real number r_v to each vertex v and there is a real number θ such that for any two distinct vertices u and v, uv is an edge if and only if $r_u + r_v \geq \theta$ (see [12]). These indicate that threshold graphs are very rare. Indeed, the number of distinct threshold graphs on n labeled vertices is at most $n!\,2^{n-1}$, while the number of all distinct graphs with the same vertex set is $2^{\binom{n}{2}}$. Therefore, most graphs are not threshold graphs.

Recently, Jamison and Sprague [9] first introduced *multithreshold graphs* as a generalization of the well-studied threshold graphs as follows. A graph G is a k-*threshold* graph with *thresholds* $\theta_1, \theta_2, \theta_3, \ldots, \theta_k$ if we can assign a real number r_v, called a *rank*, to each vertex v such that for any two distinct vertices u and v, uv is an edge if and only if the number of thresholds not exceeding $r_u + r_v$ is odd. Note that the case of one threshold is the classical case introduced by Chvátal and Hammer [3].

It is natural to ask for the existence of a positive number k for which a graph is a k-threshold graph. Jamison and Sprague [9] showed that any graph of order n is a k-threshold graph for some $k \leq \binom{n}{2}$. The smallest k for which a graph G is a k-threshold graph is said to be the *threshold number* of G, denoted by $\Theta(G)$.

A graph is a threshold graph if and only if its complement is a threshold graph since all ranks and a threshold of the complement of a threshold graph can be obtained from those of its complement by multiplying the ranks and the threshold by -1. Thus, $\Theta(G) = 1 = \Theta(G^c)$ for any nontrivial threshold graph G. However, it is not obvious how $\Theta(G)$ and $\Theta(G^c)$ are related for general k-threshold graphs when $k > 1$, for example, $\Theta(K_{2,2,2,\ldots,2}) = 3$, while $\Theta(K^c_{2,2,2,\ldots,2}) = 2$ when the number of 2's is at least 3. A relationship between the threshold numbers of a graph and its complement was found by Jamison and Sprague [9] stating that for any graph G, either $\Theta(G^c) = \Theta(G)$ or $\{\Theta(G), \Theta(G^c)\} = \{2k, 2k+1\}$ for some $k \in \mathbb{N}$. This inspired them to put forward the following conjecture.

Conjecture 1.1 ([9]). *For all $k \geq 1$, there is a graph G with $\Theta(G) = 2k$ and $\Theta(G^c) = 2k + 1$.*

They then observed that, by assigning 3^i to be the rank for each vertex of the i^{th} part of $K_{m_1,m_2,m_3,\ldots,m_n}$ and by taking 3^i and $2 \cdot 3^i$ as thresholds for $1 \leq i \leq n$, the rank sum of an edge is preceded by $2i-1$ thresholds, and the rank sum of a nonedge is preceded by $2i$ thresholds. Thus, the threshold number of $K_{m_1,m_2,m_3,\ldots,m_n}$ is at most $2n$. Note that this bound is not necessarily best possible. For example, if each part has size 2, we can assign $i-1$ and $2n-1-i$ to be the ranks of the two vertices in the i^{th} part for $1 \leq i \leq n$, and then $1, 2n-2$ and $2n-1$ are three thresholds of the graph. They put forward the following problem.

Problem 1.2 ([9])**.** *Determine the exact threshold number of the complete multipartite graph $K_{m_1,m_2,m_3,\ldots,m_n}$.*

Recently, Chen and Hao [2] gave a partial solution of Problem 1.2 which also confirmed Conjecture 1.1.

Theorem 1.3 ([2])**.** *Let $m_1, m_2, m_3, \ldots, m_n$ be positive integers and $n \geq 2$. If $m_i \geq n+1$ for $i = 1, 2, 3, \ldots, n$, then*

$$\Theta(K_{m_1,m_2,m_3,\ldots,m_n}) = 2n-2 \ \ \textit{and} \ \ \Theta(K^c_{m_1,m_2,m_3,\ldots,m_n}) = 2n-1.$$

However, their result is far from the truth when m_i are small. For example, the threshold number of $K_{1,1,1,\ldots,1}$ is 1 and the threshold number of $K_{2,2,2,\ldots,2}$ is 3. Chen and Hao [2] mentioned that it would be interesting to know the value of $\Theta(K_{3,3,3,\ldots,3})$.

As a tool for answering a question of Jamison asked in the 2019 Spring Sectional AMS Meeting, Puleo [17] proved that $\Theta(K^c_{3,3,3,\ldots,3})$ is at least $n^{1/3}$ where n is the number of its components, which in turn provides a lower bound for $\Theta(K_{3,3,3,\ldots,3})$.

In this dissertation, we determine the exact threshold numbers of $K_{3,3,3,\ldots,3}$, $K_{4,4,4,\ldots,4}$ and their complements.

The rest of this dissertation is organized as follows. Chapter II provides some background knowledge and useful results. In Chapter III, we determine the exact threshold numbers of $K_{3,3,3,\ldots,3}$ and their complements. Chapter IV is devoted to determine the exact threshold numbers of $K_{4,4,4,\ldots,4}$ and their complements. The conclusions and open problems are given in Chapter V.

CHAPTER II

BACKGROUND KNOWLEDGE

Throughout this dissertation, we denote by $K_{n\times m}$ the complete n-partite graph with m vertices in each part, and by nK_m the complement of $K_{n\times m}$.

2.1 Threshold graphs

A graph G is a *threshold graph* if we can assign a real number r_v to each vertex v and there is a real number θ such that for any vertex subset U of G, $\sum_{v\in U} r_v \leq \theta$ if and only if U is independent in G (see [3]). A vertex subset U of G is *independent* in G if no two vertices from U are adjacent in G. Figure 2.1 illustrates an example of a threshold graph along with an appropriate assignment for each vertex satisfying the inequality when $\theta = 4$.

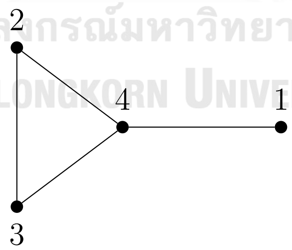


Figure 2.1: A threshold graph along with an appropriate assignment when $\theta = 4$

Threshold graphs can be characterized in a number of equivalent ways. The basic characterizations of the graphs were given in [3] and [12], some of which are stated below. We include a proof for completeness.

Theorem 2.1 ([3] and [12])**.** *For a graph G, the followings are equivalent.*

(i) *G is a threshold graph.*

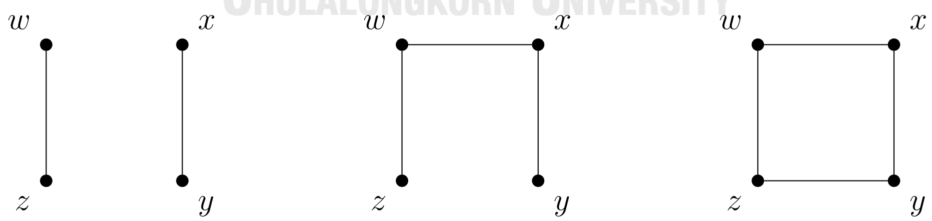
(ii) *G has no induced subgraph isomorphic to $2K_2, P_4$ or C_4.*

(iii) *G can be obtained from the one-vertex graph by repeatedly adding an isolated vertex or a universal vertex.*

(iv) *We can assign a real number r_v to each vertex v in G and there is a real number θ such that for any two distinct vertices u and v, uv is an edge if and only if $r_u + r_v \geq \theta$.*

Proof. $(i) \Rightarrow (ii)$: We first show that $2K_2, P_4$ and C_4 are not threshold graphs. Suppose to the contrary that these are threshold graphs. Let w, x, y and z be the vertices in the graphs as shown in Figure 2.2. Since wz, xy are edges and wy, xz are nonedges, by (i), there is a real number θ such that $r_w + r_z, r_x + r_y > \theta$ and $r_w + r_y, r_x + r_z \leq \theta$ respectively. Thus, $2\theta < r_w + r_x + r_y + r_z \leq 2\theta$, a contradiction.

Observe that every induced subgraph H of G is also a threshold graph since the restriction of the assignment r_v and the threshold θ of G also work for H. Hence, (ii) holds.



(a) $2K_2$ (b) P_4 (c) C_4

Figure 2.2: $2K_2$, P_4 and C_4

$(ii) \Rightarrow (iii)$: First, we show that the vertex set $V(G)$ of G can be partitioned into an independent set and a clique. Let K be a largest clique in G and let

$W = V(G) - V(K)$. To show that W is independent in G, we suppose to the contrary that W is dependent in G. Let uv be an edge in the induced subgraph $G[W]$. We claim that there exist distinct vertices x and y in K such that ux and vy are nonedges in G. Since $G[V(K) \cup \{u\}]$ is not a clique by the maximality of K, there exists x in K such that ux is a nonedge. Suppose not, that is vz is an edge for all $z \in K - x$. Since $G[V(K) \cup \{v\}]$ is not a clique by the maximality of K, vx is a nonedge. By the assumption, uz is an edge for all $z \in K - x$. Thus, $G\left[V(K-x) \cup \{u, v\}\right]$ is a clique larger than K contradicting the maximality of K. Since uv and xy are edges, $G[\{u, v, x, y\}]$ is isomorphic to $2K_2$, P_4 or C_4, a contradiction.

Next, we show that a graph that can be partitioned into an independent set and a clique must contain an isolated vertex or a universal vertex. Let H be a graph that can be partitioned into an independent set U and a clique C. Clearly, if U is an empty set, then every vertex in K is universal. Suppose that U is a nonempty set and H has no isolated vertex. Let u be a vertex in U with the smallest degree, and let u' be a neighbor of u in C. Suppose to the contrary that u' is not a universal vertex. Thus, there is a vertex v in U such that $u'v$ is a nonedge. Since u has the smallest degree, there must be a vertex v' in C such that vv' is an edge but uv' is a nonedge. Thus, $G\left[\{u, u', v, v'\}\right]$ is isomorphic to P_4, a contradiction.

Observe that after removing an isolated vertex or a universal vertex from G, the remaining graph can still be partitioned into an independent set and a clique. By repeatedly removing an isolated vertex or a universal vertex from the remaining graph, we will obtain a one-vertex graph at the end. Hence, (iii) holds.

$(iii) \Rightarrow (iv)$: Let v_i be the i^{th} vertex for the construction in (iii) for $i = 1, 2, 3, \ldots, |G|$. We say a vertex v_i is *isolated in the construction* if $v_i v_j$ is a nonedge for all $j < i$, and *universal in the construction* if v_iv_j is an edge for all $j < i$. Let $r_{v_1} = 0$. For $i = 2, 3, 4, \ldots, |G|$, we assign

$$r_{v_i} = \begin{cases} -\max_{j<i} |r_{v_j}| - 1 & \text{if } v_i \text{ is isolated in the construction,} \\ \max_{j<i} |r_{v_j}| & \text{if } v_i \text{ is universal in the construction.} \end{cases}$$

Note that $r_{v_i} + r_{v_j} \geq 0$ if and only if v_iv_j is an edge. Hence, (iv) holds with $\theta = 0$.

$(iv) \Rightarrow (i)$: We will show a stronger statement that for a graph G satisfying (iv), there exists an assignment $v \mapsto r_v$ of positive integers to the vertices and there is a positive real number θ such that for any vertex subset U of G, $\sum_{v \in U} r_v \leq \theta$ if and only if U is independent in G. We will prove by induction on $|G|$. Clearly, the statement holds for G with $|G| \leq 1$. Consider G satisfying (iv) with $|G| \geq 2$. Let $x, y \in G$ be such that $r_x = \min\{r_v : v \in G\}$ and $r_y = \max\{r_v : v \in G\}$.

Case 1. xy is a nonedge.

Then, $r_x + r_y < \theta$. Since $r_y = \max\{r_v : v \in G\}$, we have $r_x + r_v \leq r_x + r_y < \theta$ for all $v \in G - x$. Thus, xv is a nonedge for all $v \in G - x$ by (iv), that is x is an isolated vertex. Let $G' = G - x$. By the induction hypothesis, there is an assignment $v \mapsto r'_v$ of positive integers to the vertices and there is a positive real number θ' such that for any vertex subset U' of G', $\sum_{v \in U} r'_v \leq \theta'$ if and only if U' is independent in G'. Now, we assign 1 to x and assign $2r'_v$ to $v \in G'$. Take $\theta'' = 2\theta' + 1$. Let U be a vertex subset of G. If U contains x, then by the induction hypothesis, $2\sum_{v \in U \setminus \{x\}} r'_v \leq 2\theta'$ if and only if $U \setminus \{x\}$ is independent in G, and hence, $\sum_{v \in U \setminus \{x\}} 2r'_v + 1 \leq \theta''$ if and only if U is independent in G. If U does not contain x, then by the induction hypothesis, $2\sum_{v \in U} r'_v \geq 2(\theta' + 1)$ if and only if U is dependent in G, and hence, $\sum_{v \in U} 2r'_v > \theta''$ if and only if U is dependent in G.

Case 2. xy is an edge.

Then, $r_x + r_y \geq \theta$. Since $r_x = \min\{r_v : v \in G\}$, we have $\theta \leq r_x + r_y \leq r_v + r_y$

for all $v \in G - y$. Thus, vy is an edge for all $v \in G - y$ by (iv). Therefore, y is a universal vertex. Let $G' = G - y$. By the induction hypothesis, there is an assignment $v \mapsto r'_v$ of positive integers to the vertices of G' and there is a positive real number θ' such that for any vertex subset U of G', $\sum_{v \in U} r'_v \leq \theta'$ if and only if U is independent in G'. Now, we assign θ' to y. Let U be a vertex subset of G containing y. Since $r'_v > 0$ for all $v \in G'$, we have $\sum_{v \in U} r'_v = \sum_{v \in U \setminus \{y\}} r'_v + \theta' \geq \theta'$. Thus, $\sum_{v \in U} r'_v \leq \theta'$ if and only if U is independent in G. □

The followings are examples of threshold graphs which are characterized in different ways.

Example 2.2. All complete graphs, empty graphs and stars are threshold graphs since they can be obtained from the one-vertex graph by repeatedly adding an isolated vertex or a universal vertex (see Theorem 2.1 (iii)). Alternatively, it is easy to see that they have no induced subgraph isomorphic to $2K_2, P_4$ or C_4 (see Theorem 2.1 (ii)).

Example 2.3. We can also see that any complete graph and empty graph are threshold graphs by assigning a nonnegative real number to each vertex in the complete graph and assign a negative real number to each vertex in the empty graph, and we then take $\theta = 0$ (see Theorem 2.1 (iv)).

Example 2.4. We can also see that any star is a threshold graph by assigning -1 to each leaf and assign 1 to the universal vertex in the star, and we then take $\theta = -1$ (see Theorem 2.1 (iv)).

2.2 Multithreshold graphs

The equivalent statement (iv) of the definition of threshold graphs in Theorem 2.1 was generalized to define multithreshold graphs by Jamison and Sprague [9] as follows. A graph G is a *k-threshold* graph with *thresholds* $\theta_1, \theta_2, \theta_3, \ldots, \theta_k$ if we can assign a real number r_v, called a *rank*, to each vertex v such that for any two distinct vertices u and v, uv is an edge if and only if the number of thresholds not exceeding $r_u + r_v$ is odd. Equivalently,

$$uv \in E(G) \iff r_u + r_v \in [\theta_{2i-1}, \theta_{2i}) \text{ for some } i \in \left\{1, 2, 3, \ldots, \left\lceil \frac{k}{2} \right\rceil \right\}$$

provided $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_k$ and $\theta_{k+1} = \infty$. We call such an assignment r of ranks a *$(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of G*. By a *rank sum of an edge/nonedge uv*, we mean $r_u + r_v$.

The followings are examples of multithreshold graphs.

Example 2.5. The complete bipartite graph $K_{m,n}$ is a 2-threshold graph with thresholds 0 and $2a$ by assigning a positive real number a to each vertex of the first part and $-a$ to each vertex of the second part. Observe that the rank sum of each edge is 0 and the rank sum of each nonedge is either $2a$ or $-2a$.

Example 2.6. A path P_n is a 2-threshold graph with thresholds $-a$ and a where $a \in (1, 3)$ by providing the sequence of ranks $-1, 2, -3, 4, -5, \ldots$. Observe that the rank sum of each edge is either -1 or 1, while the rank sum of each nonedge is either at most -3 or at least 3.

Example 2.7. $K_{n\times 2}$ is a 3-threshold graph with thresholds 1, $2n-2$ and $2n-1$ by assigning the ranks $i-1$ and $2n-1-i$ to vertices of the i^{th} part. Observe that the rank sum of each edge is either less than or greater than $2n-2$, while the rank sum of each nonedge is $2n-2$.

We can see that $K_{n\times 2}$ has an induced subgraph isomorphic to C_4. Therefore, it is not a 1-threshold graph or a threshold graph by Theorem 2.1 (*ii*). We will prove that $K_{n\times 2}$ is not a 2-threshold graph whenever $n \geq 3$.

Proposition 2.8. *For $n \geq 3$, $K_{n\times 2}$ is not a 2-threshold graph.*

Proof. Suppose to the contrary that $K_{n\times 2}$ is a 2-threshold graph with thresholds $\theta_1 < \theta_2$. Let a_i and b_i be the ranks of vertices in the i^{th} part. Note that all edge rank sums are in $[\theta_1, \theta_2)$, while nonedge rank sums are in either $(-\infty, \theta_1)$ or $[\theta_2, \infty)$. Since the number of parts is at least three, there are two nonedge rank sums in the same interval, say $a_1 + b_1, a_2 + b_2 < \theta_1$. Thus, $a_1 + b_1 + a_2 + b_2 < 2\theta_1$. Since $a_1 + b_2$ and $a_2 + b_1$ are edge rank sums, $a_1 + b_2 + a_2 + b_1 \geq 2\theta_1$, a contradiction. □

Remark 2.9. For $n \geq 2$, nK_2 is a 2-threshold graph with thresholds $2n-2$ and $2n-1$ by applying the assignment in Example 2.7 for it. Moreover, nK_2 is not a 1-threshold graph since it has an induced subgraph isomorphic to $2K_2$.

The existence of a positive number k for which a graph is a k-threshold graph was proved by Jamison and Sprague [9]. We give a proof for completeness.

Theorem 2.10 ([9]). *Any graph of order n is a k-threshold graph for some $k \leq \binom{n}{2}$.*

Proof. Let G be a graph on n vertices $v_1, v_2, v_3, \ldots, v_n$. We assign the rank 2^i to v_i for $i \in [n]$, where $[n] = \{1, 2, 3, \ldots, n\}$. Note that $2^q + 2^r \neq 2^s + 2^t$ for any subset $\{q, r, s, t\} \subset [n]$ of size 4, and

$$2^1 + 2^2 < 2^1 + 2^3 < 2^2 + 2^3 < 2^1 + 2^4 < 2^2 + 2^4 < 2^3 + 2^4 < \cdots < 2^{n-1} + 2^n.$$

We will take the rank sum $2^i + 2^j$ as a threshold for some distinct $i, j \in [n]$ as follows. We take $2^1 + 2^2$ as a threshold when v_1v_2 is an edge. For $\{i, j\} \neq \{1, 2\}$, if v_iv_j is an edge and the greatest rank sum less than $2^i + 2^j$ is a nonedge rank sum, then we take $2^i + 2^j$ as a threshold. Similarly, if v_iv_j is a nonedge and the greatest rank sum less than $2^i + 2^j$ is an edge rank sum, then we take $2^i + 2^j$ as a threshold. These thresholds partition the real line into several intervals alternating between an interval of nonedge rank sums and an interval of edge rank sums. Therefore, G is a k-threshold graph for some $k \leq \binom{n}{2}$. □

Observe that any k-threshold graph is also a $(k+1)$-threshold graph by adding a threshold larger than all rank sums. Hence, a k-threshold graph is an ℓ-threshold graph for any integer $\ell \geq k$.

The *threshold number* of a graph G is the smallest k for which G is a k-threshold graph, denoted by $\Theta(G)$. Therefore, $\Theta(G)$ exists for every graph G by Theorem 2.10.

The followings are examples of the threshold numbers of some multithreshold graphs.

Example 2.11. By Example 2.5, $\Theta(K_{m,n}) \leq 2$. Note that $K_{m,n}$ has an induced subgraph isomorphic to C_4. By Theorem 2.1 (ii), $K_{m,n}$ is not a 1-threshold graph. Thus, $\Theta(K_{m,n}) \geq 2$. Hence, $\Theta(K_{m,n}) = 2$.

Example 2.12. By Example 2.6, $\Theta(P_n) \leq 2$. We can see that P_n has an induced subgraph isomorphic to P_4 whenever $n \geq 4$, and hence, P_n is not a 1-threshold graph by Theorem 2.1 (ii). Thus, $\Theta(P_n) \geq 2$ for all $n \geq 4$. Hence, $\Theta(P_n) = 2$ provided $n \geq 4$.

Example 2.13. By Example 2.7, $\Theta(K_{n\times 2}) \leq 3$. For $n \geq 3$, $\Theta(K_{n\times 2}) \geq 3$ by Proposition 2.8. Thus, $\Theta(K_{n\times 2}) = 3$ for all $n \geq 3$.

Example 2.14. For $n \geq 2$, $\Theta(nK_2) = 2$ by Remark 2.9.

Jamison and Sprague [9] found a relationship between the threshold numbers of a graph and its complement. We include a proof for completeness.

Proposition 2.15 ([9])**.** *For any graph G, either*

$$\Theta(G^c) = \Theta(G) \ \textit{or} \ \{\Theta(G), \Theta(G^c)\} = \{2k, 2k+1\} \ \textit{for some} \ k \in \mathbb{N}.$$

Proof. Let k and k' be such that $\Theta(G) \in \{2k, 2k+1\}$ and $\Theta(G^c) \in \{2k', 2k'+1\}$. Take a $\big(\theta_1, \theta_2, \theta_3, \ldots, \theta_{\Theta(G)}\big)$-representation r of G. We may assume that no rank sum equals a threshold by perturbing each threshold to the left. We may further assume that r has $2k+1$ thresholds by adding a sufficiently large threshold θ_{2k+1} if neccesary. We then obtain a $(-\theta_{2k+1}, -\theta_{2k}, -\theta_{2k-1}, \ldots, -\theta_1)$-representation of G^c from r by reversing the values of the ranks and the thresholds of G. Thus, $\Theta(G^c) \leq 2k+1$, and hence, $k' \leq k$. Similarly, $\Theta(G) \leq 2k'+1$, and therefore, $k \leq k'$. Now, we have $k = k'$, and hence, $\Theta(G), \Theta(G^c) \in \{2k, 2k+1\}$. □

CHAPTER III

THRESHOLD NUMBERS OF $K_{n\times 3}$ AND nK_3

In this chapter, we determine the values of $\Theta(K_{n\times 3})$ and $\Theta(nK_3)$. To outline the proofs, we will need five lemmas. Lemmas 3.1 to 3.4 are for the lower bounds where the key idea is in Lemma 3.2. We apply Lemmas 3.1 and 3.2 to prove Lemma 3.3, which determines the maximum number of triangles and parts in terms of the number of colors. Lemma 3.4 helps improve the lower bounds obtained from Lemma 3.3. On the other hand, Lemma 3.7 is a tool to prove the upper bounds.

Using an idea of Puleo [17], we start by assigning a color to each edge of nK_3 and each nonedge of $K_{n\times 3}$ as follows. In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of nK_3 where $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_k$, we color an edge uv with *color* i, for $i \in \{1, 2, 3, \ldots, \lceil \frac{k}{2} \rceil\}$, if $r_u + r_v \in [\theta_{2i-1}, \theta_{2i})$ where $\theta_{k+1} = \infty$. We say that a triangle has a *color* $ij\ell$ if the colors appearing on its edges are i, j and ℓ.

Similarly, in a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of $K_{n\times 3}$ where $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_k$, we color a nonedge xy with *color* i, for $i \in \{1, 2, 3, \ldots, \lceil \frac{k+1}{2} \rceil\}$, if $r_x + r_y \in [\theta_{2i-2}, \theta_{2i-1})$ where $\theta_0 = -\infty$. We say that a part has a *color* $ij\ell$ if the colors appearing on its nonedges are i, j and ℓ.

First, we need a result of Puleo [17] which says that no two triangles in nK_3 have the same color. Interchanging edges and nonedges, no two parts in $K_{n\times 3}$ have the same color. We include a proof for completeness.

Lemma 3.1 ([17]). *(i) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of nK_3, no two triangles have the same color.*

(ii) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of $K_{n\times 3}$, no two parts have the same color.

Proof. (i) Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of nK_3 where $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_k$. Suppose to the contrary that there are two triangles T_x and T_y in nK_3 having the same color $ij\ell$. Thus, if $V(T_x) = \{x_1, x_2, x_3\}$ and $V(T_y) = \{y_1, y_2, y_3\}$, then without loss of generality let their edge rank sums be as follows:

$$\begin{aligned} & r_{x_1} + r_{x_3}, r_{y_1} + r_{y_3} \in [\theta_{2i-1}, \theta_{2i}), \\ & r_{x_1} + r_{x_2}, r_{y_1} + r_{y_2} \in [\theta_{2j-1}, \theta_{2j}) \text{ and} \\ & r_{x_2} + r_{x_3}, r_{y_2} + r_{y_3} \in [\theta_{2\ell-1}, \theta_{2\ell}). \end{aligned}$$

Note that at least two ranks out of $\max\{r_{x_1}, r_{y_1}\}$, $\max\{r_{x_2}, r_{y_2}\}$ and $\max\{r_{x_3}, r_{y_3}\}$ are from the same triangle. Without loss of generality, let $r_{x_1} \le r_{y_1}$ and $r_{x_3} \le r_{y_3}$. Write $r_{x_p} = \min\{r_{x_1}, r_{x_3}\}$ and $r_{y_q} = \max\{r_{y_1}, r_{y_3}\}$. Observe that $r_{x_p} \le r_{y_1}, r_{y_3}$ and $r_{y_q} \ge r_{x_1}, r_{x_3}$. Therefore,

$$\theta_{2i-1} \le r_{x_1} + r_{x_3} \le r_{x_p} + r_{y_q} \le r_{y_1} + r_{y_3} < \theta_{2i}.$$

By the definition of thresholds, $x_p y_q$ is an edge of color i, which contradicts the fact that $x_p y_q$ is a nonedge in nK_3.

(ii) Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of $K_{n\times 3}$ where $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_k$. Suppose to the contrary that there are two parts S_x and S_y in $K_{n\times 3}$ having the same color $ij\ell$. Thus, if $V(S_x) = \{x_1, x_2, x_3\}$ and $V(S_y) = \{y_1, y_2, y_3\}$, then without loss of generality let their nonedge rank sums be as follows:

$$\begin{aligned} & r_{x_1} + r_{x_3}, r_{y_1} + r_{y_3} \in [\theta_{2i-2}, \theta_{2i-1}), \\ & r_{x_1} + r_{x_2}, r_{y_1} + r_{y_2} \in [\theta_{2j-2}, \theta_{2j-1}) \text{ and} \\ & r_{x_2} + r_{x_3}, r_{y_2} + r_{y_3} \in [\theta_{2\ell-2}, \theta_{2\ell-1}). \end{aligned}$$

Note that at least two ranks out of $\max\{r_{x_1}, r_{y_1}\}$, $\max\{r_{x_2}, r_{y_2}\}$ and $\max\{r_{x_3}, r_{y_3}\}$ are from the same part. Without loss of generality, let $r_{x_1} \le r_{y_1}$ and $r_{x_3} \le r_{y_3}$. Write $r_{x_p} = \min\{r_{x_1}, r_{x_3}\}$ and $r_{y_q} = \max\{r_{y_1}, r_{y_3}\}$. Observe that $r_{x_p} \le r_{y_1}, r_{y_3}$ and $r_{y_q} \ge r_{x_1}, r_{x_3}$. Therefore,

$$\theta_{2i-2} \le r_{x_1} + r_{x_3} \le r_{x_p} + r_{y_q} \le r_{y_1} + r_{y_3} < \theta_{2i-1}.$$

By the definition of thresholds, $x_p y_q$ is a nonedge of color i, which contradicts the fact that $x_p y_q$ is an edge in $K_{n\times 3}$. □

The next lemma is the key idea for obtaining the lower bounds for the threshold numbers.

Lemma 3.2. *(i) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of nK_3, and colors $i, j, \ell \in \left[\left\lceil \frac{k}{2} \right\rceil\right]$, colors ijj and $i\ell\ell$ cannot appear on two triangles simultaneously.*

(ii) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of $K_{n\times 3}$, and colors $i, j, \ell \in \left[\left\lceil \frac{k+1}{2} \right\rceil\right]$, colors ijj and $i\ell\ell$ cannot appear on two parts simultaneously.

Proof. We only prove (i) as the proof of (ii) is similar. Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of nK_3 where $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_k$. Suppose to the contrary that there are two triangles T_x and T_y in nK_3 of colors ijj and $i\ell\ell$ respectively. Thus, if $V(T_x) = \{x_1, x_2, x_3\}$ and $V(T_y) = \{y_1, y_2, y_3\}$, then without loss of generality let their edge rank sums be as follows:

$$a_1 = r_{x_1} + r_{x_3} \in [\theta_{2i-1}, \theta_{2i}), \quad b_1 = r_{x_1} + r_{x_2} \in [\theta_{2j-1}, \theta_{2j}), \quad b_2 = r_{x_2} + r_{x_3} \in [\theta_{2j-1}, \theta_{2j}),$$

$$a_2 = r_{y_1} + r_{y_3} \in [\theta_{2i-1}, \theta_{2i}), \quad c_1 = r_{y_1} + r_{y_2} \in [\theta_{2\ell-1}, \theta_{2\ell}), \quad c_2 = r_{y_2} + r_{y_3} \in [\theta_{2\ell-1}, \theta_{2\ell}).$$

From these rank sums, we can compute the ranks as follows:

$$r_{x_1} = \frac{a_1 + b_1 - b_2}{2}, \qquad r_{x_2} = \frac{b_1 + b_2 - a_1}{2}, \qquad r_{x_3} = \frac{a_1 + b_2 - b_1}{2},$$
$$r_{y_1} = \frac{a_2 + c_1 - c_2}{2}, \qquad r_{y_2} = \frac{c_1 + c_2 - a_2}{2}, \qquad r_{y_3} = \frac{a_2 + c_2 - c_1}{2}.$$

Without loss of generality, let $a_1 \le a_2$, $b_1 \le b_2$ and $c_1 \le c_2$. Let $D = a_2 - a_1 \ge 0$ and let

$$\begin{aligned} A &= b_1 - b_2 + c_1 - c_2, \\ B &= -b_1 + b_2 + c_1 - c_2 \text{ and} \\ C &= -b_1 + b_2 - c_1 + c_2. \end{aligned}$$

Note that $A \le B \le C$ and $A \le 0 \le C$. Since $D \ge 0 \ge A$, either $D \in [A, B]$, $D \in [B, C]$ or $D \in [C, \infty)$. We obtain a contradiction by the following three claims.

Claim. $D \notin [A, B]$.

Since x_2y_3 is a nonedge, we cannot have $b_1 \le r_{x_2} + r_{y_3} \le b_2$; otherwise, $r_{x_2} + r_{y_3} \in [\theta_{2j-1}, \theta_{2j})$. Observe that

$$\begin{aligned} b_1 \le r_{x_2} + r_{y_3} \le b_2 &\iff b_1 \le \frac{b_1 + b_2 - a_1}{2} + \frac{a_2 + c_2 - c_1}{2} \le b_2 \\ &\iff 2b_1 \le b_1 + b_2 - a_1 + a_2 + c_2 - c_1 \le 2b_2 \\ &\iff b_1 - b_2 + c_1 - c_2 \le a_2 - a_1 \le -b_1 + b_2 + c_1 - c_2 \\ &\iff A \le D \le B. \end{aligned}$$

Claim. $D \notin [B, C]$.

Since x_3y_2 is a nonedge, we cannot have $c_1 \le r_{x_3} + r_{y_2} \le c_2$; otherwise, $r_{x_3} + r_{y_2} \in [\theta_{2\ell-1}, \theta_{2\ell})$. Note that

$$\begin{aligned} c_1 \le r_{x_3} + r_{y_2} \le c_2 &\iff c_1 \le \frac{a_1 + b_2 - b_1}{2} + \frac{c_1 + c_2 - a_2}{2} \le c_2 \\ &\iff 2c_1 \le a_1 + b_2 - b_1 + c_1 + c_2 - a_2 \le 2c_2 \\ &\iff -2c_2 \le -a_1 - b_2 + b_1 - c_1 - c_2 + a_2 \le -2c_1 \\ &\iff -b_1 + b_2 + c_1 - c_2 \le a_2 - a_1 \le -b_1 + b_2 - c_1 + c_2 \\ &\iff B \le D \le C. \end{aligned}$$

Claim. $D \notin [C, \infty)$.

Since x_3y_3 is a nonedge, we cannot have $a_1 \le r_{x_3} + r_{y_3} \le a_2$; otherwise, $r_{x_3} + r_{y_3} \in [\theta_{2i-1}, \theta_{2i})$. Observe that

$$\begin{aligned} a_1 \le r_{x_3} + r_{y_3} \le a_2 &\iff a_1 \le \frac{a_1 + b_2 - b_1}{2} + \frac{a_2 + c_2 - c_1}{2} \le a_2 \\ &\iff 2a_1 \le a_1 + b_2 - b_1 + a_2 + c_2 - c_1 \le 2a_2 \\ &\iff a_1 - a_2 \le -b_1 + b_2 - c_1 + c_2 \le a_2 - a_1 \\ &\iff -D \le C \le D \\ &\iff C \le D, \end{aligned}$$

since $-D \le 0 \le C$ is trivially true. $\square$

We apply Lemmas 3.1 and 3.2 to determine the maximum number of triangles and parts in terms of the number of colors, which in turn gives lower bounds for the threshold numbers.

Lemma 3.3. *(i) If there are at most m colors of edges in nK_3, then $n \le m + \binom{m}{3}$. In particular, if nK_3 is a k-threshold graph, then $n \le \lceil \frac{k}{2} \rceil + \binom{\lceil k/2 \rceil}{3}$.*

(ii) If there are at most m colors of nonedges in $K_{n\times 3}$, then $n \le m + \binom{m}{3}$. In particular, if $K_{n\times 3}$ is a k-threshold graph, then $n \le \lceil \frac{k+1}{2} \rceil + \binom{\lceil (k+1)/2 \rceil}{3}$.

Proof. We will only prove (i) as the proof of (ii) is similar. Suppose that there are at most m colors of edges in nK_3. By Lemma 3.1, no two triangles in nK_3 have the same color. Thus, there are at most $\binom{m}{3}$ triangles in nK_3 whose edges are colored with 3 colors. It is sufficient to show that there are at most m triangles in nK_3 whose edges are colored with 1 or 2 colors. Indeed, for each color $i \in [m]$, there is at most one triangle of color of the form ijj where $j \in [m]$ by Lemma 3.2.

Thus, $n \leq m + \binom{m}{3}$. Note that if nK_3 is a k-threshold graph, then there are at most $\lceil \frac{k}{2} \rceil$ colors of edges in nK_3, and hence, $n \leq \lceil \frac{k}{2} \rceil + \binom{\lceil k/2 \rceil}{3}$. □

The lower bounds for the threshold numbers obtained from Lemma 3.3 are not sharp. We require another observation which states roughly that the first and last colors appear in at most one triangle or part.

Lemma 3.4. *(i) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m-1})$-representation of nK_3, an edge of color m appears in at most one triangle.*

(ii) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_m)$-representation of $K_{n\times 3}$, a nonedge of color 1 appears in at most one part.

(iii) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of $K_{n\times 3}$, a nonedge of color $m+1$ appears in at most one part.

Proof. (i) Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m-1})$-representation of nK_3. Suppose to the contrary that there are two triangles T_x and T_y in nK_3 with an edge of color m. Let $V(T_x) = \{x_1, x_2, x_3\}$ and $V(T_y) = \{y_1, y_2, y_3\}$. Suppose that x_1x_2 and y_1y_2 are edges of color m. By the definition of colors of edges, $r_{x_1} + r_{x_2}, r_{y_1} + r_{y_2} \in [\theta_{2m-1}, \infty)$, that is $r_{x_1} + r_{x_2}, r_{y_1} + r_{y_2} \geq \theta_{2m-1}$. Assume without loss of generality that $r_{x_1}, r_{y_1} \geq \frac{\theta_{2m-1}}{2}$. Thus, $r_{x_1} + r_{y_1} \geq \theta_{2m-1}$. By the definition of colors of edges, x_1y_1 is an edge in nK_3, a contradiction.

(ii) Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_m)$-representation of $K_{n\times 3}$. Suppose to the contrary that there are two parts P_x and P_y in $K_{n\times 3}$ with a nonedge of color 1. Let $V(P_x) = \{x_1, x_2, x_3\}$ and $V(P_y) = \{y_1, y_2, y_3\}$. Suppose that x_1x_2 and y_1y_2 are nonedges of color 1. By the definition of colors of nonedges, $r_{x_1} + r_{x_2}, r_{y_1} + r_{y_2} \in (-\infty, \theta_1)$, that is $r_{x_1} + r_{x_2}, r_{y_1} + r_{y_2} < \theta_1$. Assume without loss of generality that $r_{x_1}, r_{y_1} < \frac{\theta_1}{2}$. Thus, $r_{x_1} + r_{y_1} < \theta_1$. By the definition of colors of nonedges, x_1y_1 is a nonedge in $K_{n\times 3}$, a contradiction.

(iii) Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of $K_{n\times 3}$. Suppose to the contrary that there are two parts P_x and P_y in $K_{n\times 3}$ with a nonedge of color $m+1$. Let $V(P_x) = \{x_1, x_2, x_3\}$ and $V(P_y) = \{y_1, y_2, y_3\}$. Suppose that x_1x_2 and y_1y_2 are nonedges of color $m+1$. By the definition of colors of nonedges, $r_{x_1} + r_{x_2}, r_{y_1} + r_{y_2} \in [\theta_{2m}, \infty)$, that is $r_{x_1} + r_{x_2}, r_{y_1} + r_{y_2} \geq \theta_{2m}$. Assume without loss of generality that $r_{x_1}, r_{y_1} \geq \frac{\theta_{2m}}{2}$. Thus, $r_{x_1} + r_{y_1} \geq \theta_{2m}$. By the definition of colors of nonedges, x_1y_1 is a nonedge in $K_{n\times 3}$, a contradiction. □

The upper bounds for the threshold numbers will be obtained by rank assignments of the following forms. A rank assignment r of nK_3 is said to be an $\{a_1, a_2, a_3, \ldots, a_m\}$-*assignment* if each triangle has edge rank sums of the form a_i, a_i, a_i or a_i, a_j, a_k for distinct $i, j, k \in [m]$, and no two triangles have the same multiset of edge rank sums.

Remarks 3.5. (i) In an $\{a_1, a_2, a_3, \ldots, a_m\}$-assignment of nK_3, there are at most m triangles having edge rank sums of the form a_i, a_i, a_i, and there are at most $\binom{m}{3}$ triangles having edge rank sums of the form a_i, a_j, a_k for distinct $i, j, k \in [m]$.

(ii) A triangle has edge rank sums a_i, a_j and a_k if and only if its ranks are $\frac{a_i+a_j-a_k}{2}$, $\frac{a_i+a_k-a_j}{2}$ and $\frac{a_j+a_k-a_i}{2}$ (see Figure 3.1).

(iii) If $n \leq m + \binom{m}{3}$, then an $\{a_1, a_2, a_3, \ldots, a_m\}$-assignment of nK_3 exists since we can assign any edge rank sums for each triangle.

In the same fasion, a rank assignment r of $K_{n\times 3}$ is an $\{a_1, a_2, a_3, \ldots, a_m\}$-*assignment* if each part has nonedge rank sums of the form a_i, a_i, a_i or a_i, a_j, a_k for distinct $i, j, k \in [m]$, and no two parts have the same multiset of nonedge rank

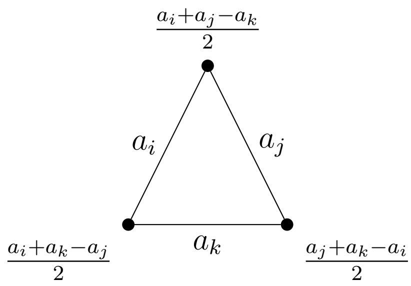


Figure 3.1: A triangle having edge rank sums a_i, a_j and a_k

sums.

Remarks 3.6. (i) In an $\{a_1, a_2, a_3, \ldots, a_m\}$-assignment of $K_{n\times 3}$, there are at most m parts having nonedge rank sums of the form a_i, a_i, a_i, and there are at most $\binom{m}{3}$ parts having nonedge rank sums of the form a_i, a_j, a_k for distinct $i, j, k \in [m]$.

(ii) A part has nonedge rank sums a_i, a_j and a_k if and only if its ranks are $\frac{a_i+a_j-a_k}{2}$, $\frac{a_i+a_k-a_j}{2}$ and $\frac{a_j+a_k-a_i}{2}$.

(iii) If $n \leq m + \binom{m}{3}$, then an $\{a_1, a_2, a_3, \ldots, a_m\}$-assignment of $K_{n\times 3}$ exists since we can assign any nonedge rank sums for each part.

The linear independence of $\{a_1, a_2, a_3, \ldots, a_m\}$ over $\mathbb{Q}$ is a sufficient condition for the edge and nonedge rank sums in an $\{a_1, a_2, a_3, \ldots, a_m\}$-assignment not to coincide.

Lemma 3.7. *Let* $\{a_1, a_2, a_3, \ldots, a_m\} \subset \mathbb{R}$ *be a linearly independent set over* $\mathbb{Q}$.

(i) In an $\{a_1, a_2, a_3, \ldots, a_m\}$*-assignment of* nK_3*, the edge and nonedge rank sums do not coincide.*

(ii) In an $\{a_1, a_2, a_3, \ldots, a_m\}$*-assignment of* $K_{n\times 3}$*, the edge and nonedge rank sums do not coincide.*

Proof. We only prove (i) as the proof of (ii) is similar. Let $A = \{a_1, a_2, a_3, \ldots, a_m\} \subset \mathbb{R}$ be a linearly independent set over $\mathbb{Q}$. Let r be an A-assignment of nK_3. Then, each triangle in nK_3 has edge rank sums of the form a_i, a_j, a_k where $i, j, k \in [m]$ are all equal or all distinct, and no two triangles in nK_3 have the same multiset of edge rank sums. Note that the rank of each vertex in nK_3 is of the form $\frac{a_i+a_j-a_k}{2}$. Suppose to the contrary that there exists a nonedge xy in nK_3 such that $r_x+r_y = a_\ell$ for some $\ell \in [m]$. Let $r_x = \frac{a_i+a_j-a_k}{2}$ and $r_y = \frac{a_r+a_s-a_t}{2}$ where $i, j, k \in [m]$ are all equal or all distinct, $r, s, t \in [m]$ are all equal or all distinct, and $\{i, j, k\} \neq \{r, s, t\}$. Hence, $r_x + r_y = a_\ell$ becomes

$$a_i + a_j - a_k + a_r + a_s - a_t = 2a_\ell.$$

Since $\{a_i, a_j, a_k\} \neq \{a_r, a_s, a_t\}$, there exists an element in one set not appearing in the other set, say $a_i \notin \{a_r, a_s, a_t\}$. Since i, j, k are all equal or all distinct, the coefficient of a_i after simplifying the left hand side of the equation is 1. Since A is a linearly independent set over $\mathbb{Q}$, the left hand side cannot equal $2a_\ell$, a contradiction. $\square$

We are now ready to determine the exact threshold numbers of nK_3.

Theorem 3.8. *Let* $q_m = m + \binom{m}{3} + 1$. *For* $n \geq 1$,

$$\Theta(nK_3) = \begin{cases} 2m-1 & \text{if } n = q_{m-1}, \\ 2m & \text{if } q_{m-1} < n < q_m. \end{cases}$$

Proof. Let m be a positive integer such that $q_{m-1} \leq n < q_m$. Suppose to the contrary that $\Theta(nK_3) \le 2m-2$. By Lemma 3.3 (i),

$$\begin{aligned} n &\le \left\lceil \frac{\Theta(nK_3)}{2} \right\rceil + \binom{\left\lceil \frac{\Theta(nK_3)}{2} \right\rceil}{3} \\ &\le \left\lceil \frac{2m-2}{2} \right\rceil + \binom{\left\lceil \frac{2m-2}{2} \right\rceil}{3} \\ &= m-1+\binom{m-1}{3} \\ &= q_{m-1}-1, \end{aligned}$$

contradicting the definition of m. Thus, $\Theta(nK_3) \ge 2m-1$.

To prove that $\Theta(nK_3) \le 2m$, let $A = \{a_1, a_2, a_3, \ldots, a_m\} \subset \mathbb{R}^+$ be a linearly independent set over $\mathbb{Q}$, for example, let $a_i = \sqrt{p_i}$ where p_i is the i^{th} prime number. Since $n \le q_m - 1 = m + \binom{m}{3}$, we can pick an A-assignment for nK_3. By Lemma 3.7 (i), the edge and nonedge rank sums do not coincide. We separate the edge and nonedge rank sums by putting two thresholds around each edge rank sum. For $i = 1, 2, 3, \ldots, m$, let $\theta_{2i-1} = a_i$ and $\theta_{2i} = a_i + \varepsilon$ be thresholds of nK_3 where ε is a sufficiently small positive real number, for example, take ε smaller than any distance between two distinct rank sums of nK_3. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of nK_3. Hence, nK_3 is a $2m$-threshold graph, that is $\Theta(nK_3) \le 2m$ as desired.

We suppose that $n = q_{m-1}$. To prove that $\Theta(nK_3) \le 2m-1$, let $A = \{a_1, a_2, a_3, \ldots, a_m\} \subset \mathbb{R}^+$ be a linearly independent set over $\mathbb{Q}$ such that $a_1 < a_2 < a_3 < \cdots < a_{m-1} \le \frac{a_m}{2}$. We then pick an $A \setminus \{a_m\}$-assignment for the first $m - 1 + \binom{m-1}{3}$ triangles in nK_3, and let the last triangle have edge rank sums a_m, a_m, a_m. Note that this is an A-assignment of nK_3. By Lemma 3.7 (i), the edge and nonedge rank sums do not coincide. We separate the edge and nonedge rank sums by putting two thresholds around each edge rank sum. For $i = 1, 2, 3, \ldots, m$, let $\theta_{2i-1} = a_i$ and $\theta_{2i} = a_i + \varepsilon$ be thresholds of nK_3 where ε is a sufficiently small positive real number. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of nK_3. In fact, we will show that we do not need the last threshold θ_{2m} by proving that no rank sum exceeds θ_{2m-1}. It is sufficient to show that the rank of each vertex is at most $\frac{\theta_{2m-1}}{2} = \frac{a_m}{2}$. This is clear for the last triangle with edge rank sums a_m, a_m, a_m since the rank of each vertex is $\frac{a_m}{2}$. For the other triangles, the rank of each vertex is of the form $\frac{a_i+a_j-a_k}{2}$ for some $i, j, k \in [m-1]$. Since $a_i, a_j \leq \frac{a_m}{2}$ and $a_k > 0$, we have $\frac{a_i+a_j-a_k}{2} \leq \frac{\frac{a_m}{2}+\frac{a_m}{2}+0}{2} = \frac{a_m}{2}$. Thus, all rank sums are at most $a_m = \theta_{2m-1}$. Then, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m-1})$-representation of nK_3. Hence, nK_3 is a $(2m-1)$-threshold graph, that is $\Theta(nK_3) \leq 2m-1$ as desired.

Suppose that $n > q_{m-1}$. To prove that $\Theta(nK_3) \geq 2m$, we suppose that $\Theta(nK_3) \leq 2m-1$. Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m-1})$-representation of nK_3. Then, there are at most m colors of edges in nK_3. By Lemma 3.3 (i), there are at most $m-1+\binom{m-1}{3} = q_{m-1}-1$ triangles without color m. By Lemma 3.4 (i), an edge of color m appears in at most one triangle. Thus, $n \leq (q_{m-1}-1)+1$, a contradiction. Therefore, $\Theta(nK_3) \geq 2m$. □

By applying Theorem 3.8 together with Proposition 2.15, we can narrow down the possible values of $\Theta(K_{n\times 3})$ to just two numbers.

Theorem 3.9. *Let* $p_m = m + \binom{m}{3} + 2$. *For* $n \geq 2$,

$$\Theta(K_{n\times 3}) = \begin{cases} 2m & \text{if } n = p_{m-1}, \\ 2m+1 & \text{if } p_{m-1} < n < p_m. \end{cases}$$

Proof. Let m be a positive integer such that $p_{m-1} \leq n < p_m$. Observe that $p_m = q_m + 1$. Thus, m is such that $q_{m-1} < n \leq q_m$. By Theorem 3.8,

$$\Theta(nK_3) = \begin{cases} 2m+1 & \text{if } n = q_m, \\ 2m & \text{if } q_{m-1} < n < q_m. \end{cases}$$

By Proposition 2.15, $\Theta(K_{n\times 3}) \in \{2m, 2m+1\}$.

Suppose that $n = p_{m-1}$. To prove that $\Theta(K_{n\times 3}) \leq 2m$, we will let $A = \{a_1, a_2, a_3, \ldots, a_{m+1}\} \subset \mathbb{R}$ be a linearly independent set over $\mathbb{Q}$ such that $a_1 < a_2 < a_3 < \cdots < a_{m+1}$, $-|a_i| \geq \frac{a_1}{3}$ and $|a_i| \leq \frac{a_{m+1}}{3}$ for all $i \in [m] \smallsetminus \{1\}$. We pick an $A \smallsetminus \{a_1, a_{m+1}\}$-assignment for the first $m - 1 + \binom{m-1}{3}$ parts in $K_{n\times 3}$, and let the last two parts have nonedge rank sums a_1, a_1, a_1 and $a_{m+1}, a_{m+1}, a_{m+1}$. Note that this is an A-assignment of $K_{n\times 3}$. By Lemma 3.7 (ii), the edge and nonedge rank sums do not coincide. Let θ_1 be smaller than all rank sums. We then separate the edge and nonedge rank sums by putting two thresholds around each interval of nonedge rank sums. For $i = 1, 2, 3, \ldots, m+1$, let $\theta_{2i} = a_i$ and $\theta_{2i+1} = a_i + \varepsilon$ where ε is a sufficiently small positive real number. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m+3})$-representation of $K_{n\times 3}$. In fact, we will show that we do not need the thresholds θ_1, θ_2 and θ_{2m+3} by proving that no rank sum is smaller than θ_2 or larger than θ_{2m+2}. It is sufficient to show that the rank of each vertex is at least $\frac{\theta_2}{2} = \frac{a_1}{2}$ and at most $\frac{\theta_{2m+2}}{2} = \frac{a_{m+1}}{2}$. This is clear for the last two parts with nonedge rank sums a_1, a_1, a_1 and $a_{m+1}, a_{m+1}, a_{m+1}$ since the rank of each vertex is either $\frac{a_1}{2}$ or $\frac{a_{m+1}}{2}$. For the other parts, the rank of each vertex is of the form $\frac{a_i + a_j - a_k}{2}$ for some $i, j, k \in [m] \smallsetminus \{1\}$. Since $\frac{a_1}{3} \leq a_i, a_j, -a_k \leq \frac{a_{m+1}}{3}$, we have

$$\frac{a_1}{2} = \frac{\frac{a_1}{3} + \frac{a_1}{3} + \frac{a_1}{3}}{2} \leq \frac{a_i + a_j - a_k}{2} \leq \frac{\frac{a_{m+1}}{3} + \frac{a_{m+1}}{3} + \frac{a_{m+1}}{3}}{2} = \frac{a_{m+1}}{2}.$$

Thus, all rank sums are at least $a_1 = \theta_2$ and at most $a_{m+1} = \theta_{2m+2}$. Then, the above rank assignment is a $(\theta_3, \theta_4, \theta_5, \ldots, \theta_{2m+2})$-representation of $K_{n\times 3}$. Therefore, $K_{n\times 3}$ is a $2m$-threshold graph, that is $\Theta(K_{n\times 3}) \leq 2m$ as desired.

Suppose that $n > p_{m-1}$. To prove that $\Theta(K_{n\times 3}) \geq 2m + 1$, we suppose that $\Theta(K_{n\times 3}) \leq 2m$. Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of $K_{n\times 3}$. Then, there are at most $m+1$ colors of nonedges in $K_{n\times 3}$. By Lemma 3.3 (ii), there are at most $m - 1 + \binom{m-1}{3} = p_{m-1} - 2$ parts without colors 1 and $m + 1$. By Lemma 3.4 (ii)

and 3.4 (*iii*), a nonedge of color 1 appears in at most one part and a nonedge of color $m+1$ also appears in at most one part. Therefore, $n \leq (p_{m-1} - 2) + 1 + 1$, a contradiction. $\square$

CHAPTER IV

THRESHOLD NUMBERS OF $K_{n\times 4}$ AND nK_4

In this chapter, we determine the exact threshold numbers of $K_{n\times 4}$ and nK_4. We will need Lemmas 3.1 and 3.2 as well as five new lemmas. Lemma 4.1 identifies all sets of edge rank sums that can appear in a K_4. Lemmas 4.2 and 4.3 are for the lower bounds where the key idea is in Lemma 4.2. We apply Lemmas 3.1 and 3.2 to prove Lemma 4.2, which provide the maximum number of K_4's and parts in terms of the number of colors. Lemma 4.3 improves the lower bounds obtained from Lemma 4.2. On the other hand, Lemma 4.5 which is a tool to prove the upper bounds utilizes Lemma 4.4 in its proof.

We start by assigning a color to each edge of nK_4 and each nonedge of $K_{n\times 4}$ as follows. In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of nK_4 where $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_k$, we color an edge uv with *color* i, for $i \in \left\{1, 2, 3, \ldots, \left\lceil \frac{k}{2} \right\rceil\right\}$, if $r_u + r_v \in [\theta_{2i-1}, \theta_{2i})$ where $\theta_{k+1} = \infty$.

Similarly, in a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$-representation of $K_{n\times 4}$ where $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_k$, we color a nonedge xy with *color* i, for $i \in \left\{1, 2, 3, \ldots, \left\lceil \frac{k+1}{2} \right\rceil\right\}$, if $r_x + r_y \in [\theta_{2i-2}, \theta_{2i-1})$ where $\theta_0 = -\infty$.

We denote by $K_4(a_1, b_1, a_2, b_2, a_3, b_3)$ a K_4 each of whose vertices is assigned a rank so that the edge rank sums are a_1, b_1, a_2, b_2, a_3 and b_3 where a_i and b_i belong to a perfect matching for each i as shown in Figure 4.1. For convenience, we write $K_4(c)$ for $K_4(c, c, c, c, c, c)$. Observe that $K_4(b_1, a_1, a_2, b_2, a_3, b_3)$, $K_4(a_1, b_1, b_2, a_2, a_3, b_3)$ and $K_4(a_1, b_1, a_2, b_2, b_3, a_3)$ are isomorphic, while

$K_4(a_1, b_1, a_2, b_2, b_3, a_3)$ and $K_4(a_1, b_1, a_2, b_2, a_3, b_3)$ are not isomorphic.

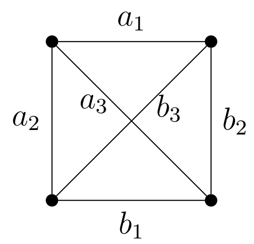


Figure 4.1: $K_4(a_1, b_1, a_2, b_2, a_3, b_3)$

In the same fasion, we denote by $E_4(a_1, b_1, a_2, b_2, a_3, b_3)$ an empty graph on four vertices having nonedge rank sums a_1, b_1, a_2, b_2, a_3 and b_3 where a_i and b_i belong to an independent nonedges for each i.

It is easy to determine which edge rank sums a_1, b_1, a_2, b_2, a_3 and b_3 can appear in a K_4.

Proposition 4.1. *The following statements are equivalent:*

(i) $K_4(a_1, b_1, a_2, b_2, a_3, b_3)$ *exists.*

(ii) $E_4(a_1, b_1, a_2, b_2, a_3, b_3)$ *exists.*

(iii) $a_1 + b_1 = a_2 + b_2 = a_3 + b_3$.

Proof. *(i)* $\Rightarrow$ *(iii)*: Suppose that $K_4(a_1, b_1, a_2, b_2, a_3, b_3)$ exists, that is we can assign a rank to each vertex so that the edge rank sums are a_1, b_1, a_2, b_2, a_3 and b_3 where a_i and b_i belong to a perfect matching for each i. Since each perfect matching spans all vertices of the graph, the summation of all ranks is equal to $a_i + b_i$ for each i. Thus, $a_1 + b_1 = a_2 + b_2 = a_3 + b_3$ as desired.

(iii) $\Rightarrow$ *(i)*: Let $\{w, x, y, z\}$ be the vertex set of K_4. We will provide an assignment r of ranks so that the graph is $K_4(a_1, b_1, a_2, b_2, a_3, b_3)$ as shown in Figure 4.2.

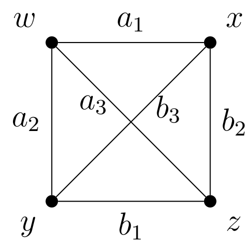


Figure 4.2

To obtain edge rank sums b_1, b_2, b_3 for the triangle xyz, we let

$$r(x) = \frac{b_2 + b_3 - b_1}{2},\ r(y) = \frac{b_1 + b_3 - b_2}{2} \text{ and } r(z) = \frac{b_1 + b_2 - b_3}{2}.$$

We immediately obtain $r(y) + r(z) = b_1$, $r(x) + r(z) = b_2$ and $r(x) + r(y) = b_3$. Now, let $r(w) = \frac{a_1+a_2-b_3}{2}$. Thus,

$$r(w) + r(x) = \frac{a_1 + a_2 - b_3}{2} + \frac{b_2 + b_3 - b_1}{2} = a_1 \text{ since } a_1 + b_1 = a_2 + b_2,$$

$$r(w) + r(y) = \frac{a_1 + a_2 - b_3}{2} + \frac{b_1 + b_3 - b_2}{2} = a_2 \text{ since } a_1 + b_1 = a_2 + b_2 \text{ and}$$

$$r(w) + r(z) = \frac{a_1 + a_2 - b_3}{2} + \frac{b_1 + b_2 - b_3}{2} = a_3 \text{ since } a_1 + b_1 = a_2 + b_2 = a_3 + b_3.$$

For $(ii) \Leftrightarrow (iii)$, the proof is similar. □

The following key lemma for the lower bounds for the threshold number, determines the maximum numbers of K_4's and parts in terms of the number of colors. The crux of the proof is an observation that each K_4 must contain a particular kind of K_3.

Lemma 4.2. *(i) If there are at most m colors of edges in nK_4, then $n \le m + \binom{\lfloor m/2 \rfloor}{3} + \binom{\lceil m/2 \rceil}{3}$. In particular, if nK_4 is a k-threshold graph, then $n \le \lceil \frac{k}{2} \rceil + \binom{\lfloor (k+1)/4 \rfloor}{3} + \binom{\lceil k/4 \rceil}{3}$.*

(ii) If there are at most m colors of nonedges in $K_{n\times 4}$, then $n \le m + \binom{\lfloor m/2 \rfloor}{3} + \binom{\lceil m/2 \rceil}{3}$. In particular, if $K_{n\times 4}$ is a k-threshold graph, then $n \le \lceil \frac{k+1}{2} \rceil + \binom{\lfloor (k+2)/4 \rfloor}{3} + \binom{\lceil (k+1)/4 \rceil}{3}$.

Proof. We will only prove (i) as the proof of (ii) is similar. Let r be a representation of nK_4 such that there are at most m colors of edges. We decompose nK_4 into two subgraphs $G_1 = n_1K_4$ and $G_2 = n_2K_4$ with $n = n_1 + n_2$ such that G_1 consists of all K_4's containing a triangle whose edges are colored with 1 or 2 colors and G_2 consists of all K_4's with four triangles whose edges are colored with 3 colors. First, we show that $n_1 \leq m$. Consider a subgraph n_1K_3 of G_1 consisting of triangles whose edges are colored with 1 or 2 colors. Since n_1K_3 is an induced subgraph of nK_4, we have r is also a representation of n_1K_3. Applying Lemma 3.2 with the representation r of n_1K_3, for each color $i \in [m]$, there is at most one triangle in n_1K_3 of color of the form ijj where $j \in [m]$. Thus, $n_1 \leq m$.

It remains to show that $n_2 \leq \binom{\lfloor m/2 \rfloor}{3} + \binom{\lceil m/2 \rceil}{3}$. Let $\mathcal{L}$ be the set of triangles in nK_4 of colors $ij\ell$ where $i, j, \ell \in \{1, 2, 3, \ldots, \lfloor \frac{m}{2} \rfloor\}$ are all distinct. Let $\mathcal{U}$ be the set of triangles in nK_4 of colors $ij\ell$ where $i, j, \ell \in \{\lfloor \frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor + 2, \lfloor \frac{m}{2} \rfloor + 3, \ldots, m\}$ are all distinct. Note that $|\mathcal{L}| \leq \binom{\lfloor m/2 \rfloor}{3}$ and $|\mathcal{U}| \leq \binom{\lceil m/2 \rceil}{3}$.

Claim. *Each K_4 in G_2 contains at least one triangle in $\mathcal{L} \cup \mathcal{U}$.*

Proof of Claim. Let $\{v_1, v_2, v_3, v_4\}$ be the vertex set of K_4. Suppose without loss of generality that $r_{v_1} \leq r_{v_2} \leq r_{v_3} \leq r_{v_4}$. Thus,

$$r_{v_1} + r_{v_2} \leq r_{v_1} + r_{v_3} \leq r_{v_2} + r_{v_3} \leq r_{v_2} + r_{v_4} \leq r_{v_3} + r_{v_4}.$$

If v_2v_3 have color i, then i is in either

$$\left\{1, 2, 3, \ldots, \left\lfloor \frac{m}{2} \right\rfloor\right\} \text{ or } \left\{\left\lfloor \frac{m}{2} \right\rfloor + 1, \left\lfloor \frac{m}{2} \right\rfloor + 2, \left\lfloor \frac{m}{2} \right\rfloor + 3, \ldots, m\right\}.$$

Hence, either $v_1v_2v_3$ is in $\mathcal{L}$ or $v_2v_3v_4$ is in $\mathcal{U}$. □

By Claim, there exists a subgraph n_2K_3 of G_2 consisting of triangles in $\mathcal{L} \cup \mathcal{U}$. Since n_2K_3 is an induced subgraph of nK_4, we have r is also a representation of

n_2K_3. Applying Lemma 3.1 with the representation r of n_2K_3, no two triangles in n_2K_3 have the same color. Thus,

$$n_2 \leq |\mathcal{L} \cup \mathcal{U}| \leq \binom{\lfloor \frac{m}{2} \rfloor}{3} + \binom{\lceil \frac{m}{2} \rceil}{3}.$$

Observe that if nK_4 is a k-threshold graph, then there are at most $\lceil \frac{k}{2} \rceil$ colors of edges in nK_4, and hence,

$$n \leq \left\lceil \frac{k}{2} \right\rceil + \binom{\left\lfloor \frac{\lceil k/2 \rceil}{2} \right\rfloor}{3} + \binom{\left\lceil \frac{\lceil k/2 \rceil}{2} \right\rceil}{3} = \left\lceil \frac{k}{2} \right\rceil + \binom{\lfloor \frac{k+1}{4} \rfloor}{3} + \binom{\lceil \frac{k}{4} \rceil}{3}. \qquad \square$$

Similarly to the case of K_3, the lower bounds for the threshold numbers obtained from Lemma 4.2 are not sharp. We again need another observation which says roughly that the first and last colors appear in at most one K_4 or part.

Lemma 4.3. *(i) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m-1})$-representation of nK_4, an edge of color m appears in at most one K_4.*

(ii) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_m)$-representation of $K_{n\times 4}$, a nonedge of color 1 appears in at most one part.

(iii) In a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of $K_{n\times 4}$, a nonedge of color $m+1$ appears in at most one part.

Proof. The proof is similar to that of Lemma 3.4. $\square$

The upper bounds for the threshold numbers will be obtained from rank assignments of the following forms. Let $A = \{a_1, a_2, a_3, \ldots, a_M\}$, $B = \{b_1, b_2, b_3, \ldots, b_M\}$ be such that $a_i + b_i = N$ for some $N \in \mathbb{R}$ and for all $i \in [M]$. For $n = 2M + 2\binom{M}{3}$, the (A, B)-*assignment* is the rank assignment of nK_4 consisting of the following

K_4's:

$K_4(a_i)$ for each $i \in [M]$,

$K_4(b_i)$ for each $i \in [M]$,

$K_4(a_i, b_i, a_j, b_j, a_k, b_k)$ for each subset $\{i, j, k\} \subset [M]$ of size 3 and

$K_4(a_i, b_i, a_j, b_j, b_k, a_k)$ for each subset $\{i, j, k\} \subset [M]$ of size 3,

where each of them appears exactly once. Note that the numbers of K_4's in each line are M, M, $\binom{M}{3}$ and $\binom{M}{3}$ respectively, and they exist by Proposition 4.1.

Let $\varepsilon > 0$. For $n = 2M + 1 + \binom{M}{3} + \binom{M+1}{3}$, the (A, B, ε)-*assignment* is the rank assignment of nK_4 consisting of the following K_4's:

$K_4(a_i)$ for each $i \in [M]$,

$K_4(b_i)$ for each $i \in [M]$,

$K_4(a_i, b_i, a_j, b_j, a_k, b_k)$ for each subset $\{i, j, k\} \subset [M]$ of size 3,

$K_4(a_i, b_i, a_j, b_j, b_k, a_k)$ for each subset $\{i, j, k\} \subset [M]$ of size 3,

$K_4\left(\frac{N}{2} + \varepsilon\right)$ and

$K_4\left(a_i + \varepsilon, b_i + \varepsilon, a_j + \varepsilon, b_j + \varepsilon, \frac{N}{2} + \varepsilon, \frac{N}{2} + \varepsilon\right)$ for distinct i and j in $[M]$,

where each of them appears exactly once. Note that the numbers of K_4's in each line are M, M, $\binom{M}{3}$, $\binom{M}{3}$, 1 and $\binom{M}{2}$ respectively, and they exist by Proposition 4.1.

Occasionally, we say that a K_4 is of

- *type I* if it is a $K_4(a_i)$ or $K_4(b_i)$ for some $i \in [M]$,
- *type II* if it is a $K_4(a_i, b_i, a_j, b_j, a_k, b_k)$ or $K_4(a_i, b_i, a_j, b_j, b_k, a_k)$ for some subset $\{i, j, k\} \subset [M]$ of size 3,
- *type III* if it is a $K_4\left(\frac{N}{2} + \varepsilon\right)$ and

- *type IV* if it is a $K_4\left(a_i+\varepsilon, b_i+\varepsilon, a_j+\varepsilon, b_j+\varepsilon, \frac{N}{2}+\varepsilon, \frac{N}{2}+\varepsilon\right)$ for some distinct $i,j\in[M]$.

In the same fasion, we can define the (A,B)-*assignment* and the (A,B,ε)-*assignment* of $K_{n\times 4}$ by replacing K_4 with E_4.

The following lemma will be used repeatedly in the proof of Lemma 4.5.

Lemma 4.4. *Let* $\{N, a_1, a_2, a_3, \ldots, a_M\}\subset\mathbb{R}$ *be a linearly independent set over* $\mathbb{Q}$ *and* $b_i = N - a_i$ *for* $i = 1,2,3,\ldots,M$. *Let* $A=\{a_1,a_2,a_3,\ldots,a_M\}$ *and* $B=\{b_1,b_2,b_3,\ldots,b_M\}$. *If*

$$\sum_{i=1}^{S}\alpha_i x_i + \beta N = 0,$$

where $\alpha_i\in\mathbb{Z}$, $x_i\in A\cup B$ *for all* $i\in[S]$ *and* $\beta\in\mathbb{Q}$, *then* $\sum_{i=1}^{S}\alpha_i$ *is even.*

Proof. Suppose that $\sum_{i=1}^{S}\alpha_i x_i+\beta N=0$ where $\alpha_i\in\mathbb{Z}$, $x_i\in A\cup B$ for all $i\in[S]$ and $\beta\in\mathbb{Q}$. Observe that x_i is either a_{j_i} or $b_{j_i}=N-a_{j_i}$ where $j_i\in[M]$. Then, we can write $x_i=\delta_i a_{j_i}+\beta_i N$ where $\delta_i\in\{-1,1\}$ and $\beta_i\in\{0,1\}$. The equation becomes

$$\sum_{i=1}^{S}\delta_i\alpha_i a_{j_i}+\sum_{i=1}^{S}\beta_i\alpha_i N+\beta N=0.$$

Since $\{N,a_1,a_2,a_3,\ldots,a_M\}$ is linearly independent over $\mathbb{Q}$, we have $\sum_{i=1}^{S}\delta_i\alpha_i=0$. Hence,

$$\sum_{i=1}^{S}\alpha_i=\sum_{i=1}^{S}\delta_i\alpha_i+2\sum_{\delta_i=-1}\alpha_i=2\sum_{\delta_i=-1}\alpha_i$$

is even. □

The linear independence of $\{N,a_1,a_2,a_3,\ldots,a_M\}$ over $\mathbb{Q}$ is a sufficient condition for the edge and nonedge rank sums in the (A,B)-assignment and in the (A,B,ε)-assignment not to coincide. For the (A,B,ε)-assignment, we prove further that there are small intervals without nonedge rank sums that cover all edge rank sums.

Lemma 4.5. *Let* $\{N, a_1, a_2, a_3, \ldots, a_M\} \subset \mathbb{R}$ *be a linearly independent set over* $\mathbb{Q}$ *and* $b_i = N - a_i$ *for* $i = 1, 2, 3, \ldots, M$. *Let* $A = \{a_1, a_2, a_3, \ldots, a_M\}$ *and* $B = \{b_1, b_2, b_3, \ldots, b_M\}$.

(i) *Let* $n = 2M + 2\binom{M}{3}$. *In the* (A, B)*-assignment of* nK_4*, the edge and nonedge rank sums do not coincide.*

(ii) *Let* $n = 2M + 1 + \binom{M}{3} + \binom{M+1}{3}$. *Then, there exists a positive real number* ε *such that, in the* (A, B, ε)*-assignment of* nK_4*, no nonedge rank sum lies in either* $[a_i, a_i + \varepsilon]$, $[b_i, b_i + \varepsilon]$ *or* $\left\{\frac{N}{2} + \varepsilon\right\}$ *for all* $i \in [M]$. *Moreover, the sets of the form* $[a_i, a_i + \varepsilon]$, $[b_i, b_i + \varepsilon]$ *and* $\left\{\frac{N}{2} + \varepsilon\right\}$ *for all* $i \in [M]$ *are pairwise disjoint.*

(iii) *Let* $n = 2M + 2\binom{M}{3}$. *In the* (A, B)*-assignment of* $K_{n\times 4}$*, the edge and nonedge rank sums do not coincide.*

(iv) *Let* $n = 2M + 1 + \binom{M}{3} + \binom{M+1}{3}$. *Then, there exists a positive real number* ε *such that, in the* (A, B, ε)*-assignment of* $K_{n\times 4}$*, no edge rank sum lies in either* $[a_i, a_i + \varepsilon]$, $[b_i, b_i + \varepsilon]$ *or* $\left\{\frac{N}{2} + \varepsilon\right\}$ *for all* $i \in [M]$. *Moreover, the sets of the form* $[a_i, a_i + \varepsilon]$, $[b_i, b_i + \varepsilon]$ *and* $\left\{\frac{N}{2} + \varepsilon\right\}$ *for all* $i \in [M]$ *are pairwise disjoint.*

Proof. For (i) and (ii), it is sufficient to prove (ii) since every K_4 in the (A, B)-assignment appears in the (A, B, ε)-assignment and each edge rank sum in the (A, B)-assignment is either a_i or b_i. The proofs of (iii) and (iv) are similar to those of (i) and (ii).

To prove (ii), let $n = 2M + 1 + \binom{M}{3} + \binom{M+1}{3}$. We first consider the (A, B, ε)-assignment of nK_4 in the case when $\varepsilon = 0$.

Claim. *For the* $(A, B, 0)$*-assignment* r' *of* nK_4*, no nonedge rank sum lies in* $A \cup B$.

Proof of Claim. Suppose to the contrary that there exists a nonedge xy in nK_4 such that $r'_x + r'_y$ lies in $A \cup B$, say $r'_x + r'_y = e_t \in \{a_t, b_t\}$ for some $t \in [M]$. We divide into cases according to the four possible types of K_4 that x and y are in as shown in Table 4.1.

x \ y	I K_4	II K_4	III K_4	IV K_4
I K_4	Case 1	Case 2	Case 4	Case 5
II K_4		Case 3	Case 6	Case 7
III K_4			Case 8	Case 9
IV K_4				Case 10

Table 4.1: Ten cases according to the four possible types of K_4 that x and y are in.

Observe that the rank of each vertex in a type I K_4 is of the form $\frac{c_i}{2}$ where $i \in [M]$ and $c_i \in \{a_i, b_i\}$, that in a type II K_4 is of the form $\frac{c_i+c_j-c_k}{2}$ where $i, j, k \in [M]$ are all distinct and $c_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j, k\}$, that in a type III K_4 is of the form $\frac{N}{4}$, and that in a type IV K_4 is of the form $\frac{c_i+c_j-N/2}{2}$ where $i, j \in [M]$ are distinct and $c_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j\}$.

Case 1. $x, y \in$ type I K_4.

Then, $x \in K_4(c_i)$ and $y \in K_4(d_j)$ where $i, j \in [M]$ and $c_i \in \{a_i, b_i\}$, $d_j \in \{a_j, b_j\}$. Thus, $r'_x = \frac{c_i}{2}$ and $r'_y = \frac{d_j}{2}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i + d_j = 2e_t.$$

First, suppose that $i \neq j$. One of i or j cannot equal to t, say $i \neq t$. By writing the equation in terms of the basis $\{N, a_1, a_2, a_3, \ldots, a_M\}$, we can see that the equality cannot occur since c_i is the only term in the equation involving a_i, a contradiction. Now, suppose that $i = j$. Since x and y are in different K_4's, we have $\{c_i, d_j\} = \{a_i, b_i\}$, and hence, the equation becomes $N = 2e_t$, a contradiction.

Case 2. $x \in$ type I K_4 and $y \in$ type II K_4.

Then, $x \in K_4\,(c_i)$ where $i \in [M]$, $c_i \in \{a_i, b_i\}$, y is in either $K_4\,(a_p, b_p, a_q, b_q, a_s, b_s)$ or $K_4\,(a_p, b_p, a_q, b_q, b_s, a_s)$ where $p, q, s \in [M]$ are all distinct. Thus, $r'_x = \frac{c_i}{2}$ and $r'_y = \frac{d_p + d_q - d_s}{2}$ where $d_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{p, q, s\}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i + d_p + d_q - d_s = 2e_t.$$

Since p, q, s are all distinct, there is an index in $\{p, q, s\}$ not appearing in $\{i, t\}$, say $p \notin \{i, t\}$. Thus, the equality cannot occur since d_p is the only term in the equation involving a_p, a contradiction.

Case 3. $x, y \in$ type II K_4.

Then, x is in either $K_4\,(a_i, b_i, a_j, b_j, a_k, b_k)$ or $K_4\,(a_i, b_i, a_j, b_j, b_k, a_k)$ where $i, j, k \in [M]$ are all distinct, and y is in either $K_4\,(a_p, b_p, a_q, b_q, a_s, b_s)$ or $K_4\,(a_p, b_p, a_q, b_q, b_s, a_s)$ where $p, q, s \in [M]$ are all distinct. Thus, $r'_x = \frac{c_i + c_j - c_k}{2}$ where $c_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j, k\}$ and $r'_y = \frac{d_p + d_q - d_s}{2}$ where $d_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{p, q, s\}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i + c_j - c_k + d_p + d_q - d_s = 2e_t.$$

Case 3.1. $\{i, j, k\} \neq \{p, q, s\}$.

Then, there is an index in $\{i, j, k\}$ not appearing in $\{p, q, s\}$, say $i \notin \{p, q, s\}$. Similarly, there is an index in $\{p, q, s\}$ not appearing in $\{i, j, k\}$, say $p \notin \{i, j, k\}$. One of i or p cannot equal to t, say $i \neq t$. Thus, the equality cannot occur since c_i is the only term in the equation involving a_i, a contradiction.

Case 3.2. $\{i, j, k\} = \{p, q, s\}$.

Without loss of generality, let $i = p$, $j = q$ and $k = s$. Since x, y are in different

c_i	c_j	c_k
a_i	b_j	a_k
b_i	b_j	b_k
b_i	a_j	a_k
a_i	a_j	b_k

(a) $K_4(a_i, b_i, a_j, b_j, a_k, b_k)$

d_i	d_j	d_k
a_i	b_j	b_k
b_i	b_j	a_k
b_i	a_j	b_k
a_i	a_j	a_k

(b) $K_4(a_i, b_i, a_j, b_j, b_k, a_k)$

Table 4.2: The possible values of c_i, c_j, c_k and d_i, d_j, d_k.

K_4's, we can assume without loss of generality that $x \in K_4(a_i, b_i, a_j, b_j, a_k, b_k)$ and $y \in K_4(a_i, b_i, a_j, b_j, b_k, a_k)$. By considering the edge rank sum of each triangle in $K_4(a_i, b_i, a_j, b_j, a_k, b_k)$, each row in Table 4.2a shows the possible values of c_i, c_j, c_k, and by considering the edge rank sum of each triangle in $K_4(a_i, b_i, a_j, b_j, b_k, a_k)$, each row in Table 4.2b shows the possible values of d_i, d_j, d_k.

By comparing a row in Table 4.2a with a row in Table 4.2b, we observe that either none or two of $c_i = d_i$, $c_j = d_j$ and $c_k = d_k$ hold. If none holds, then $\{c_i, d_i\} = \{a_i, b_i\}, \{c_j, d_j\} = \{a_j, b_j\}$ and $\{c_k, d_k\} = \{a_k, b_k\}$. Thus, the above equation becomes

$$N + N - N = 2e_t$$

which is a contradiction. If two of $c_i = d_i, c_j = d_j$ and $c_k = d_k$ hold, then we assume without loss of generality that $\{c_i, d_i\} = \{a_i, b_i\}$ and $c_j = d_j$, $c_k = d_k$. Thus, the original equation becomes

$$N + 2c_j - 2c_k = 2e_t.$$

Since $j \neq k$, one of j or k cannot equal to t, say $j \neq t$. Hence, the equality cannot occur since c_j is the only term in the equation involving a_j, a contradiction.

Case 4. $x \in$ type I K_4 and $y \in$ type III K_4.

Then, $x \in K_4(c_i)$ where $i \in [M]$ and $c_i \in \{a_i, b_i\}$, and $y \in K_4\left(\frac{N}{2}\right)$. Thus, $r'_x = \frac{c_i}{2}$ and $r'_y = \frac{N}{4}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i - 2e_t + \frac{N}{2} = 0.$$

By Lemma 4.4, the sum of the coefficients of c_i and e_t must be even, a contradiction.

Case 5. $x \in$ type I K_4 and $y \in$ type IV K_4.

Then, $x \in K_4(c_i)$ where $i \in [M]$ and $c_i \in \{a_i, b_i\}$, and $y \in K_4\left(a_p, b_p, a_q, b_q, \frac{N}{2}, \frac{N}{2}\right)$ where $p, q \in [M]$ are distinct. Thus, $r'_x = \frac{c_i}{2}$ and $r'_y = \frac{d_p + d_q - N/2}{2}$ where $d_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{p, s\}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i + d_p + d_q - 2e_t - \frac{N}{2} = 0.$$

By Lemma 4.4, the sum of the coefficients of c_i, d_p, d_q and e_t must be even, a contradiction.

Case 6. $x \in$ type II K_4 and $y \in$ type III K_4.

Then, x is in either $K_4(a_i, b_i, a_j, b_j, a_k, b_k)$ or $K_4(a_i, b_i, a_j, b_j, b_k, a_k)$ where $i, j, k \in [M]$ are all distinct, and $y \in K_4\left(\frac{N}{2}\right)$. Thus, $r'_x = \frac{c_i + c_j - c_k}{2}$ where $c_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j, k\}$, and $r'_y = \frac{N}{4}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i + c_j - c_k + \frac{N}{2} - 2e_t = 0.$$

By Lemma 4.4, the sum of the coefficients of c_i, c_j, c_k and e_t must be even, a contradiction.

Case 7. $x \in$ type II K_4 and $y \in$ type IV K_4.

Then, x is in either $K_4(a_i, b_i, a_j, b_j, a_k, b_k)$ or $K_4(a_i, b_i, a_j, b_j, b_k, a_k)$ where $i, j, k \in [M]$ are all distinct, and $y \in K_4\left(a_p, b_p, a_q, b_q, \frac{N}{2}, \frac{N}{2}\right)$ where $p, q \in [M]$ are distinct. Thus, $r'_x = \frac{c_i + c_j - c_k}{2}$ where $c_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j, k\}$, and $r'_y = \frac{d_p + d_q - N/2}{2}$ where

$d_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{p, q\}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i + c_j - c_k + d_p + d_q - 2e_t - \frac{N}{2} = 0.$$

By Lemma 4.4, the sum of the coefficients of c_i, c_j, c_k, d_p, d_q and e_t must be even, a contradiction.

Case 8. $x, y \in$ type III K_4.

This case cannot occur since x and y are in different K_4's, but there is only one $K_4\left(\frac{N}{2}\right)$.

Case 9. $x \in$ type III K_4 and $y \in$ type IV K_4.

Then, $x \in K_4\left(\frac{N}{2}\right)$ and $y \in K_4\left(a_i, b_i, a_j, b_j, \frac{N}{2}, \frac{N}{2}\right)$ where $i, j \in [M]$ are distinct. Thus, $r'_x = \frac{N}{4}$ and $r'_y = \frac{c_i + c_j - N/2}{2}$ where $c_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j\}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i + c_j = 2e_t.$$

We obtain a contradiction similar to Case 1.

Case 10. $x, y \in$ type IV K_4.

Then, $x \in K_4\left(a_i, b_i, a_j, b_j, \frac{N}{2}, \frac{N}{2}\right)$ where $i, j \in [M]$ are distinct, and y is in $K_4\left(a_p, b_p, a_q, b_q, \frac{N}{2}, \frac{N}{2}\right)$ where $p, q \in [M]$ are distinct. Thus, $r'_x = \frac{c_i + c_j - N/2}{2}$ where $c_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j\}$, and $r'_y = \frac{d_p + d_q - N/2}{2}$ where $d_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{p, q\}$. The equation $r'_x + r'_y = e_t$ becomes

$$c_i + c_j + d_p + d_q - N = 2e_t.$$

Since x and y are in different K_4's, we have $\{i, j\} \neq \{p, q\}$. Thus, there is an index in one set not appearing in the other set, say $i \notin \{p, q\}$ and $p \notin \{i, j\}$. One of i or p cannot equal to t, say $i \neq t$. Therefore, the equality cannot occur since c_i is the only term in the equation involving a_i, a contradiction. $\square$

Let ε be a positive real number smaller than any distance between two distinct rank sums in the $(A, B, 0)$-assignment of nK_4. Note that the set of edge rank sums in the $(A, B, 0)$-assignment of nK_4 is $A \cup B \cup \left\{\frac{N}{2}\right\}$. By the definition of ε, the sets of the form $[a_i, a_i + \varepsilon]$, $[b_i, b_i + \varepsilon]$ and $\left\{\frac{N}{2} + \varepsilon\right\}$ for all $i \in [M]$ are pairwise disjoint. Let r be the (A, B, ε)-assignment of nK_4. Then,, for any vertex $u \in nK_4$,

$$r_u = \begin{cases} r'_u & \textit{if } u \text{ is in a type I or II } K_4, \\ r'_u + \frac{\varepsilon}{2} & \textit{if } u \text{ is in a type III or IV } K_4. \end{cases}$$

Let xy be a nonedge in nK_4 and consider $a_i \in A$. Observe that

$$r_x + r_y \in \left\{ r'_x + r'_y, r'_x + r'_y + \frac{\varepsilon}{2}, r'_x + r'_y + \varepsilon \right\}.$$

By Claim, $r'_x + r'_y \neq a_i$. Since a_i is a rank sum in the $(A, B, 0)$-assignment, the distance between $r'_x + r'_y$ and a_i exceeds ε by the definition of ε. If $r'_x + r'_y > a_i$, then $a_i + \varepsilon < r'_x + r'_y \leq r_x + r_y$. If $r'_x + r'_y < a_i$, then $r_x + r_y \leq r'_x + r'_y + \varepsilon < a_i$. Thus, $r_x + r_y \notin [a_i, a_i + \varepsilon]$. Similarly, $r_x + r_y \notin [b_i, b_i + \varepsilon]$.

It remains to show that $r_x + r_y \neq \frac{N}{2} + \varepsilon$. Note that $\frac{N}{2}$ is a rank sum in the $(A, B, 0)$-assignment. Thus, the distance between $r'_x + r'_y$ and $\frac{N}{2}$ is either 0 or more than ε by the definition of ε. If x or y is in a type I or II K_4, then $r_x + r_y \in \left\{r'_x + r'_y, r'_x + r'_y + \frac{\varepsilon}{2}\right\}$. If $r_x + r_y = \frac{N}{2} + \varepsilon$, then the distance between $r'_x + r'_y$ and $\frac{N}{2}$ is either ε or $\frac{\varepsilon}{2}$, a contradiction. Thus, we may suppose that both x and y are in a a type III or IV K_4. Since there is only one K_4 of type III, we may suppose further that x is in a type IV K_4. Then, $x \in K_4\left(a_i + \varepsilon, b_i + \varepsilon, a_j + \varepsilon, b_j + \varepsilon, \frac{N}{2} + \varepsilon, \frac{N}{2} + \varepsilon\right)$ for some distinct $i, j \in [M]$. Thus, $r_x = \frac{c_i + c_j - N/2 + \varepsilon}{2}$ where $c_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j\}$.

If y is in a type III K_4, then $y \in K_4\left(\frac{N}{2} + \varepsilon\right)$. Thus, $r_y = \frac{N}{4} + \frac{\varepsilon}{2}$. Hence,

$$r_x + r_y = \left(\frac{c_i + c_j - N/2 + \varepsilon}{2}\right) + \left(\frac{N}{4} + \frac{\varepsilon}{2}\right) = \frac{c_i + c_j}{2} + \varepsilon.$$

Suppose to the contrary that $r_x + r_y = \frac{N}{2} + \varepsilon$, that is $c_i + c_j = N$. Since $i \neq j$, we have c_i is the only term in the equation involving a_i. Thus, the equality cannot occur, a contradiction.

If y is in a type IV K_4, then $y \in K_4\left(a_p+\varepsilon, b_p+\varepsilon, a_q+\varepsilon, b_q+\varepsilon, \frac{N}{2}+\varepsilon, \frac{N}{2}+\varepsilon\right)$ for some distinct $p, q \in [M]$. Thus, $r_y = \frac{d_p+d_q-N/2+\varepsilon}{2}$ where $d_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{p, q\}$. Suppose to the contrary that $r_x + r_y = \frac{N}{2} + \varepsilon$, i.e.,

$$c_i + c_j + d_p + d_q - 2N = 0.$$

Since x and y are in different K_4's, we have $\{i, j\} \neq \{p, q\}$. Thus, there exists an index in one set not appearing in the other set, say $i \notin \{p, q\}$. Recall that $i \neq j$. Hence, the equality cannot occur since c_i is the only term in the equation involving a_i, a contradiction. □

Now, we are ready to determine the exact threshold numbers of nK_4 and $K_{n\times 4}$. Its proof follows the same line of argument as in the proof of Theorems 3.8 and 3.9, nevertheless, those of Theorems 4.6 and 4.7 are significantly more complicated.

Theorem 4.6. *Let* $t_m = m + \binom{\lfloor m/2\rfloor}{3} + \binom{\lceil m/2\rceil}{3} + 1$. *For* $n \geq 1$,

$$\Theta(nK_4) = \begin{cases} 2m-1 & \text{if } n = t_{m-1}, \\ 2m & \text{if } t_{m-1} < n < t_m. \end{cases}$$

Proof. Let m be a positive integer such that $t_{m-1} \leq n < t_m$. Suppose to the contrary that $\Theta(nK_4) \leq 2m-2$. By Lemma 4.2 (i),

$$\begin{aligned} n &\leq \left\lceil \frac{\Theta(nK_4)}{2} \right\rceil + \binom{\left\lfloor \frac{\Theta(nK_4)+1}{4} \right\rfloor}{3} + \binom{\left\lceil \frac{\Theta(nK_4)}{4} \right\rceil}{3} \\ &\leq \left\lceil \frac{2m-2}{2} \right\rceil + \binom{\left\lfloor \frac{2m-2+1}{4} \right\rfloor}{3} + \binom{\left\lceil \frac{2m-2}{4} \right\rceil}{3} \\ &= m-1 + \binom{\left\lfloor \frac{m-1}{2} \right\rfloor}{3} + \binom{\left\lceil \frac{m-1}{2} \right\rceil}{3} \\ &= t_{m-1} - 1, \end{aligned}$$

contradicting the definition of m. Hence, $\Theta(nK_4) \geq 2m-1$.

To prove that $\Theta(nK_4) \leq 2m$, let $\{N, a_1, a_2, a_3, \ldots, a_{\lfloor m/2 \rfloor}\} \subset \mathbb{R}^+$ be a linearly independent set over $\mathbb{Q}$ such that $a_i < N$ for all $i \in \{1, 2, 3, \ldots, \lfloor \frac{m}{2} \rfloor\}$ and let $b_i = N - a_i$ for $i \in \{1, 2, 3, \ldots, \lfloor \frac{m}{2} \rfloor\}$. Write $A = \{a_1, a_2, a_3, \ldots, a_{\lfloor m/2 \rfloor}\}$ and $B = \{b_1, b_2, b_3, \ldots, b_{\lfloor m/2 \rfloor}\}$.

Case 1. m is even.

Let $n' = t_m - 1 = m + 2\binom{m/2}{3}$. It is sufficient to show that $\Theta(n'K_4) \leq 2m$ since $\Theta(nK_4) \leq \Theta(n'K_4)$ as nK_4 is an induced subgraph of $n'K_4$. Consider the (A, B)-assignment of $n'K_4$. By Lemma 4.5 (i), the edge and nonedge rank sums do not coincide. Note that the set of edge rank sums of $n'K_4$ is $A \cup B$. Let $A \cup B = \{c_1, c_2, c_3, \ldots, c_m\}$. We separate the edge and nonedge rank sums by putting two thresholds around each edge rank sum. For $i = 1, 2, 3, \ldots, m$, let $\theta_{2i-1} = c_i$ and $\theta_{2i} = c_i + \varepsilon'$ be thresholds of $n'K_4$ where ε' is a sufficiently small positive real number, for example, take ε' smaller than any distance between two distinct rank sums of $n'K_4$. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of $n'K_4$, and hence, $n'K_4$ is a $2m$-threshold graph, that is $\Theta(n'K_4) \leq 2m$.

Case 2. m is odd.

Let $n' = t_m - 1 = m + \binom{\lfloor m/2 \rfloor}{3} + \binom{\lceil m/2 \rceil}{3}$. It is sufficient to show that $\Theta(n'K_4) \leq 2m$ since nK_4 is an induced subgraph of $n'K_4$. By Lemma 4.5 (ii), there is a positive real number ε such that, in the (A, B, ε)-assignment of $n'K_4$, no nonedge rank sum lies in either $[a_i, a_i + \varepsilon]$, $[b_i, b_i + \varepsilon]$ or $\{\frac{N}{2} + \varepsilon\}$ for all $i \in \{1, 2, 3, \ldots, \lfloor \frac{m}{2} \rfloor\}$, and moreover, the sets of the form $[a_i, a_i + \varepsilon]$, $[b_i, b_i + \varepsilon]$ and $\{\frac{N}{2} + \varepsilon\}$ for all $i \in \{1, 2, 3, \ldots, \lfloor \frac{m}{2} \rfloor\}$ are pairwise disjoint. Let $A \cup B \cup \{\frac{N}{2} + \varepsilon\} = \{c_1, c_2, c_3, \ldots, c_m\}$. We separate the edge and nonedge rank sums by putting two thresholds around each interval of edge rank sums of the form $[c_i, c_i + \varepsilon]$ and $\{\frac{N}{2} + \varepsilon\}$. For $i =$

$1, 2, 3, \ldots, m$, let $\theta_{2i-1} = c_i$ and

$$\theta_{2i} = \begin{cases} c_i + \varepsilon + \varepsilon' & \text{if } c_i \in A \cup B, \\ c_i + \varepsilon' & \text{if } c_i = \frac{N}{2} + \varepsilon \end{cases}$$

be thresholds of $n'K_4$ where ε' is a sufficiently small positive real number. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of $n'K_4$, and hence, $n'K_4$ is a $2m$-threshold graph, that is $\Theta(n'K_4) \leq 2m$.

Suppose that $n = t_{m-1}$. To prove that $\Theta(nK_4) \leq 2m-1$, we write $M = \left\lfloor \frac{m+1}{2} \right\rfloor$ and let $\{N, a_1, a_2, a_3, \ldots, a_M\} \subset \mathbb{R}^+$ be a linearly independent set over $\mathbb{Q}$ such that $a_i < N \leq \frac{a_M}{2}$ for all $i \in [M-1]$. Let $b_i = N - a_i$ for $i = 1, 2, 3, \ldots, M-1$. Write $A = \{a_1, a_2, a_3, \ldots, a_{M-1}\}$ and $B = \{b_1, b_2, b_3, \ldots, b_{M-1}\}$.

Case 1. $m-1$ is even.

We take the (A, B)-assignment for the first $m - 1 + 2\binom{(m-1)/2}{3}$ K_4's in nK_4, and let every edge in the last K_4 have edge rank sum a_M. Note that these K_4's appear in the $(A \cup \{a_M\}, B \cup \{b_M\})$-assignment of $(t_{m+1} - 1)\, K_4$. By Lemma 4.5 (i), the edge and nonedge rank sums do not coincide. Observe that the set of edge rank sums of nK_4 is $A \cup B \cup \{a_M\}$. Let $A \cup B \cup \{a_M\} = \{c_1, c_2, c_3, \ldots, c_m\}$ where $c_1 < c_2 < c_3 < \cdots < c_m$. We separate the edge and nonedge rank sums by putting two thresholds around each edge rank sum. For $i = 1, 2, 3, \ldots, m$, let $\theta_{2i-1} = c_i$ and $\theta_{2i} = c_i + \varepsilon'$ be thresholds of nK_4 where ε' is a sufficiently small positive real number. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of nK_4. In fact, we will show that we do not need the last threshold θ_{2m} by proving that no rank sum exceeds θ_{2m-1}. It is sufficient to show that the rank of each vertex is at most $\frac{\theta_{2m-1}}{2} = \frac{c_m}{2} = \frac{a_M}{2}$. This is clear for the last K_4 with the set of edge rank sums $\{a_M\}$. For the other K_4's, the rank of each vertex is of the form $\frac{c_i + c_j - c_k}{2}$ for some $i, j, k \in [m-1]$, which is at most $\frac{a_M}{2}$ since $c_i, c_j \leq \frac{a_M}{2}$ and $c_k > 0$. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m-1})$-representation of nK_4, and hence, nK_4 is a $(2m-1)$-threshold graph, that is $\Theta(nK_4) \leq 2m-1$.

Case 2. $m-1$ is odd.

We choose ε such that the $(A \cup \{a_M\}, B \cup \{b_M\}, \varepsilon)$-assignment of $(t_{m+1}-1)\,K_4$ satisfies the properties in Lemma 4.5 (ii). We then take the (A, B, ε)-assignment for the first $m-1+\binom{\lfloor (m-1)/2 \rfloor}{3}+\binom{\lceil (m-1)/2 \rceil}{3}$ K_4's in nK_4, and let every edge in the last K_4 have edge rank sum a_M. Note that these K_4's appear in the $(A \cup \{a_M\}, B \cup \{b_M\}, \varepsilon)$-assignment of $(t_{m+1}-1)\,K_4$. By the choice of ε, no nonedge rank sum lies in either $[a_i, a_i+\varepsilon]$, $[b_i, b_i+\varepsilon]$ or $\left\{\frac{N}{2}+\varepsilon\right\}$ for all $i \in [M-1]$, and moreover, the sets of the form $[a_i, a_i+\varepsilon]$, $[b_i, b_i+\varepsilon]$ and $\left\{\frac{N}{2}+\varepsilon\right\}$ for all $i \in [M-1]$ are pairwise disjoint. Let $A \cup B \cup \left\{a_M, \frac{N}{2}+\varepsilon\right\} = \{c_1, c_2, c_3, \ldots, c_m\}$ where $c_1 < c_2 < c_3 < \cdots < c_m$. We claim that $c_m = a_M$. Indeed, it is clear that $a_M > a_i, b_i$ for all $i \in [M-1]$. Since $\frac{N}{2}+\varepsilon$ lies between the intervals $[a_1, a_1+\varepsilon]$ and $[b_1, b_1+\varepsilon]$ by the choice of ε, we have $\frac{N}{2}+\varepsilon < \max\{a_1, b_1\} < a_M$. We separate the edge and nonedge rank sums by putting two thresholds around each interval of edge rank sums. For $i = 1, 2, 3, \ldots, m$, let $\theta_{2i-1} = c_i$ and

$$\theta_{2i} = \begin{cases} c_i + \varepsilon + \varepsilon' & \text{if } c_i \in A \cup B \cup \{a_M\}, \\ c_i + \varepsilon' & \text{if } c_i = \frac{N}{2} + \varepsilon \end{cases}$$

be thresholds of nK_4 where ε' is a sufficiently small positive real number. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of nK_4. In fact, we will show that we do not need the last threshold θ_{2m} by proving that no rank sum is greater than or equal to $\theta_{2m} = a_M + \varepsilon + \varepsilon'$. It is sufficient to show that the rank of each vertex is at most $\frac{a_M+\varepsilon}{2}$. This is clear for the last K_4 with the set of edge rank sums $\{a_M\}$. For the other K_4's, the rank of each vertex is of the form $\frac{d_i+d_j-d_k}{2}$, $\frac{N}{4}+\frac{\varepsilon}{2}$ or $\frac{d_i+d_j-N/2+\varepsilon}{2}$ where $i, j, k \in [M-1]$ and $d_\ell \in \{a_\ell, b_\ell\}$ for

$\ell \in \{i,j,k\}$, which is at most $\frac{a_M+\varepsilon}{2}$ since $0 < d_i, d_j, d_k, \frac{N}{2} \leq \frac{a_M}{2}$. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m-1})$-representation of nK_4, and hence, nK_4 is a $(2m-1)$-threshold graph, that is $\Theta(nK_4) \leq 2m-1$.

Suppose that $n > t_{m-1}$. To prove that $\Theta(nK_4) \geq 2m$, we suppose that $\Theta(nK_4) \leq 2m-1$. Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m-1})$-representation of nK_4. Then, there are at most m colors of edges in nK_4. By Lemma 4.2 (i), there are at most $t_{m-1}-1$ K_4's without color m. By Lemma 4.3 (i), an edge of color m appears in at most one K_4. Thus, $n \leq (t_{m-1}-1)+1$, a contradiction. Therefore, $\Theta(nK_4) \geq 2m$. □

Theorem 4.7. *Let* $s_m = m + \binom{\lfloor m/2 \rfloor}{3} + \binom{\lceil m/2 \rceil}{3} + 2$. *For* $n \geq 2$,

$$\Theta(K_{n\times 4}) = \begin{cases} 2m & \text{if } n = s_{m-1}, \\ 2m+1 & \text{if } s_{m-1} < n < s_m. \end{cases}$$

Proof. Let m be a positive integer such that $s_{m-1} \leq n < s_m$. By Theorem 4.6, $\Theta(nK_4) \in \{2m, 2m+1\}$, and hence, $\Theta(K_{n\times 4}) \in \{2m, 2m+1\}$ by Proposition 2.15.

Suppose that $n = s_{m-1}$. To prove that $\Theta(K_{n\times 4}) \leq 2m$, we write $M = \left\lfloor \frac{m+1}{2} \right\rfloor$ and let $\{N, a_1, a_2, a_3, \ldots, a_M\} \subset \mathbb{R}$ be a linearly independent set over $\mathbb{Q}$ such that $\frac{a_M}{3} \leq -N < -a_i < 0$ for all $i \in [M-1]$. Let $b_i = N - a_i$ for $i = 1, 2, 3, \ldots, M$. Then, $\frac{a_M}{3} \leq a_i, b_i, -N, N \leq \frac{b_M}{3}$ for all $i \in [M-1]$. Write $A = \{a_1, a_2, a_3, \ldots, a_{M-1}\}$ and $B = \{b_1, b_2, b_3, \ldots, b_{M-1}\}$.

Case 1. $m-1$ is even.

We take the (A,B)-assignment for the first $m - 1 + 2\binom{(m-1)/2}{3}$ parts in $K_{n\times 4}$, and let the last two parts have the sets of nonedge rank sums $\{a_M\}$ and $\{b_M\}$. Note that these parts appear in the $(A \cup \{a_M\}, B \cup \{b_M\})$-assignment of $K_{(s_{m+1}-2)\times 4}$. By Lemma 4.5 (iii), the edge and nonedge rank sums do not coincide. Observe that the set of nonedge rank sums of $K_{n\times 4}$ is $A \cup B \cup \{a_M, b_M\}$. Let $A \cup B \cup \{a_M, b_M\} =$

$\{c_1, c_2, c_3, \ldots, c_{m+1}\}$ where $c_1 < c_2 < c_3 < \cdots < c_{m+1}$. Let θ_1 be smaller than all rank sums. We then separate the edge and nonedge rank sums by putting two thresholds around each nonedge rank sum. For $i = 1, 2, 3, \ldots, m+1$, let $\theta_{2i} = c_i$ and $\theta_{2i+1} = c_i + \varepsilon'$ where ε' is a sufficiently small positive real number. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m+3})$-representation of $K_{n\times 4}$. In fact, we will show that we do not need the thresholds θ_1, θ_2 and θ_{2m+3} by proving that no rank sum is smaller than θ_2 or larger than θ_{2m+2}. It is sufficient to show that the rank of each vertex is at least $\frac{\theta_2}{2} = \frac{c_1}{2} = \frac{a_M}{2}$ and at most $\frac{\theta_{2m+2}}{2} = \frac{c_{m+1}}{2} = \frac{b_M}{2}$. This is clear for the last two parts with the sets of nonedge rank sums $\{a_M\}$ and $\{b_M\}$. For the other parts, the rank of each vertex is of the form $\frac{c_i+c_j-c_k}{2}$ for some $i, j, k \in [m] \setminus \{1\}$, which is at least $\frac{a_M}{2}$ and at most $\frac{b_M}{2}$ since $\frac{a_M}{3} \leq c_i, c_j, -c_k \leq \frac{b_M}{3}$. Thus, the above rank assignment is a $(\theta_3, \theta_4, \theta_5, \ldots, \theta_{2m+2})$-representation of $K_{n\times 4}$, and hence, $K_{n\times 4}$ is a $2m$-threshold graph, that is $\Theta(K_{n\times 4}) \leq 2m$.

Case 2. $m-1$ is odd.

We choose ε such that the $(A \cup \{a_M\}, B \cup \{b_M\}, \varepsilon)$-assignment of $K_{(s_{m+1}-2)\times 4}$ satisfies the properties in Lemma 4.5 (iv). We then take the (A, B, ε)-assignment for the first $m-1+\binom{\lfloor (m-1)/2 \rfloor}{3}+\binom{\lceil (m-1)/2 \rceil}{3}$ parts in $K_{n\times 4}$, and let the last two parts have the sets of nonedge rank sums $\{a_M\}$ and $\{b_M\}$. Note that these parts appear in the $(A \cup \{a_M\}, B \cup \{b_M\}, \varepsilon)$-assignment of $K_{(s_{m+1}-2)\times 4}$. By the choice of ε, no edge rank sum lies in either $[a_i, a_i+\varepsilon]$, $[b_i, b_i+\varepsilon]$ or $\left\{\frac{N}{2}+\varepsilon\right\}$ for all $i \in [M-1]$, and moreover, the sets of the form $[a_i, a_i+\varepsilon]$, $[b_i, b_i+\varepsilon]$ and $\left\{\frac{N}{2}+\varepsilon\right\}$ for all $i \in [M-1]$ are pairwise disjoint. Let $A \cup B \cup \left\{a_M, b_M, \frac{N}{2}+\varepsilon\right\} = \{c_1, c_2, c_3, \ldots, c_{m+1}\}$ where $c_1 < c_2 < c_3 < \cdots < c_{m+1}$. We claim that $c_1 = a_M$ and $c_{m+1} = b_M$. Indeed, it is clear that $a_M < a_i, b_i < b_M$ for all $i \in [M-1]$. Since $\frac{N}{2}+\varepsilon$ lies between the intervals $[a_1, a_1+\varepsilon]$ and $[b_1, b_1+\varepsilon]$ by the choice of ε, we have $a_M < \min\{a_1, b_1\} < \frac{N}{2}+\varepsilon < \max\{a_1, b_1\} < b_M$. Let θ_1 be smaller than all rank sums. We then separate the edge and nonedge rank sums by putting two thresholds around each interval of nonedge rank sums. For $i = 1, 2, 3, \ldots, m+1$, let $\theta_{2i} = c_i$ and

$$\theta_{2i+1} = \begin{cases} c_i + \varepsilon + \varepsilon' & \text{if } c_i \in A \cup B \cup \{a_M, b_M\}, \\ c_i + \varepsilon' & \text{if } c_i = \frac{N}{2} + \varepsilon \end{cases}$$

be thresholds of $K_{n\times 4}$ where ε' is a sufficiently small positive real number. Thus, the above rank assignment is a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m+3})$-representation of $K_{n\times 4}$. In fact, we will show that we do not need the thresholds θ_1, θ_2 and θ_{2m+3} by proving that no rank sum is smaller than θ_2, or larger than or equal to θ_{2m+3}. It is sufficient to show that the rank of each vertex is at least $\frac{\theta_2}{2} = \frac{c_1}{2} = \frac{a_M}{2}$ and at most $\frac{\theta_{2m+3}-\varepsilon'}{2} = \frac{c_{m+1}+\varepsilon}{2} = \frac{b_M+\varepsilon}{2}$. This is clear for the last two parts with the sets of nonedge rank sums $\{a_M\}$ and $\{b_M\}$. For the other parts, the rank of each vertex is of the form $\frac{d_i+d_j-d_k}{2}, \frac{N}{4} + \frac{\varepsilon}{2}$ or $\frac{d_i+d_j-N/2+\varepsilon}{2}$ where $i, j, k \in [M-1]$ are all distinct and $d_\ell \in \{a_\ell, b_\ell\}$ for $\ell \in \{i, j, k\}$, which is at least $\frac{a_M}{2}$ and at most $\frac{b_M+\varepsilon}{2}$ since $\frac{a_M}{3} \leq d_i, d_j, -d_k, -N, N \leq \frac{b_M}{3}$. Thus, the above rank assignment is a $(\theta_3, \theta_4, \theta_5, \ldots, \theta_{2m+2})$-representation of $K_{n\times 4}$, and hence, $K_{n\times 4}$ is a $2m$-threshold graph, that is $\Theta(K_{n\times 4}) \leq 2m$.

Suppose that $n > s_{m-1}$. To prove that $\Theta(K_{n\times 4}) \geq 2m+1$, we suppose that $\Theta(K_{n\times 4}) \leq 2m$. Let r be a $(\theta_1, \theta_2, \theta_3, \ldots, \theta_{2m})$-representation of $K_{n\times 4}$. Then, there are at most $m+1$ colors of nonedges in $K_{n\times 4}$. By Lemma 4.2 (ii), there are at most $s_{m-1} - 2$ parts without colors 1 and $m+1$. By Lemma 4.3 (ii) and 4.3 (iii), a nonedge of color 1 appears in at most one part and a nonedge of color $m+1$ also appears in at most one part. Therefore, $n \leq (s_{m-1}-2)+1+1$, a contradiction. $\square$

CHAPTER V

CONCLUSIONS AND OPEN PROBLEMS

In this dissertation, we determine the exact threshold numbers of $K_{n\times 3}$, $K_{n\times 4}$ and their complements, nK_3 and nK_4. Theorems 3.8 and 3.9 in Chapter III indicate the threshold numbers of $K_{n\times 3}$ and nK_3, while Theorems 4.6 and 4.7 in Chapter IV indicate the threshold numbers of $K_{n\times 4}$ and nK_4. These results can be summarized in the following theorems.

Theorem 5.1. *Let* $q_m = m + \binom{m}{3} + 1$.

(i) For $n \geq 2$,

$$\Theta(K_{n\times 3}) = \begin{cases} 2m & \text{if } n = q_{m-1} + 1, \\ 2m+1 & \text{if } q_{m-1} + 1 < n < q_m, \\ 2m+1 & \text{if } n = q_m \text{ and } m \geq 3. \end{cases}$$

(ii) For $n \geq 1$,

$$\Theta(nK_3) = \begin{cases} 2m & \text{if } n = q_{m-1} + 1 \text{ and } m \geq 3, \\ 2m & \text{if } q_{m-1} + 1 < n < q_m, \\ 2m+1 & \text{if } n = q_m. \end{cases}$$

Theorem 5.2. *Let* $t_m = m + \binom{\lfloor m/2 \rfloor}{3} + \binom{\lceil m/2 \rceil}{3} + 1$.

(i) For $n \geq 2$,

$$\Theta(K_{n\times 4}) = \begin{cases} 2m & \text{if } n = t_{m-1} + 1, \\ 2m+1 & \text{if } t_{m-1} + 1 < n < t_m, \\ 2m+1 & \text{if } n = t_m \text{ and } m \geq 5. \end{cases}$$

(ii) For $n \geq 1$,

$$\Theta(nK_4) = \begin{cases} 2m & \text{if } n = t_{m-1} + 1 \text{ and } m \geq 5, \\ 2m & \text{if } t_{m-1} + 1 < n < t_m, \\ 2m + 1 & \text{if } n = t_m. \end{cases}$$

We recall Conjecture 1.1 as shown.

Conjecture 5.3 ([9])**.** *For all* $k \geq 1$*, there is a graph* G *with* $\Theta(G) = 2k$ *and* $\Theta(G^c) = 2k + 1$.

This conjecture was confirmed by Chen and Hao [2] (see Theorem 1.3). Note that Theorem 5.1 gives more examples satisfying the conjecture except for $k = 3$, while Theorem 5.2 gives more examples satisfying the conjecture except for $k \in \{5, 6\}$. In addition, Theorem 5.1 also improves the result of Puleo [17] providing an upper bound for $\Theta(K_{n\times 3})$.

Chen and Hao [2] gave the value of $\Theta(K_{m_1,m_2,m_3,\ldots,m_n})$ for $m_i > n \geq 2$, while our main results give the values of $\Theta(K_{n\times 3})$ and $\Theta(K_{n\times 4})$. Therefore, Problem 1.2 remains unsolved for other complete multipartite graphs. The followings could be the next goals.

Problem 5.4. *Determine the exact threshold numbers of* $n_3K_3 \cup n_4K_4$ *and their complements.*

Problem 5.5. *Determine the exact threshold numbers of* $n_1K_1 \cup n_2K_2 \cup n_3K_3$ *and their complements.*

Problem 5.6. *Determine the exact threshold numbers of* $n_1K_1 \cup n_2K_2 \cup n_3K_3 \cup n_4K_4$ *and their complements.*

Problem 5.7. *Determine the exact threshold numbers of* $K_{n\times m}$ *for* $m \geq 5$ *and their complements.*

The method we used can be generalized to give some bounds for $\Theta(K_{n\times m})$, but new ideas seem to be required in order to find the exact value.

REFERENCES

[1] Calamoneri, T., Monti, A., Petreschi, R.: On dynamic threshold graphs and related classes, *Theoret. Comput. Sci.* **718**, 46–57 (2018).

[2] Chen, G., Hao, Y.: Multithreshold multipartite graphs, *J. Graph Theory.* **100**, 727–732 (2022).

[3] Chvátal, V., Hammer, P. L.: Aggregation of inequalities in integer programming, *Ann. Discrete Math.* **1**, 145–162 (1977).

[4] Golumbic, M. C.: *Algorithmic graph theory and perfect graphs*, Academic Press, New York, NY. 2nd ed., 1980.

[5] Golumbic, M. C., Jamison, R. E.: Rank tolerance graph classes, *J. Graph Theory.* **52**, 317–340 (2006).

[6] Hammer, P. L., Mahadev, N. V. R., Peled, U. N.: Some properties of 2-threshold graphs, *Networks.* **19**, 17–23 (1989).

[7] Henderson, P.B., Zalcstein, Y.: A graph-theoretic characterization of the pv chunk class of synchronizing primitives, *SIAM J. Comput.* **6**(1), 88–108 (1977).

[8] Jamison, R. E., Sprague, A. P.: Double-threshold permutation graphs, *Algebraic Combin.* **56**, 23–41 (2022).

[9] Jamison, R. E., Sprague, A. P.: Multithreshold graphs, *J. Graph Theory.* **94**, 518–530 (2020).

[10] Kobayashi, Y., Okamoto, Y., Otachi, Y., Uno, Y.: Linear-time recognition of double-threshold graphs, *Algorithmica.* **84**, 1163–1181 (2022).

[11] Koop, G.J.: Cyclic scheduling of offweekends, *Oper. Res. Lett.* **4**, 259–263 (1986).

[12] Mahadev, N. V. R., Peled, U. N.: Threshold graphs and related topics, *Ann. Discrete Math.* **56** (1995).

[13] Ordman, E. T.: Threshold coverings and resource allocation, *16th Southeastern Conference on Combinatorics, Graph Theory and Computing.* 99–113 (1986).

[14] Orlin, J.: The minimal integral separator of a threshold graph, *Ann. Discrete Math.* **1**, 415–419 (1977).

[15] Petreschi, R., Sterbini, A.: Recognizing strict 2-threshold graphs in O(m) time, *Inform. Process. Lett.* **54**, 193–198 (1995).

[16] Petreschi, R., Sterbini, A.: Threshold graphs and synchronization protocols, *Lecture Notes in Computer Science.* **1120**, 378–395 (1996).

[17] Puleo, G. J.: Some results on multithreshold graphs, *Graphs Combin.* **36**(3), 913–919 (2020).

[18] Reilly, E., Scheinerman, E., Zhang, Y.: Random threshold digraphs, *Electron. J. Combin.* **21**(2), P2.48 (2014).

VITA

Name	:	Miss Thanaporn Sumalroj
Date of Birth	:	1 July 1991
Place of Birth	:	Nakhon Pathom, Thailand
Education	:	B.Sc. (Mathematics), Silpakorn University, 2014
		M.Sc. (Mathematics), Chulalongkorn University, 2017
Scholarship	:	The Development and Promotion of Science and Technology Talents Project (DPST)