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นางสาวบุษยพัชร์ ใจห้าว

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GLOBAL ANALYSIS OF A DISCRETE SIRS EPIDEMIC MODEL WITH NONLINEAR INCIDENCE RATE

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ในวิทยานิพนธ์ฉบับนี้ ได้พิสูจน์พฤติกรรมบางประการของแบบจำลองโรคระบาดแบบเอสไออาร์เอสวิยุต ที่มีอัตราอุบัติการณ์แบบไม่เชิงเส้น และมีตัวหน่วงเวลาแบบกระจาย แบบจำลองนี้ถูกสร้างขึ้นจากแบบจำลองต่อเนื่องที่สอดคล้องกัน โดยการดิสครีตไตซ์ด้วยระเบียบวิธีผลต่างอันตะแบบไม่มาตรฐาน สมบัติพื้นฐาน ได้แก่ ความเป็นบวกและการมีขอบเขตของผลเฉลยได้รับการยืนยัน เราได้หาจุดสมดุลที่ปลอดโรคและจุดสมดุลที่ติดเชื้อของแบบจำลอง นอกจากนั้นโดยการสร้างฟังก์ชันเลียปูนอฟ สามารถพิสูจน์ได้ว่าจุดสมดุลที่ปลอดโรคมีสมบัติดึงดูดวงกว้าง ยิ่งไปกว่านั้นเรายังได้เงื่อนไขที่เพียงพอที่ทำให้แบบจำลองมีความถาวร และสุดท้ายเราได้นำเสนอการจำลองเชิงตัวเลขเพื่อเป็นตัวอย่างให้เห็นภาพของผลลัพธ์เชิงวิเคราะห์ที่ได้

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BUTSAYAPAT CHAIHAO : GLOBAL ANALYSIS OF A DISCRETE SIRS EPIDEMIC MODEL WITH NONLINEAR INCIDENCE RATE
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In this thesis, we prove some behaviors of a discrete SIRS epidemic model with a nonlinear incidence rate and a distributed time-delay. This model is constructed from the discretization of the corresponding continuous model by using a nonstandard finite difference method. The basic properties including the positivity and the boundedness of the solutions are established. We derive the existence of the disease-free equilibrium and the endemic equilibrium of the model. In addition, by applying Lyapunov function techniques, we prove that the disease-free equilibrium is globally attractive. Moreover, we give a sufficient condition for the permanence of the model. In order to illustrate our analytical results, finally, some numerical simulations are also included.

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CONTENTS

CHAPTER I
INTRODUCTION

Mathematical epidemic models have been extensively studied in order to understand the spread of epidemic diseases and disease control in human population. One of the classical epidemic models is an SIR model, which was proposed by Kermack and McKendrick in 1927. In this model, the total population is divided into three classes:

- **Susceptible class:** individuals who are healthy but can contract the disease.
- **Infectious class:** individuals who have contracted the disease and can transmit the disease. Infected individuals may not be infectious during entire time of being infected.
- **Recovery class:** individuals who have recovered from the disease.

After a susceptible individual undergoes the disease by contacting with an infectious individual, he or she moves into the infectious class. At the same time, infectious individuals who recover from the disease and have immunity transfer into the recovery class. The SIR model assumes that those recovered individuals have permanent immunity to the disease. However, in most communicable diseases such as cholera, pertussis, influenza and malaria, recovered individuals can return to the susceptible class [15]. This situation can be described by the epidemic model which is so-called "SIRS model". In this work, we consider the following continuous

SIRS epidemic model with distributed delays:

$$\begin{cases} \dot{S}(t) = \Lambda - \int_0^\tau p(\xi)g(I(t-\xi))S(t)d\xi - \mu_1 S(t) + \gamma R(t), \\ \dot{I}(t) = \int_0^\tau p(\xi)g(I(t-\xi))S(t)d\xi - (\mu_2 + k)I(t), \\ \dot{R}(t) = kI(t) - (\mu_3 + \gamma)R(t), \end{cases} \tag{1.1}$$

where $\Lambda > 0$ denotes the recruitment rate into the population. The constants $\mu_i > 0\,(i = 1, 2, 3)$ are the death rate of susceptible, infectious and recovered individuals, respectively. The constant $k > 0$ is the recovery rate of infectious individuals. The recovered individual loses immunity and returns to the susceptible class with a constant rate $\gamma \geq 0$. The function $g(I)S$ is called the incidence rate: the number of individuals who become infectious per unit of time (e.g. one month, one year), and $g(I)$ measures the infection force of a disease. In the model, the incidence rate is used with the form $\int_0^\tau p(\xi)g(I(t-\xi))S(t)d\xi$, which includes the distributed delays ξ, where ξ is the incubation time and $p(\xi)$ is the distributed proportion of the population taking time ξ to become infectious. The constant $\tau > 0$ is the maximum incubation period. The function $p(\xi)$ is assumed to be nonnegative and continuous on $[0, \tau]$ and that satisfies $\int_0^\tau p(\xi)d\xi = 1$ [3]. The following figure shows the flow diagram of a simple SIRS model:

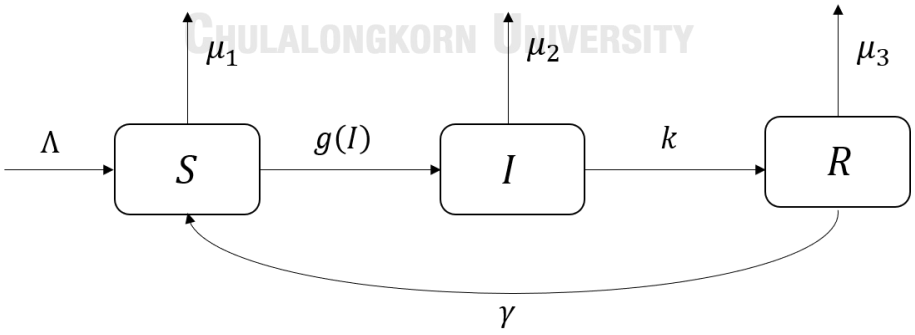

Figure 1.1: Flow diagram of a simple SIRS epidemic model

In the modeling of epidemic diseases, an incidence rate plays a significant role in describing the evolution of infectious disease. A bilinear incidence rate has been

often used in early epidemic models. The rate is defined as

$$g(I)S = \delta IS, \tag{1.2}$$

where δ is a parameter for infectivity. Because of the effect concerning the nonlinearity of incidence rate for some disease transmissions, many researchers have suggested various kinds of nonlinear incidence rates. Capasso and Serio [5] introduced a saturated incidence rate of the form

$$g(I)S = \frac{\delta IS}{1 + \alpha I}, \tag{1.3}$$

where α measures the inhibition effect from the behavioral change of the susceptible individuals when the number of infectious individuals increases or from the crowding effect of the infective individuals. Xiao and Ruan [30] have given an assumption that the incidence rate takes the nonlinear form

$$g(I)S = \frac{\delta IS}{1 + \alpha I^2}, \tag{1.4}$$

which can be interpreted as a nonmonotone incidence rate. In [31], Xiao and Zhou studied continuous SIRS epidemic models with a complete form of the nonmonotone incidence rate,

$$g(I)S = \frac{\delta IS}{1 + \beta I + \alpha I^2}, \tag{1.5}$$

where δ is the infection rate, $\alpha > 0$ measures the psychological or inhibitory effect and β is a parameter such that $1 + \beta I + \alpha I^2 > 0$ for all $I \geq 0$, which holds if $\beta > -2\sqrt{\alpha}$. When a serious disease emerges, the infection force increases because people have very little knowledge about the disease. However, when I is large and the disease becomes more serious, psychological factor leads people to adjust their behavior and control the spread of the disease (See Figure 1.2(a)). For example, in the outbreak of coronavirus disease 2019 (COVID-19), public heath interventions

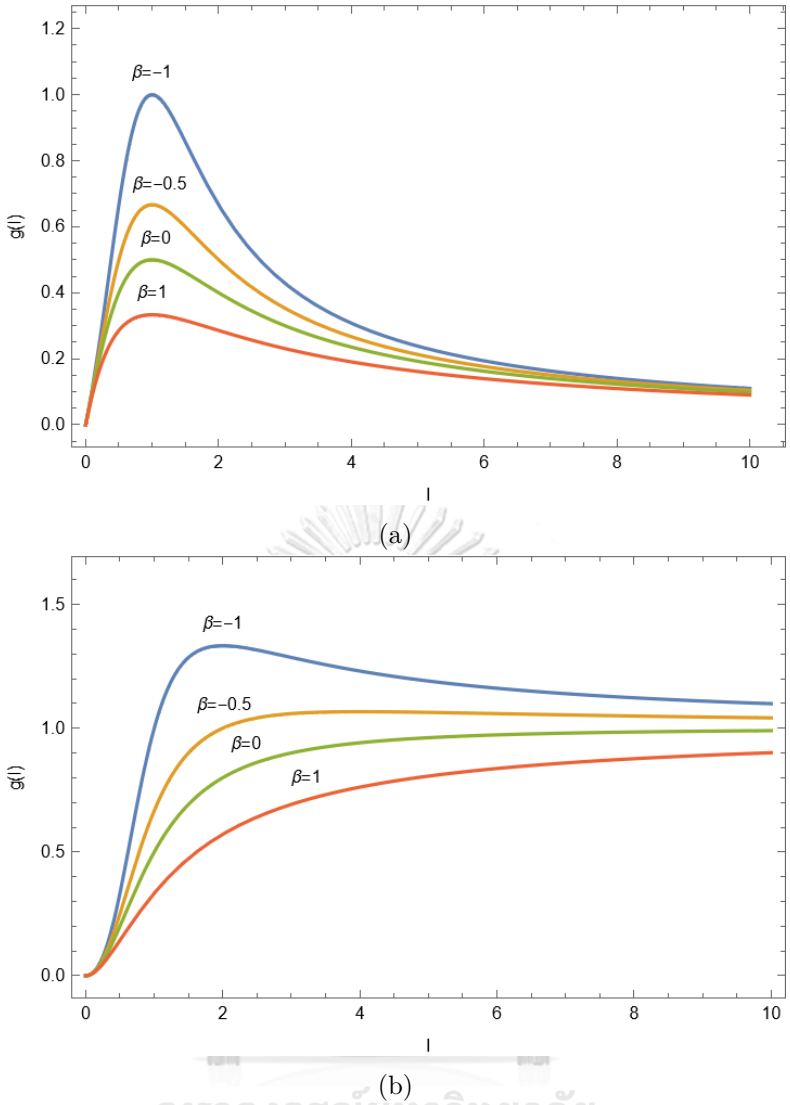

Figure 1.2: The graphs of nonlinear infection functions. (a) $g(I) = \frac{\delta I}{1+\beta I+\alpha I^2}$; (b) $g(I) = \frac{\delta I^2}{1+\beta I+\alpha I^2}$ when $\delta = \alpha = 1$

including borders measures, quarantine, isolation, city lockdown, mask wearing, social distancing, etc. were proved to be very effective in decreasing the transmission [2]. Note that both nonmonotone incidence functions (1.4) and (1.5) tend to zero when I goes to infinity. In [16], the authors proposed a more general incidence rate which has a combination of monotonicity, nonmonotonicity and saturation properties as follows:

$$g(I)S = \frac{\delta I^2 S}{1 + \beta I + \alpha I^2}, \tag{1.6}$$

which exhibits complicated dynamical behaviors. When $\beta \geq 0$, the infection function $g(I) = \frac{\delta I^2}{1+\beta I+\alpha I^2}$ is monotonic, which always increases and tends to a saturated level $\frac{\delta}{\alpha}$ as I goes to infinity. When $-2\sqrt{\alpha} < \beta < 0$, $g(I)$ is nonmonotonic, which increases when I is small, decreases when I is large and finally tends to $\frac{\delta}{\alpha}$ as I goes to infinity (See Figure 1.2(b)). Figure 1.2 also shows that if δ and α are fixed, $g(I)$ in (1.5) and (1.6) increases faster for smaller β when I is small.

Usually, there are two types of epidemic models: continuous-time models and discrete-time models. The continuous-time epidemic models described by differential equations have been widely investigated in many articles (for example, [10,11,21,25,30,31] and the references therein). In recent years, there has been an increasing interest in the study of discrete-time models (see [6,8,9,23,24,26,32] and the references therein). Because statistical data on epidemics are reported daily, monthly or yearly etc., it is more convenient and accurate to describe epidemics by employing discrete models. In order to obtain discrete-time epidemic models, discretizing a continuous-time employing numerical schemes such as the Euler, Huen and Rung-Kutta has been frequently used (see [8, 9, 23]). However, one approach to avoid numerical instabilities and excessively small step sizes is to apply a nonstandard finite difference (NSFD) scheme. This method was first proposed by Mickens [18] and have received much attention up to now (for example, [6, 24, 26, 32]). For solving an n-dimensional autonomous dynamical systems, Wood and Kojouharov [29] demonstrate that applying NSFD method preserves positivity of solutions and local stability of equilibria for any time step size, while Heun's method does not preserve those, especially when the time step size is large. The authors in [8, 9] discussed the local and global stability of the disease-free equilibrium and endemic equilibrium for some discrete SIR and SIRS epidemic models. In [24], Sekiguchi and Ishiwata derived a discretized SIRS epidemic model with time delay from Mickens' NSFD scheme and obtained the global stability of the disease-free equilibrium and the permanence of the disease.

In [26], a discrete SIRS epidemic model with distributed delays and general nonlinear incidence rates was constructed from the discretization of the corre-

sponding continuous epidemic model by applying Mickens's NSFD scheme. Under the crucial assumption about monotonicity of the incidence function, the authors obtained the global stability of the disease-free equilibrium and the endemic equilibrium. However, the results cannot be applied to any discrete SIRS epidemic models with nonlinear incidence rates, especially, the model with a nonmonotone incidence rate like (1.5) and (1.6).

Motivated by the above-mentioned works, applying Mickens's NSFD scheme to the continuous model (1.1), we give the following discrete SIRS epidemic model with a nonlinear incidence rate and distributed delays:

$$\begin{cases} S_{n+1} = h\left(\Lambda - \sum_{j=0}^{m} p_j g(I_{n-j}) S_{n+1} - \mu_1 S_{n+1} + \gamma R_{n+1}\right) + S_n, \\ I_{n+1} = h\left(\sum_{j=0}^{m} p_j g(I_{n-j}) S_{n+1} - (\mu_2 + k) I_{n+1}\right) + I_n, \\ R_{n+1} = h\left(k I_{n+1} - (\mu_3 + \gamma) R_{n+1}\right) + R_n, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.7}$$

where $h > 0$ is a time step size; S_n, I_n and R_n denote the numbers of susceptible, infectious and recovered individuals at time n, respectively; $\mu_i > 0\,(i = 1, 2, 3)$ are the death rate of susceptible, infectious and recovered individuals, respectively; $k > 0$ is the recovery rate of infectious individuals; $\gamma \geq 0$ is the rate at which recovered individuals lose immunity and return to the susceptible class. The incidence rate $g(I)S$ satisfies (1.5) or (1.6), where $\delta > 0$ is the infection rate; $\alpha > 0$ measures the psychological or inhibitory effect; $\beta > -2\sqrt{\alpha}$ such that $1 + \beta I + \alpha I^2 > 0$ for all $I \geq 0$. The constant $m \geq 0$ is the maximum incubation period and $p_j \geq 0\,(j = 0, 1, \ldots, m)$ represent the distributed proportion of the population taking time j to become infectious, and we always assume that $\sum_{j=0}^{m} p_j = 1$. All of parameters of the model are also described in Table 1.1.

This work focuses on the dynamical behaviors of solutions to the discrete SIRS epidemic model (1.7) with nonlinear incidence rates (1.5) and (1.6). The global stability of the disease-free equilibrium of (1.7) is investigated. Furthermore, we establish the sufficient conditions for permanence, which is an important property

Table 1.1: Model parameters and their descriptions.

Parameter	Description
Λ	Recruitment rate
μ_1	Death rate of susceptible individuals
μ_2	Death rate of infectious individuals
μ_3	Death rate of recovered individuals
k	Recovery rate
γ	The loss of immunity rate of recovered individuals
δ	Transmission rate
α	The parameter measuring the psychological or inhibitory effect
β	A parameter satisfying $\beta > -2\sqrt{\alpha}$
m	Maximum incubation period
p_j	The distributed proportion of the population taking time j to become infectious

in epidemic model because it implies that the disease continues to exist, regardless of initial conditions.

The organization of this work is as follows. In Chapter II, we give some basic definitions and results to be used in this work. The basic assumptions of model (1.7) are introduced in Chapter III. Also, the basic properties about the positivity and the ultimate boundedness of solutions for model (1.7) are stated and proved. In Chapter IV, we concentrate on the discrete SIRS model (1.7) with incidence rate (1.6). The existence of the disease-free equilibrium and endemic equilibrium and results on the global stability of the disease-free equilibrium are established. Moreover, we discuss the permanence of the disease for the model. In Chapter V, the dynamical properties of the discrete SIRS model (1.7) with incidence rate (1.6) including the existence of equilibria and the global stability of the disease-free equilibrium are shown. In addition, we give some examples and simulations to illustrate the dynamical behaviors of the model by using MATLAB. Finally, we present some ideas for further research in Chapter VI.

CHAPTER II
PRELIMINARIES

In this chapter, we give an introduction to the nonstandard finite difference method. Some basic knowledges for studying the behaviors of solutions to the system of difference equations are recalled. We also present some basic results that will be useful.

2.1 Nonstandard Finite Difference Method

Consider a system of ordinary differential equation

$$\frac{dx}{dt} = f(x); \quad x(t_0) = x_0, \tag{2.1}$$

where $x := (x^{(1)}, x^{(2)}, \ldots, x^{(m)}) : [t_0, \infty) \to \mathbb{R}^m$ and $f := (f^{(1)}, f^{(2)}, \ldots, f^{(m)}) : \mathbb{R}^m \to \mathbb{R}^m$ is differentiable and $x_0 \in \mathbb{R}^m$. A finite difference method to approximate the system (2.1) can be written as

$$D_h(x_k) = F_h(f; x_k), \tag{2.2}$$

where $D_h(x_k) \approx \frac{dx}{dt}\Big|_{t=t_k}$, $x_k \approx x(t_k)$, $F_h(f; x_k)$ approximates $f(x(t_k))$ in system (2.1) and $t_k = t_0 + kh$, where $h > 0$. The finite difference method employed in this work is **nonstandard finite difference (NSFD) method** [18–20, 22, 28]. In general, any finite difference method, which is not standard can be considered as nonstandard. The construction of NSFD schemes is not always straightforward and there are no general criteria for them. However, several important rules for constructing NSFD schemes were discovered by Mickens as follows:

Rule 1. The orders of the discrete derivatives should be equal to the orders of the

corresponding derivatives of the differential equations.

Rule 2. Denominator functions for the discrete derivative must, in general, be expressed in terms of more complicated functions of the step-sizes than those conventionally used. For example, the first-order derivative is replaced by a discrete representation of the form

$$\frac{dx(t_k)}{dt} \to \frac{x_{k+1} - x_k}{\phi(h)},$$

where the denominator function $\phi(h)$ has the property that

$$\phi(h) = h + \mathcal{O}(h^2) \quad \text{as } h \to 0$$

such as $e^h - 1$, $1 - e^{-h}$, $\sin(h)$, etc. These functions may also depend on the various parameters that appear in the differential equations. Note that the conventional discrete representation for the first derivative takes $\phi(h) = h$.

Rule 3. Nonlinear terms should, in general, be replaced by nonlocal discrete representations using more than one mesh point. For example, the nonlinear term x^2 can be replaced by a nonlocal representation evaluated at two mesh points such as $x_{k+1}x_k$ and $2x_k^2 - x_{k+1}x_k$. This is in contrast to standard methods which use a local representation applying only one mesh point, that is x_k^2.

Rule 4. Special conditions that hold for the solutions of the differential equations should also hold for the solutions of the finite difference scheme. An important example is the condition of positivity that must be satisfied by many systems in the real-world problems. If the discrete equations allow their solutions to become negative, then numerical instabilities will occur.

Rule 5. The finite difference scheme should not introduce extraneous or spurious solutions that do not correspond to any solution of the corresponding differential equations.

According to Anguelove and Luboma [1], the following definition of NSFD method was also proposed:

Definition 2.1. The finite difference scheme (2.2) for solving system (2.1) is an NSFD method if at least one of the following conditions is satisfied:

- In the first-order derivative $D_h(x_k)$, the traditional denominator h is replaced by a nonnegative function $\phi(h)$ such that $\phi(h) = h + \mathcal{O}(h^2)$ as $h \to \infty$.
- Nonlinear term in $f(x)$ are approximated in a nonlocal way, i.e., by a suitable function of several points of the mesh. For example, $x^2(t_k) \approx x_k x_{k+1}$ and $x^3(t_k) \approx 2x_{k+1}^2 x_k^2/(x_{k+1} + x_k)$.

In this work, when we refer to NFSD method, it means those, which at least one of the above Mickens's rules is satisfied.

2.2 System of Difference Equations

Given a map $T : \mathbb{R}^m \to \mathbb{R}^m$, the following system:

$$x_{n+1} = T(x_n) \quad \text{for all } n \geq 0, \tag{2.3}$$

is referred to as a system of m-dimensional first-order difference equations. A sequence $\{x_n\}_{n=0}^{\infty}$ is the solution of (2.3) together with an initial condition $x_0 \in \mathbb{R}^m$ if $x_n = T^n(x_0)$ for all $n \geq 0$. Similarly, for any positive number k, a map $g : \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k} \to \mathbb{R}^m$ defines a system of m-dimensional k^{th}-order difference equations of the form:

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-k+1}). \tag{2.4}$$

Proposition 2.2 ([17]). *Any one-dimensional m^{th}-order difference equation $x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-m+1})$ is equivalent to a system of m-dimensional first-order difference equations.*

2.2.1 Stability and permanence

We first present the definition of equilibria and the concepts of stability for (2.3). From now on, $\|\cdot\|$ denote the Euclidean norm.

Definition 2.3. An **equilibrium point** of (2.3) is a point $\bar{x} \in \mathbb{R}^m$ such that $\bar{x} = T(\bar{x})$.

Definition 2.4. Let $\bar{x}$ be an equilibrium point of (2.3).

(i) $\bar{x}$ is called **locally stable** if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x_0 - \bar{x}\| < \delta$ implying

$$\|x_n - \bar{x}\| < \epsilon \quad \text{for all } n \geq 0.$$

(ii) $\bar{x}$ is called **locally asymptotically stable** if it is locally stable and there exists $\gamma > 0$ such that $\|x_0 - \bar{x}\| < \gamma$ implying

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = 0.$$

(iii) $\bar{x}$ is called a **global attractor** if for every $\{x_n\}_{n=0}^{\infty}$ satisfying (2.3), we have

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = 0.$$

(iv) $\bar{x}$ is called **unstable** if it is not locally stable.

Suppose that the function T is a continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Then, the linearized equation of (2.3) about the equilibrium point $\bar{x}$ is the equation of the form

$$y_{n+1} = J_T(\bar{x})y_n, \tag{2.5}$$

where $J_T(\bar{x})$ is the Jacobian matrix of T at the equilibrium point $\bar{x}$ and the char-

acteristic equation of (2.5) about $\bar{x}$ is

$$\det(J_T(\bar{x}) - \lambda I_m) = 0. \tag{2.6}$$

The following theorems will be useful for studying the local stability character of an equilibrium point $\bar{x}$ of (2.3).

Theorem 2.5 ([13]). *Let $\bar{x}$ be an equilibrium point of (2.3) and assume that T is a continuously differentiable in $\mathbb{R}^m$. Then, the following statements are true:*

(i) If all eigenvalues of the Jacobian matrix $J_T(\bar{x})$ lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\bar{x}$ of (2.3) is locally asymptotically stable.

(ii) If at least one eigenvalue of $J_T(\bar{x})$ has absolute value greater than one, then the equilibrium point $\bar{x}$ of (2.3) is unstable.

Definition 2.6. The equation (2.3) is said to be **permanent** provided there exist constants $m_i > 0$, $M_i > 0$ $(i = 1, 2, \ldots, m)$ such that any solution $\{x_n = (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)})\}$ satisfies

$$m_i \leq \liminf_{n\to\infty} x_n^{(i)} \leq \limsup_{n\to\infty} x_n^{(i)} \leq M_i \quad \text{for all } i = 1, 2, \ldots, m.$$

2.2.2 Lyapunov functions and LaSalle invariance principle

Definition 2.7. Given a map $T : \mathbb{R}^m \to \mathbb{R}^m$ and a set $H \subset \mathbb{R}^m$, define the set $T(H) = \{y \in \mathbb{R}^m |\ y = T(x) \text{ for some } x \in H\}$. The set H is called **invariant under** T if $T(H) = H$.

Definition 2.8. For any point $x \in \mathbb{R}^m$ and set $S \subset \mathbb{R}$. The distance between x and S is defined by $\text{dist}(x, S) = \inf_{y\in S} \|x - y\|$. We say that $T^n(x)$ converges to S if $\lim_{n\to\infty} \text{dist}(T^n(x), S) = 0$.

Definition 2.9. Let G be any set in $\mathbb{R}^m$. A **Lyapunov function** of (2.3) on G is a continuous function $V : \mathbb{R}^m \to \mathbb{R}$ satisfying

$$\Delta V(x) := V(T(x)) - V(x) \leq 0 \quad \text{for all } x \in G.$$

The following theorems will be useful for studying the global stability character of equilibria in Chapters IV and V.

Theorem 2.10 (LaSalle invariance principle [14])**.** *Let G be any set in $\mathbb{R}^m$ and $V : \mathbb{R}^m \to \mathbb{R}$ be a Lyapunov function of (2.3) on G. Suppose $\{x_n\}_{n=0}^{\infty}$ is a bounded solution of (2.3) in G. Then there exists $c \in \mathbb{R}$ such that*

$$x_n \to M \cap V^{-1}(c) \quad as\ n \to \infty,$$

i.e. $dist(x_n, M \cap V^{-1}(c)) \to 0$, where $V^{-1}(c) = \{x \in \mathbb{R}^m \,|V(x) = c\}$ and M is the largest invariant set in $E = \{x \in \overline{G} \,|\, \Delta V(x) = 0\}$.

CHAPTER III
POSITIVITY AND BOUNDEDNESS OF SOLUTIONS

The equation (1.7) is supplied with the following initial conditions:

$$S_j \geq 0, \quad I_j \geq 0, \quad R_j \geq 0 \quad (-m \leq j \leq 0) \quad \text{and} \quad S_0 > 0, \quad I_0 > 0. \tag{3.1}$$

In this work, we assume that $g(I) = f(I)I$; in addition, the following notations are defined:

- $S^0 = \dfrac{\Lambda}{\mu_1}$.
- $G_n = G_n(I) = \displaystyle\sum_{j=0}^{m} p_j g(I_{n-j})$.
- $\Pi_1 = 1 + h\mu_1, \quad \Pi_2 = 1 + h(\mu_2 + k) \quad \text{and} \quad \Pi_3 = 1 + h(\mu_3 + \gamma)$.
- $f^M = \max\limits_{I \geq 0} f(I)$.

For the positivity of solutions of model (1.7) with initial condition (3.1), we have the following result.

Proposition 3.1. *Any solution of the model (1.7) with initial condition (3.1) is positive, that is* $S_n > 0, I_n > 0, R_n > 0$ *for all* $n > 0$.

Proof. The first equation of (1.7) yields

$$S_{n+1} = \frac{h\Lambda + S_n + h\gamma R_{n+1}}{\Pi_1 + hG_n}. \tag{3.2}$$

By the second and third equations, we obtain

$$I_{n+1} = \frac{hG_nS_{n+1} + I_n}{\Pi_2} \quad \text{and} \quad R_{n+1} = \frac{hkI_{n+1} + R_n}{\Pi_3}. \tag{3.3}$$

This implies that

$$\begin{aligned} \Pi_3 R_{n+1} &= hk\frac{hG_nS_{n+1}+I_n}{\Pi_2} + R_n \\ &= \frac{hk(hG_nS_{n+1}+I_n)+\Pi_2 R_n}{\Pi_2} \\ &= \frac{h^2kG_nS_{n+1}+hkI_n+\Pi_2 R_n}{\Pi_2} \end{aligned} \tag{3.4}$$

By equations (3.2) and (3.4), we obtain

$$\begin{aligned} \Pi_2\Pi_3(\Pi_1+hG_n)S_{n+1} &= \Pi_2\Pi_3(h\Lambda+S_n)+h\gamma\Pi_2\Pi_3 R_{n+1} \\ &= \Pi_2\Pi_3(h\Lambda+S_n)+h\gamma(h^2kG_nS_{n+1}+hkI_n+\Pi_2 R_n) \\ &= \Pi_2\Pi_3(h\Lambda+S_n)+h^3\gamma kG_nS_{n+1}+h^2\gamma kI_n+h\gamma\Pi_2 R_n. \end{aligned}$$

Thus,

$$S_{n+1} = \frac{\Pi_2\Pi_3(h\Lambda+S_n)+h\gamma\Pi_2 R_n+h^2\gamma kI_n}{\Pi_1\Pi_2\Pi_3+(\Pi_2\Pi_3-h^2\gamma k)hG_n}.$$

Notice that $\Pi_i > 0$ for $i = 1, 2, 3$ and $\Pi_2\Pi_3 - h^2\gamma k > 0$. From the initial condition (3.1), it follows easily that $G_0 \geq 0$ so $S_1 > 0$. Using (3.3), we also have $I_1 > 0$ and $R_1 > 0$. Applying the same argument, we obtain $G_{n-1} > 0$ and so $S_n, I_n, R_n > 0$ for all n. This completes the proof. □

From now on we make the following assumption:

$$\mu_1 \leq \min\{\mu_2, \mu_3\}. \tag{H1}$$

This can be interpreted as that the death rate of the infected and recovered individuals may increase because of the disease. The following result shows that the solutions of (1.7) with initial condition (3.1) are bounded above.

Proposition 3.2. *Suppose (H1). For any solution (S_n, I_n, R_n) of (1.7), the total*

population $N_n := S_n + I_n + R_n$ *satisfies*

$$\limsup_{n\to\infty} N_n \leq S^0 = \frac{\Lambda}{\mu_1}.$$

Proof. Adding the equations in (1.7), we obtain

$$N_{n+1} = N_n + h\left(\Lambda - \mu_1 S_{n+1} - \mu_2 I_{n+1} - \mu_3 R_{n+1}\right), \quad n = 0, 1, 2, \ldots.$$

Applying the hypothesis (H1) and Proposition (3.1), we get

$$\begin{aligned} N_{n+1} &\leq N_n + h[\Lambda - \mu_1(S_{n+1} + I_{n+1} + R_{n+1})] \\ &= N_n + h(\Lambda - \mu_1 N_{n+1}). \end{aligned}$$

This implies that

$$N_{n+1} \leq \frac{h\Lambda}{\Pi_1} + \frac{1}{\Pi_1} N_n, \quad n = 0, 1, 2, \ldots.$$

Using iteration method, we have

$$\begin{aligned} N_{n+1} &\leq \frac{h\Lambda}{\Pi_1} + \frac{1}{\Pi_1}\left(\frac{h\Lambda}{\Pi_1} + \frac{1}{\Pi_1} N_{n-1}\right) \\ &\leq \frac{h\Lambda}{\Pi_1} + \frac{h\Lambda}{\Pi_1^2} + \frac{1}{\Pi_1^2}\left(\frac{h\Lambda}{\Pi_1} + \frac{1}{\Pi_1} N_{n-2}\right) \\ &\leq \frac{h\Lambda}{\Pi_1} + \frac{h\Lambda}{\Pi_1^2} + \cdots + \frac{h\Lambda}{\Pi_1^{n+1}} + \frac{1}{\Pi_1^{n+1}} N_0 \\ &= \frac{h\Lambda}{\Pi_1}\left(1 + \frac{1}{\Pi_1} + \cdots + \frac{1}{\Pi_1^n}\right) + \frac{1}{\Pi_1^{n+1}} N_0 \\ &= \frac{h\Lambda}{\Pi_1}\left(\frac{1 - \frac{1}{\Pi_1^{n+1}}}{1 - \frac{1}{\Pi_1}}\right) + \frac{1}{\Pi_1^{n+1}} N_0 \\ &= \frac{h\Lambda}{\Pi_1}\left(\frac{\Pi_1}{h\mu_1}\right)\left(1 - \frac{1}{\Pi_1^{n+1}}\right) + \frac{1}{\Pi_1^{n+1}} N_0 \\ &= S^0\left(1 - \frac{1}{\Pi_1^{n+1}}\right) + \frac{1}{\Pi_1^{n+1}} N_0 \to S^0. \end{aligned}$$

Thus, $\limsup_{n\to\infty} N_n \leq S^0$. □

Remark 3.3. In the case $\mu_1 = \mu_2 = \mu_3$, we have $N_n = S^0\left(1 - \frac{1}{\Pi_1^{n+1}}\right) + \frac{1}{\Pi_1^{n+1}}N_0$ and so $\lim_{n\to\infty} N_n = S^0$.

Remark 3.4. Proposition 3.1 and (3.2) may be summarized by saying that

$$\limsup_{n\to\infty} S_n \leq S^0, \quad \limsup_{n\to\infty} I_n \leq S^0 \quad \text{and} \quad \limsup_{n\to\infty} R_n \leq S^0.$$

CHAPTER IV
STABILITY AND PERMANENCE OF A DISCRETE SIRS EPIDEMIC MODEL WITH INCIDENCE RATE $(\delta IS)/(1+\beta I+\alpha I^2)$

In this chapter, we concentrate on the discrete SIRS epidemic model (1.7) with nonlinear incidence rate

$$g_1(I)S = \frac{\delta IS}{1+\beta I+\alpha I^2}.$$

Then (1.7) may be written as follows:

$$\begin{cases} S_{n+1} = h\left(\Lambda - G_n S_{n+1} - \mu_1 S_{n+1} + \gamma R_{n+1}\right) + S_n, \\ I_{n+1} = h\left(G_n S_{n+1} - (\mu_2 + k) I_{n+1}\right) + I_n, \\ R_{n+1} = h\left(kI_{n+1} - (\mu_3 + \gamma) R_{n+1}\right) + R_n, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.1}$$

and it can be rearranged to get the explicit form

$$\begin{cases} S_{n+1} = \dfrac{h\Lambda + S_n + h\gamma R_{n+1}}{\Pi_1 + hG_n}, \\ I_{n+1} = \dfrac{hG_n S_{n+1} + I_n}{\Pi_2}, \\ R_{n+1} = \dfrac{hkI_{n+1} + R_n}{\Pi_3}, \end{cases} \tag{4.2}$$

where $G_n = G_n(I) = \sum_{j=0}^{m} p_j g_1(I_{n-j})$. Here, we write

$$g_1(I) = f_1(I)I, \quad \text{where} \quad f_1(I) = \frac{\delta}{1+\beta I+\alpha I^2}.$$

Setting $S_{n+1} = S_n$, $I_{n+1} = I_n = 0$, $R_{n+1} = R_n$ yields $S_n = S^0$ and $R_n = 0$; hence, (4.1) admits a unique disease-free equilibrium $E^0 = (S^0, 0, 0)$. To find the endemic equilibria $E^* = (S^*, I^*, R^*)$, we consider the equations

$$\begin{cases} \Lambda - g_1(I^*)S^* - \mu_1 S^* + \gamma R^* = 0, \\ g_1(I^*)S^* - (\mu_2 + k)I^* = 0, \\ kI^* - (\mu_3 + \gamma)R^* = 0. \end{cases} \tag{4.3}$$

Solving the last equation for R^* and adding the first two equations, we obtain

$$R^* = \frac{kI^*}{\mu_3 + \gamma}, \quad S^* = S^0 - \frac{\mu_2(\mu_3 + \gamma) + k\mu_3}{\mu_1(\mu_3 + \gamma)} I^*. \tag{4.4}$$

By plugging S^* into the second equation of (4.3), $I^* > 0$ satisfies

$$1 + \beta I^* + \alpha(I^*)^2 - \frac{\delta}{\mu_2 + k}\left(S^0 - \frac{\mu_2(\mu_3 + \gamma) + k\mu_3}{\mu_1(\mu_3 + \gamma)} I^*\right) = 0.$$

Then I^* satisfies the equation

$$\alpha I^2 + (A + \beta)I + (1 - B) = 0, \tag{4.5}$$

where

$$A = \frac{\delta}{\mu_2 + k} \frac{\mu_2(\mu_3 + \gamma) + k\mu_3}{\mu_1(\mu_3 + \gamma)}, \quad B = \frac{\delta}{\mu_2 + k} S^0.$$

It can be seen that $A > 0$ and $B > 0$. The existence of the endemic equilibria E^* determined the roots of (4.5), which are

$$I_+ = \frac{-(A + \beta) + \sqrt{D}}{2\alpha}, \quad \text{and} \quad I_- = \frac{-(A + \beta) - \sqrt{D}}{2\alpha},$$

where $D = (A + \beta)^2 - 4\alpha(1 - B)$.

Then we have

$$I_+I_- = \frac{1-B}{\alpha} \quad \text{and} \quad I_+ + I_- = \frac{-(A+\beta)}{\alpha}.$$

Obviously, $D \geq 0$ is equivalent to $B \geq \frac{4\alpha-(A+\beta)^2}{4\alpha}$. We denote

$$B_0 = \frac{4\alpha-(A+\beta)^2}{4\alpha}.$$

It follows easily that if $B < B_0$ or, in other words, $D < 0$, then (4.1) has no endemic equilibrium. Assume that $B \geq B_0$.

- If $B < 1$ then I_+I_->0, so (4.1) has no endemic equilibria or else it has two endemic equilibria. The first circumstance occurs if and only if $A \geq -\beta$ while the second one occurs precisely when $A < -\beta$.
- If $B = 1$ then $I_+I_- = 0$. So (4.1) has no endemic equilibria if and only if $A \geq -\beta$ and it has one endemic equilibrium if and only if $A < -\beta$.
- If $B > 1$ then $I_+I_- < 0$. It follows immediately that (4.1) has exactly one endemic equilibrium.

This is precisely the assertion of the theorem.

Theorem 4.1. *(4.1) has a unique disease-free equilibrium $E^0 = (S^0, 0, 0)$. Regarding the endemic equilibria $E^* = (S^*, I^*, R^*)$ of (4.1), the following results hold.*

(a) there is no endemic equilibria provided one of the following conditions holds:

(i) $B < B_0$;

(ii) $B_0 \leq B \leq 1$ and $A \geq -\beta$;

(b) there is exactly one endemic equilibrium provided one of the following conditions holds:

(i) $B = 1$ and $A < -\beta$;

(ii) $B > 1$;

(c) there are two endemic equilibria if and only if $B_0 \leq B < 1$ *and* $A < -\beta$.

For simplicity to study the local stability of the equilibrium (4.1), we consider a special case with $m = 0$ and $\gamma = 0$, that is, model (4.1) degenerates into the following discrete SIR epidemic model without delayed time:

$$\begin{cases} S_{n+1} = \dfrac{h\Lambda + S_n}{\Pi_1 + hg_1(I_n)}, \\ I_{n+1} = \dfrac{hg_1(I_n)S_{n+1} + I_n}{\Pi_2}, \\ R_{n+1} = \dfrac{hkI_{n+1} + R_n}{\Pi_3}. \end{cases} \tag{4.6}$$

We notice that the first two equations in (4.6) do not depend on the third equation. Hence, to obtain the local stability of the equilibrium of (4.6), we can omit the third equation and define the following function:

$$\begin{aligned} F(S,I) &= \frac{h\Lambda + S}{\Pi_1 + hg_1(I)}, \\ G(S,I) &= \frac{hg_1(I)F(S,I) + I}{\Pi_2}. \end{aligned} \tag{4.7}$$

Then the Jacobian matrix of (4.7) is given by

$$J((S,I)) = \begin{pmatrix} \dfrac{\partial F(S,I)}{\partial S} & \dfrac{\partial F(S,I)}{\partial I} \\ \dfrac{\partial G(S,I)}{\partial S} & \dfrac{\partial G(S,I)}{\partial I} \end{pmatrix}, \tag{4.8}$$

where

$$\begin{aligned} \frac{\partial F(S,I)}{\partial S} &= \frac{1}{\Pi_1 + hg_1(I)}, \\ \frac{\partial F(S,I)}{\partial I} &= -\frac{hg_1'(I)(h\Lambda + S)}{(\Pi_1 + hg_1(I))^2}, \\ \frac{\partial G(S,I)}{\partial S} &= \frac{hg_1(I)}{\Pi_2(\Pi_1 + hg_1(I))}, \end{aligned}$$

$$\frac{\partial G(S,I)}{\partial I} = \frac{1}{\Pi_2}\left(1 + hg_1(I)\frac{\partial F(S,I)}{\partial I} + hF(S,I)g_1'(I)\right),$$

and

$$g_1(I) = \frac{\delta I}{1+\beta I+\alpha I^2}, \quad g_1'(I) = \frac{\delta(1-\alpha I^2)}{(1+\beta I+\alpha I^2)^2}.$$

Then, we give the following theorem about the local stability of the disease-free equilibrium of (4.6). From now on, we denote

$$\mathcal{R}_0 = \frac{\delta S^0}{\mu_2 + k}.$$

Theorem 4.2. *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium E^0 of (4.6) is locally asymptotically stable. Otherwise, it is unstable if $\mathcal{R}_0 > 1$.*

Proof. Substituting the disease-free equilibrium $E^0 = (S^0, 0)$ into the jacobian matrix (4.8), we obtain

$$J(E^0) = \begin{pmatrix} \frac{1}{1+h\mu_1} & -\frac{h\delta S^0}{1+h\mu_1} \\ 0 & \frac{1+h\delta S^0}{1+h(\mu_2+k)} \end{pmatrix}.$$

It follows easily that the eigenvalues of $J(E^0)$ are

$$\lambda_1 = \frac{1}{1+h\mu_1} \quad \text{and} \quad \lambda_2 = \frac{1+h\delta S^0}{1+h(\mu_2+k)}.$$

Obviously, $|\lambda_1| < 1$. By the hypothesis, $\mathcal{R}_0 < 1$, we get $1 + h\delta S^0 < 1 + h(\mu_2 + k)$. This implies that $|\lambda_2| < 1$. Therefore, the disease-free equilibrium E^0 is locally asymptotically stable. □

4.1 Global Attractivity of the Disease-Free Equilibrium

In this section, we obtain a sufficient condition for global attractivity of the disease-free equilibrium E^0 of (4.1) by constructing a Lyapunov function inspired by [23].

Theorem 4.3. *The disease-free equilibrium E^0 of (4.1) is globally attractive if*

$$\mathcal{R}_{0,1} := \frac{f_1^M S^0}{\mu_2 + k} < 1,$$

where $f_1^M = \max_{I \geq 0} f_1(I)$. In the case $\beta \geq 0$, if $\mathcal{R}_0 < 1$, then the disease-free equilibrium is globally attractive.

Proof. Let (S_n, I_n, R_n) be any positive solution of model (4.1) with initial condition (3.1). We introduce the following candidate for a Lyapunov function:

$$U_n = \frac{1}{h} I_n + \frac{\rho_1}{h} R_n + \rho_2 \sum_{j=0}^{m} p_j \sum_{l=n-j}^{n} g_1(I_l) + \frac{\rho_3}{2h} \left(S_n - S^0\right)^2,$$

where ρ_1, ρ_2, and ρ_3 are positive constants which will be offered later. Then $\Delta_n U = U_{n+1} - U_n$ is given by

$$\begin{aligned}
\Delta_n U &= \frac{1}{h}(I_{n+1} - I_n) + \frac{\rho_1}{h}(R_{n+1} - R_n) + \rho_2 \sum_{j=0}^{m} p_j \left(g_1(I_{n+1}) - g_1(I_{n-j})\right) \\
&\quad + \frac{\rho_3}{2h}(S_{n+1} - S_n)(S_{n+1} + S_n - 2S^0) \\
&= \frac{1}{h}(I_{n+1} - I_n) + \frac{\rho_1}{h}(R_{n+1} - R_n) + \rho_2 \left(g_1(I_{n+1}) - G_n\right) \\
&\quad + \frac{\rho_3}{2h}(S_{n+1} - S_n)(S_{n+1} + S_n - 2S^0).
\end{aligned}$$

Using (1.7) and that $(S_{n+1} - S_n)(S_{n+1} + S_n - 2S^0) \leq 2(S_{n+1} - S_n)(S_{n+1} - S^0)$, we get

$$\begin{aligned}
\Delta_n U \leq\ & G_n S_{n+1} - (\mu_2 + k) I_{n+1} + \rho_1 k I_{n+1} - \rho_1(\mu_3 + \gamma) R_{n+1} + \rho_2 g_1(I_{n+1}) \\
& - \rho_2 G_n + \rho_3 (S_{n+1} - S^0)\left(\mu_1 S^0 - G_n S_{n+1} - \mu_1 S_{n+1} + \gamma R_{n+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\rho_1(\mu_3+\gamma)R_{n+1} + \rho_3\gamma(S_{n+1}-S^0)R_{n+1} + \rho_1 k I_{n+1} + \rho_2 g_1(I_{n+1}) - \rho_2 G_n \\
&\quad - (\mu_2+k)I_{n+1} + G_n S_{n+1} - \rho_3\mu_1(S_{n+1}-S^0)^2 - \rho_3(S_{n+1}-S^0)G_n S_{n+1} \\
&= -\rho_3\mu_1(S_{n+1}-S^0)^2 - (\rho_1(\mu_3+\gamma) - \rho_3\gamma(S_{n+1}-S^0))R_{n+1} \\
&\quad + (S_{n+1} - \rho_2 - \rho_3 S_{n+1}(S_{n+1}-S^0))G_n + (\rho_1 k + \rho_2 f_1(I_{n+1}) - (\mu_2+k))I_{n+1}.
\end{aligned}
$$

Since $f_1(I_n) \le f_1^M$ for all $n \ge 0$,

$$
\begin{aligned}
\Delta_n U &\le -\rho_3\mu_1(S_{n+1}-S^0)^2 - (\rho_1(\mu_3+\gamma) - \rho_3\gamma(S_{n+1}-S^0))R_{n+1} \\
&\quad + (S_{n+1} - \rho_2 - \rho_3 S_{n+1}(S_{n+1}-S^0))G_n + (\rho_1 k + \rho_2 f_1^M - (\mu_2+k))I_{n+1}.
\end{aligned}
$$

Now we select $\rho_1, \rho_2, \rho_3 > 0$ to satisfy

$$\rho_1(\mu_3+\gamma) - \rho_3\gamma(S_{n+1}-S^0) > 0, \tag{4.9}$$

$$\rho_1 k + \rho_2 f_1^M < \mu_2 + k, \tag{4.10}$$

$$S_{n+1} - \rho_2 - \rho_3 S_{n+1}(S_{n+1}-S^0) < 0. \tag{4.11}$$

By the hypothesis $S^0 f_1^M < \mu_2 + k$, there are $\theta_1 > 0$ and $\theta_2 > 0$ such that

$$\theta_1 k + (S^0+\theta_2) f_1^M < \mu_2 + k.$$

By Remark (3.4), $\limsup_{n\to\infty} S_n \le S^0$, so we can choose $N = N(\theta_2)$ sufficiently large so that $S_{n+1} - S^0 < \theta_2/2$ for all $n \ge N$. Now we consider $n \ge N$. Then (4.9) holds provided

$$\rho_3 < \frac{2(\mu_3+\gamma)}{\gamma\theta_2}\rho_1.$$

We choose $\rho_1 = \theta_1$ and $\rho_2 = S^0 + \theta_2$. Then (4.10) immediately follows. For (4.11), note that $S_{n+1} - \rho_2 = S_{n+1} - (S^0+\theta_2) < -\theta_2/2$ and there is a constant $c > 0$

such that $|S_{n+1}(S_{n+1} - S^0)| \leq c$ for all $n \geq N$. Take $\rho_3 > 0$ to satisfying

$$\rho_3 < \min\{\frac{2(\mu_3 + \gamma)\theta_1}{\gamma\theta_2}, \frac{\theta_2}{2c}\}$$

then (4.9) follows and we get

$$S_{n+1} - \rho_2 - \rho_3 S_{n+1}(S_{n+1} - S^0) < 0,$$

that is, (4.11) holds. Finally, we obtain $\Delta_n U \leq 0$. Moreover, $\Delta_n U = 0$ implies that $(S_n, I_n, R_n) = (S^0, 0, 0)$. Applying the LaSalle invariance principle, we finally obtain that E^0 of (4.1) is a global attractor. This completes the proof. □

4.2 Permanence of Solutions

In this section, we prove the permanence of model (4.1).

Theorem 4.4. *If $\mathcal{R}_0 > 1$, then the model (4.1) is permanent for any initial conditions (3.1).*

Proof. By the positivity and boundedness of solutions, it suffices to prove a universal lower bound for S_n, I_n and R_n.

Estmate for S_n. There is a constant $m_S > 0$ such that $S_n \geq m_S$ for all $n \geq 1$.

Proof of Estimante for S_n. By Proposition3.1 (Positivity) and (4.2), we have

$$S_{n+1} = \frac{h\Lambda + S_n + h\gamma R_{n+1}}{\Pi_1 + hG_n} \geq \frac{h\Lambda}{\Pi_1 + hg_1^M} =: m_S > 0,$$

where $g_1^M = \max_{I \geq 0} g_1(I)$.

Estimate for I_n. There is a constant $m_I > 0$ such that $\liminf_{n\to\infty} I_n \geq m_I$.

Proof of Estimate for I_n. By the hypothesis, $\frac{f_1(0)}{\mu_2 + k}\frac{\Lambda}{\mu_1} > 1$, so we get by the continuity of f_1 that we can choose $\mu > 0$ (small) and $\rho \in \mathbb{N}$ (large) to satisfy

$$\lambda := \min_{I \in [0,\nu]} \frac{f_1(I)}{\mu_2 + k}\frac{\Lambda}{\mu_1 + \nu f_1^M}\left(1 - \frac{1}{(\Pi_1 + h\nu f_1^M)^{\rho m}}\right) > 1,$$

where $f_1^M = \max_{I \geq 0} f_1(I)$.
We denote

$$S^\Delta := \frac{\Lambda}{\mu_1 + \nu f_1^M} \left(1 - \frac{1}{(\Pi_1 + h\nu f_1^M)^{\rho m}}\right).$$

Then we get

$$f_1(I)S^\Delta > \lambda(\mu_2 + k) \quad \text{for all } I \in [0, \nu]. \tag{4.12}$$

Claim 1. If $I_n \leq \nu$ for $n \in [n_1, n_2]$ with $n_2 - n_1 \geq m + \rho m$ then

$$S_{n_2+1} > S^\Delta.$$

Proof of Claim 1. For $n = n_2$, we have $I_{n-j} \leq \nu$ for $j = 0, 1, 2 \ldots, m$, which implies

$$g_1(I_{n-j}) = f_1(I_{n-j})I_{n-j} \leq \nu f_1^M,$$

and so

$$G_n = \sum_{j=0}^{m} p_j g_1(I_{n-j}) \leq \nu f_1^M.$$

By Proposition 3.1 and (4.2)

$$\begin{aligned} S_{n+1} &= \frac{h\Lambda + S_n + h\gamma R_{n+1}}{\Pi_1 + hG_n} \\ &\geq \frac{h\Lambda + S_n}{\Pi_1 + hG_n} \\ &\geq \frac{h\Lambda + S_n}{\Pi_1 + h\nu f_1^M}. \end{aligned}$$

Similarly, we also obtain for $n = n_2 - 1, n_2 - 2, \ldots, n_2 - \rho m$,

$$S_{n+1} \geq \frac{h\Lambda}{\Pi_1 + h\nu f_1^M} + \frac{S_n}{\Pi_1 + h\nu f_1^M}.$$

By iterative method,

$$\begin{aligned}
S_{n_2+1} &\geq \frac{h\Lambda}{\Pi_1 + h\nu f_1^M} + \frac{1}{\Pi_1 + h\nu f_1^M}\left(\frac{h\Lambda + S_{n_2-1}}{\Pi_1 + h\nu f_1^M}\right) \\
&= \frac{h\Lambda}{\Pi_1 + h\nu f_1^M} + \frac{h\Lambda}{(\Pi_1 + h\nu f_1^M)^2} + \cdots + \frac{h\Lambda}{(\Pi_1 + h\nu f_1^M)^{\rho m+1}} \\
&\quad + \frac{S_{n_2-\rho m}}{(\Pi_1 + h\nu f_1^M)^{\rho m+1}} \\
&= \frac{h\Lambda}{(\Pi_1 + h\nu f_1^M)}\left(1 + \frac{1}{(\Pi_1 + h\nu f_1^M)} + \cdots + \frac{1}{(\Pi_1 + h\nu f_1^M)^{\rho m}}\right) \\
&\quad + \frac{S_{n_2-\rho m}}{(\Pi_1 + h\nu f_1^M)^{\rho m+1}} \\
&= \frac{h\Lambda}{(\Pi_1 + h\nu f_1^M)}\left(\frac{\Pi_1 + h\nu f_1^M - 1}{h(\mu_1 + \nu f_1^M)}\right)\left(1 - \frac{1}{(\Pi_1 + h\nu f_1^M)^{\rho m+1}}\right) \\
&\quad + \frac{S_{n_2-\rho m}}{(\Pi_1 + h\nu f_1^M)^{\rho m+1}} \\
&> \frac{\Lambda}{\mu_1 + \nu f_1^M}\left(1 - \frac{1}{(\Pi_1 + h\nu f_1^M)^{\rho m+1}}\right) \\
&> \frac{\Lambda}{\mu_1 + \nu f_1^M}\left(1 - \frac{1}{(\Pi_1 + h\nu f_1^M)^{\rho m}}\right) = S^{\Delta}.
\end{aligned}$$

Claim 2. If $I_n \leq \nu$ for $n \in [n_1, n_2]$ with $n_2 - n_1 \geq m + \rho m$ then

$$I_{n_2+1} \geq \kappa \min_{p\in[n_2-m,n_2]} I_p,$$

where $\kappa := \frac{1+\lambda h(\mu_2+k)}{1+h(\mu_2+k)} > 1$ is a constant.

Proof of Claim 2. Let us denote

$$\sigma = \min_{p\in[n_2-m,n_2]} I_p.$$

By (4.2), Claim 1, and (4.12), we get

$$\begin{aligned}
I_{n_2+1} &= \frac{I_{n_2} + h\sum_{j=0}^{m} p_j f_1(I_{n_2-j}) I_{n_2-j} S_{n_2+1}}{\Pi_1} \\
&\geq \frac{\sigma + h\sum_{j=0}^{m} p_j f_1(I_{n_2-j})\sigma S^{\Delta}}{1 + h(\mu_2 + k)}
\end{aligned}$$

$$\geq \frac{\sigma + h\sigma \sum_{j=0}^{m} p_j \lambda(\mu_2 + k)}{1 + h(\mu_2 + k)}$$
$$= \sigma \left(\frac{1 + h\lambda(\mu_2 + k)}{1 + h(\mu_2 + k)} \right) = \kappa\sigma.$$

Claim 3. It is impossible that $I_n \leq \nu$ for all sufficient large n.

Proof of Claim 3. Assume the contrary that there is n_1 such that $I_n \leq \nu$ for all $n \geq n_1$. Denote $n_2 = n_1 + m + \rho m$ and put

$$\sigma = \min_{p \in [n_2 - m, n_2]} I_p.$$

We have by Claim 2 that

$$I_{n_2+1} \geq \kappa\sigma,$$

where $\kappa > 1$. Observing that

$$\min_{p \in [n_2+1-m, n_2+1]} I_p \geq \sigma.$$

Then we get by Claim 2 once again that

$$I_{n_2+2} \geq \kappa \min_{p \in [n_2+1-m, n_2+1]} I_p \geq \kappa\sigma.$$

Continuing the argument, we obtain

$$I_n \geq \kappa\sigma \quad \text{for all } n \geq n_2 + 1.$$

Let $n_3 = n_2 + m + \rho m = n_1 + 2(m + \rho m)$. Since $I_n \leq \nu$ for all $n \in [n_2, n_3]$ and observe that

$$\min_{p \in [n_3 - m, n_3]} I_p \geq \kappa\sigma,$$

by Claim 2, we get

$$I_{n_3+1} \geq \kappa \min_{p\in[n_3-m,n_3]} I_p \geq \kappa^2\sigma.$$

Note that

$$\min_{p\in[n_3+1-m,n_3+1]} I_p \geq \kappa\sigma.$$

Then we apply Claim 2 again to get

$$I_{n_3+2} \geq \kappa \min_{p\in[n_3+1-m,n_3+1]} I_p \geq \kappa^2\sigma.$$

Continuing the argument, we have

$$I_n \geq \kappa^2\sigma \quad \text{for all } n \geq n_3 + 1.$$

Setting $n_l = n_1 + (l-1)(m+\rho m)$ and repeating the process, we obtain

$$I_n \geq \kappa^{l-1}\sigma \quad \text{for all } n \geq n_l + 1.$$

Since $\kappa > 1$, by selecting l large enough, we can choose $n \geq n_1$ such that

$$I_n \geq \kappa^{l-1}\sigma > \nu,$$

which is a contradiction. Hence, Claim 3 is true.

Now we are ready to prove the lower estimate for I_n. According to Claim 3, there are the following two possibilities.

(i) $I_n > \nu$ for all sufficiently large n.

(ii) I_n oscillates about ν for large n.

Obviously, for the case (i), we have $\liminf_{n\to\infty} I_n \geq \nu$. Therefore, we give the proof only for the case (ii). We show that there is a constant $m_I > 0$ such that $I_n \geq m_I$

for all large enough n. Suppose $n_1 < n_2$ are such that

$$I_{n_1} \geq \nu, \quad I_{n_2} \geq \nu \quad \text{and} \quad I_n < \nu \quad \text{for } n_1 < n < n_2.$$

For any $n > 0$, it follows from (4.2) that

$$I_{n+1} = \frac{I_n + hG_nS_{n+1}}{\Pi_2} \geq \frac{I_n}{\Pi_2}.$$

By iterative method, we have

$$I_n \geq \frac{I_{n-1}}{\Pi_2} \geq \frac{I_{n-2}}{\Pi_2^2} \geq \cdots \geq \frac{I_{n-p}}{\Pi_2^p}$$

provided $p \leq n$. Denote $n_3 = n_1 + 1 + m + \rho m$. If $n_1 \leq n \leq n_3$, then

$$I_n \geq \frac{I_{n_1}}{\Pi_2^{n-n_1}} \geq \frac{I_{n_1}}{\Pi_2^{1+m+\rho m}} \geq \frac{\nu}{\Pi_2^{1+m+\rho m}} =: m_I.$$

If $n_2 \leq n_3$, then we obtain

$$I_n \geq m_I \quad \text{for all } n \in [n_1, n_2].$$

Suppose $n_2 > n_3$. By the previous case, we get $I_n \geq m_I$ for all $n \in [n_1, n_3]$. Next, we will show that $I_n \geq m_I$ for all $n \in [n_3 + 1, n_2]$. Clearly, $I_{n_2} \geq \nu \geq m_I$. Since $n_3 - (n_1 + 1) = m + \rho m$ and $I_n < \nu$ for all $n \in [n_1 + 1, n_3]$, it follows by Claim 2 that

$$I_{n_3+1} \geq \kappa \min_{p \in [n_3 - m, n_3]} I_p \geq \kappa m_I \geq m_I.$$

Observe that

$$\min_{p \in [n_3+1-m, n_3+1]} I_p \geq m_I.$$

Employing $I_n < \nu$ for $n \in [n_1 + 2, n_3 + 1]$, we get by Claim 2 again that

$$I_{n_3+2} \geq \kappa \min_{p\in[n_3+1-m,n_3+1]} I_p \geq \kappa m_I$$

as long as $n_3 + 1 < n_2$. By induction, we finally conclude that

$$I_n \geq m_I \quad \text{for all } n \in [n_3 + 1, n_2].$$

Since n_1 and n_2 are chosen in an arbitrary way and m_I is independent of those, we conclude that $I_n \geq m_I$ for all large n. Therefore,

$$\liminf_{n\to\infty} I_n \geq m_I$$

as desired.

Estimate for R_n. There is a constant $m_R > 0$ such that $\liminf_{n\to\infty} R_n \geq m_R$.

Proof of Estimate for R_n. From the third equation of (4.2) and Proposition 3.1, we get

$$R_{n+1} = \frac{R_n + hkI_{n+1}}{\Pi_3} > \frac{hkI_{n+1}}{\Pi_3}.$$

Consequently,

$$\liminf_{n\to\infty} R_n \geq \frac{hk}{\Pi_3} \liminf_{n\to\infty} I_n \geq \frac{hk}{\Pi_3} m_I =: m_R.$$

Therefore, the proof is complete. □

4.3 Numerical Simulations

For the model (4.1), Theorem 4.3 implies that the disease goes to extinction if $\mathcal{R}_{0,1} < 1$. According Theorem 4.4, the disease is permanent if $\mathcal{R}_0 > 1$. In this section, the numerical simulations will be given to confirm the validity of the theoretical results obtained in the previous section. Consider the following SIRS

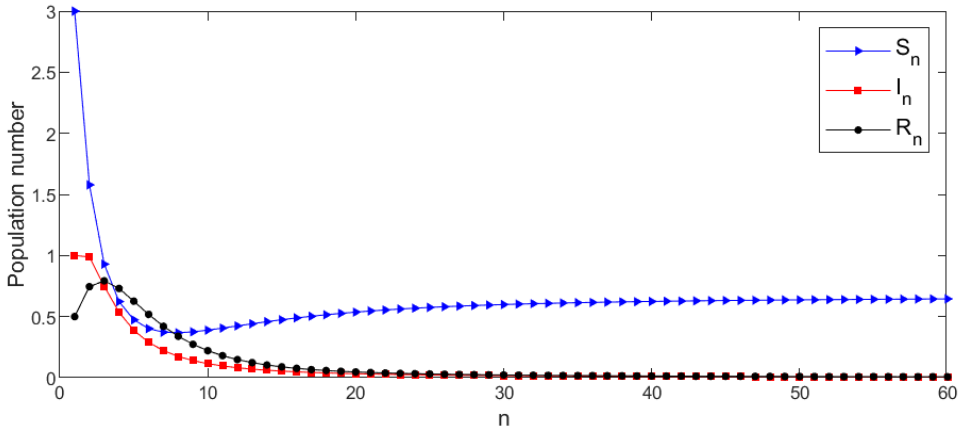

Figure 4.1: Numerical solution (S_n, I_n, R_n) of model (4.13) with $\mathcal{R}_{0,1} = 0.9524$

epidemic model:

$$\begin{cases} S_{n+1} = \Lambda - \sum_{j=0}^{2} p_j \dfrac{\delta I_{n-j} S_{n+1}}{1 + \beta I_{n-j} + \alpha I_{n-j}^2} - \mu_1 S_{n+1} + \gamma R_{n+1} + S_n, \\ I_{n+1} = \sum_{j=0}^{2} p_j \dfrac{\delta I_{n-j} S_{n+1}}{1 + \beta I_{n-j} + \alpha I_{n-j}^2} - (\mu_2 + k) I_{n+1} + I_n, \\ R_{n+1} = k I_{n+1} - (\mu_3 + \gamma) R_{n+1} + R_n, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{4.13}$$

For simplicity, some parameters are fixed: $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.5$, $k = 0.7$, $\gamma = 0.3$, $\delta = 2$, $\alpha = 1$. Now, we present the examples and the numerical simulations for model (4.13) with different parameters.

Example 4.5. We choose $\Lambda = 0.2$, $\beta = 0.3$, $\mu_1 = \mu_3 = 0.3$, $\mu_2 = 0.7$. Assuming the following initial conditions: $S_j = 3$, $I_j = 1$, $R_j = 0.5\,(j = -2, -1, 0)$, by calculation, we have that $\mathcal{R}_{0,1} = 0.9524$ and the disease-free equilibrium $E^0 = (0, 0, 0.67)$. According to Theorem 4.3, the disease-free equilibrium E^0 is globally stable, which is shown in Figure 4.1.

Example 4.6. We choose $\Lambda = 0.2$, $\beta = -0.3$, $\mu_1 = \mu_3 = 0.3$, $\mu_2 = 0.7$. Assuming the following initial conditions: $S_j = 3$, $I_j = 1$, $R_j = 0.5\,(j = -2, -1, 0)$, by calculation, we have that $\mathcal{R}_{0,1} = 0.9743$ and the disease-free equilibrium $E^0 = (0, 0, 0.67)$. According to Theorem 4.3, the disease-free equilibrium E^0 is globally stable, which is shown in Figure 4.2.

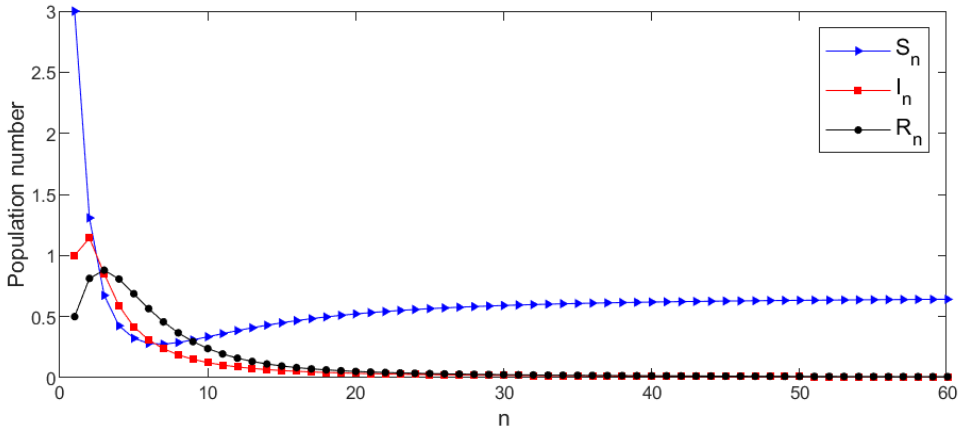

Figure 4.2: Numerical solution (S_n, I_n, R_n) of model (4.13) with $\mathcal{R}_{0,1} = 0.9743$

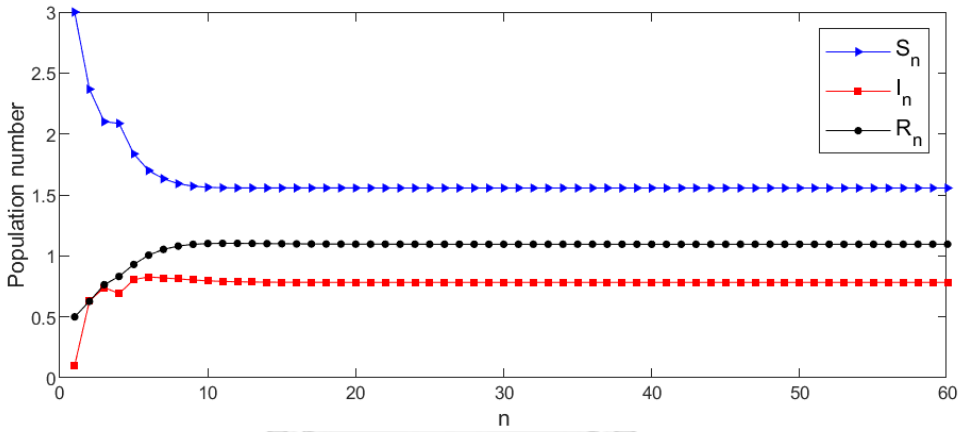

Figure 4.3: Numerical solution (S_n, I_n, R_n) of model (4.13) with $\beta > 0$ and $\mathcal{R}_0 = 7.6923$

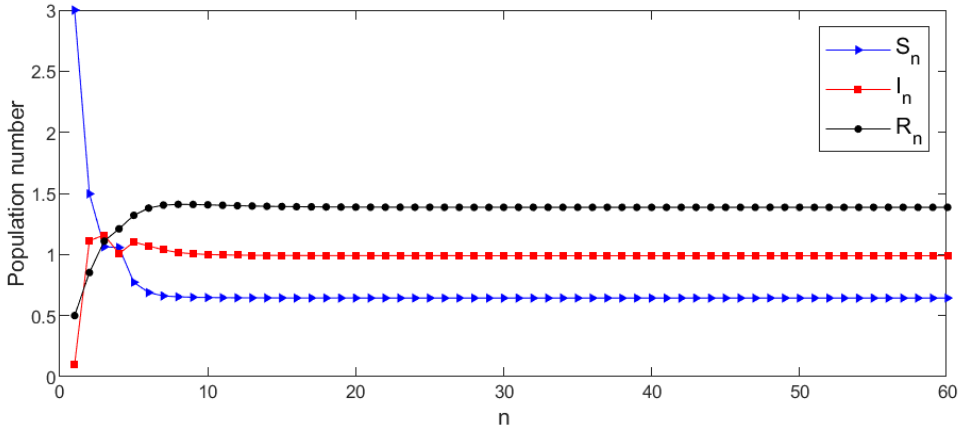

Figure 4.4: Numerical solution (S_n, I_n, R_n) of model (4.13) with $\beta < 0$ and $\mathcal{R}_0 = 7.6923$

Example 4.7. We choose $\Lambda = 1$, $\beta = 1$, $\mu_1 = \mu_3 = 0.2$, $\mu_2 = 0.6$. Setting the following initial conditions: $S_j = 3$, $R_j = 0.5\,(j = -2, -1, 0), I_{-2} = I_{-1} = 1$ and $I_0 = 0.1$, by calculation, we obtain $\mathcal{R}_0 = 7.6923$. According to Theorem 4.4, the disease is permanent as shown in Figure 4.3.

Example 4.8. We choose $\Lambda = 1$, $\beta = -1$, $\mu_1 = \mu_3 = 0.2$, $\mu_2 = 0.6$. Setting the following initial conditions: $S_j = 3$, $R_j = 0.5\,(j = -2, -1, 0), I_{-2} = I_{-1} = 1$ and $I_0 = 0.1$, by calculation, we obtain $\mathcal{R}_0 = 7.6923$. According to Theorem 4.4, the disease is permanent as shown in Figure 4.4.

CHAPTER V

STABILITY OF A DISCRETE SIRS EPIDEMIC MODEL WITH INCIDENCE RATE $(\delta I^2 S)/(1+\beta I+\alpha I^2)$

In this section, we consider the discrete SIRS epidemic model (1.7) with nonlinear incidence rate

$$g_2(I)S = \frac{\delta I^2 S}{1+\beta I+\alpha I^2}.$$

Then (1.7) may be written as follows:

$$\begin{cases} S_{n+1} = h\left(\Lambda - G_n S_{n+1} - \mu_1 S_{n+1} + \gamma R_{n+1}\right) + S_n, \\ I_{n+1} = h\left(G_n S_{n+1} - (\mu_2 + k) I_{n+1}\right) + I_n, \\ R_{n+1} = h\left(k I_{n+1} - (\mu_3 + \gamma) R_{n+1}\right) + R_n, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{5.1}$$

and it can be rearranged to obtain the explicit form

$$\begin{cases} S_{n+1} = \dfrac{h\Lambda + S_n + h\gamma R_{n+1}}{\Pi_1 + hG_n}, \\ I_{n+1} = \dfrac{hG_n S_{n+1} + I_n}{\Pi_2}, \\ R_{n+1} = \dfrac{hkI_{n+1} + R_n}{\Pi_3}, \end{cases} \tag{5.2}$$

where $G_n = G_n(I) = \sum_{j=0}^{m} p_j g_2(I_{n-j})$. Here, we write

$$g_2(I) = f_2(I)I, \quad \text{where} \quad f_2(I) = \frac{\delta I}{1+\beta I+\alpha I^2}.$$

Taking $S_{n+1} = S_n, I_{n+1} = I_n = 0, R_{n+1} = R_n$ gives $S_n = S^0$ and $R_n = 0$; hence, (5.1) admits a unique disease-free equilibrium $E^0 = (S^0, 0, 0)$. To find the endemic equilibria $E^* = (S^*, I^*, R^*)$, we consider the equations

$$\begin{cases} \Lambda - g_2(I^*)S^* - \mu_1 S^* + \gamma R^* = 0, \\ g_2(I^*)S^* - (\mu_2 + k)I^* = 0, \\ kI^* - (\mu_3 + \gamma)R^* = 0. \end{cases} \tag{5.3}$$

Solving the last equation for R^* and adding the first two's, we obtain

$$R^* = \frac{kI^*}{\mu_3 + \gamma}, \quad S^* = S^0 - \frac{\mu_2(\mu_3 + \gamma) + k\mu_3}{\mu_1(\mu_3 + \gamma)} I^*. \tag{5.4}$$

By plugging S^* into the second equation of (5.3), $I^* > 0$ satisfies

$$1 + \beta I^* + \alpha(I^*)^2 - \frac{\delta I^*}{\mu_2 + k}\left(S^0 - \frac{\mu_2(\mu_3 + \gamma) + k\mu_3}{\mu_1(\mu_3 + \gamma)} I^*\right) = 0.$$

Then I^* satisfies the equation

$$(A + \alpha)I^2 - (B - \beta)I + 1 = 0, \tag{5.5}$$

where

$$A = \frac{\delta}{\mu_2 + k} \frac{\mu_2(\mu_3 + \gamma) + k\mu_3}{\mu_1(\mu_3 + \gamma)}, \quad B = \frac{\delta}{\mu_2 + k} S^0.$$

Clearly, $A > 0$ and $B > 0$. The existence of the endemic equilibria E^* determined the roots of (5.5), which are

$$I_+ = \frac{B - \beta + \sqrt{D}}{2(A + \alpha)} \quad \text{and} \quad I_- = \frac{B - \beta - \sqrt{D}}{2(A + \alpha)},$$

where $D = (B - \beta)^2 - 4(A + \alpha)$.

Then we have

$$I_+ I_- = \frac{1}{A+\alpha} > 0 \quad \text{and} \quad I_+ + I_- = \frac{B-\beta}{A+\alpha}.$$

Note that $D \geq 0$ if and only if $A \leq \frac{(B-\beta)^2-4\alpha}{4}$. We denote

$$A_0 = \frac{(B-\beta)^2 - 4\alpha}{4}, \quad I^0 = \frac{B-\beta}{2(A+\alpha)}.$$

It follows that if $A > A_0$, in other words, $D < 0$, then (5.1) has no endemic equilibrium.

- If $B \leq \beta$ then $I_+ + I_- \leq 0$. So (5.1) has no endemic equilibrium if and only if $A \leq A_0$.
- If $B > \beta$ then $I_+ + I_- > 0$, so (5.1) has either one endemic equilibrium with $I^* = I^0$ or two endemic equilibria. The first case occurs if and only if $A = A_0$ and the second one occurs when $A < A_0$.

Theorem 5.1. *(5.1) has a unique disease-free equilibrium $E^0 = (S^0, 0, 0)$. Regarding the endemic equilibria $E^* = (S^*, I^*, R^*)$ of (5.1), the following results hold.*

(a) there is no endemic equilibrium provided one of the following conditions holds:

(i) $A > A_0$;

(ii) $A \leq A_0$ and $B \leq \beta$;

(b) there is a unique endemic equilibrium if and only if $A = A_0$ and $B > \beta$;

(c) there are two endemic equilibria if and only if $A < A_0$ and $B > \beta$.

Next, we consider a special case of model (5.2) by letting $m = 0$ and $\gamma = 0$, that is, (5.2) degenerates into the following discrete SIR epidemic model without

delay times:

$$\begin{cases} S_{n+1} = \dfrac{h\Lambda + S_n}{\Pi_1 + hg_2(I_n)}, \\ I_{n+1} = \dfrac{hg_2(I_n)S_{n+1} + I_n}{\Pi_2}, \\ R_{n+1} = \dfrac{hkI_{n+1} + R_n}{\Pi_3}. \end{cases} \tag{5.6}$$

In order to obtain the local stability of the disease-free equilibrium of (5.6), the third equation of (5.6) can be omitted without loss of generality properties. We define the following functions:

$$\begin{aligned} F(S,I) &= \frac{h\Lambda + S}{\Pi_1 + hg_2(I)}, \\ G(S,I) &= \frac{hg_2(I)F(S,I) + I}{\Pi_2}. \end{aligned} \tag{5.7}$$

Then the Jacobian matrix of (5.7) is given by

$$J((S,I)) = \begin{pmatrix} \dfrac{\partial F(S,I)}{\partial S} & \dfrac{\partial F(S,I)}{\partial I} \\ \dfrac{\partial G(S,I)}{\partial S} & \dfrac{\partial G(S,I)}{\partial I} \end{pmatrix}, \tag{5.8}$$

where

$$\begin{aligned} \frac{\partial F(S,I)}{\partial S} &= \frac{1}{\Pi_1 + hg_2(I)}, \\ \frac{\partial F(S,I)}{\partial I} &= -\frac{hg_2'(I)(h\Lambda + S)}{(\Pi_1 + hg_2(I))^2}, \\ \frac{\partial G(S,I)}{\partial S} &= \frac{hg_2(I)}{\Pi_2(\Pi_1 + hg_2(I))}, \\ \frac{\partial G(S,I)}{\partial I} &= \frac{1}{\Pi_2}\left(1 + hg_2(I)\frac{\partial F(S,I)}{\partial I} + hF(S,I)g_2'(I)\right), \end{aligned}$$

and

$$g_2(I) = \frac{\delta I^2}{1 + \beta I + \alpha I^2}, \quad g_2'(I) = \frac{\delta I(2 + \beta I)}{(1 + \beta I + \alpha I^2)^2}.$$

Now, we give the following theorem about the local stability of the disease-free equilibrium of (5.6).

Theorem 5.2. *The disease-free equilibrium E^0 of (5.6) is locally asymptotically stable.*

Proof. Substituting the disease-free equilibrium $E^0 = (S^0, 0)$ into the jacobian matrix (5.8) yields

$$J(E^0) = \begin{pmatrix} \dfrac{1}{1+h\mu_1} & 0 \\ 0 & \dfrac{1}{1+h(\mu_2+k)} \end{pmatrix}.$$

It is easily seen that the eigenvalues of $J(E^0)$ are

$$\lambda_1 = \frac{1}{1+h\mu_1} \quad \text{and} \quad \lambda_2 = \frac{1}{1+h(\mu_2+k)}.$$

Obviously, $|\lambda_1|, |\lambda_2| < 1$ for all h. Therefore, the disease-free equilibrium E^0 of (5.6) is always locally asymptotically stable. □

5.1 Global Attractivity of the Disease-Free Equilibrium

In this section, by constructing a suitable Lyapunov function, we obtain the global attractivity of the disease-free equilibrium of (5.1).

Theorem 5.3. *The disease-free equilibrium E^0 of (5.1) is globally attractive provided*

$$\mathcal{R}_{0,2} := \frac{f_2^M S^0}{\mu_2 + k} \leq 1,$$

where $f_2^M = \max_{I \geq 0} f_2(I)$.

Proof. Let (S_n, I_n, R_n) be any positive solution of (5.1) with initial condition (3.1).

Define

$$\begin{aligned}
U_1(S_n) &= \frac{1}{h}\left(S_n - S^0 - S^0 \ln\left(\frac{S_n}{S^0}\right)\right),\\
U_2(I_n) &= \frac{1}{h} I_n + (\mu_2 + k)\sum_{j=0}^{m} p_j \sum_{l=n-j}^{n} I_l,\\
U_3(R_n) &= \frac{1}{2h} R_n^2,\\
U_4(N_n, R_n) &= \frac{1}{2h}(N_n - S^0 + \eta R_n)^2,
\end{aligned}$$

where $N_n = S_n + I_n + R_n$ and $\eta = \frac{\mu_2 - \mu_1}{k} \geq 0$.

Computing $\Delta_n U_1 := U_1(S_{n+1}) - U_1(S_n)$, we have

$$\begin{aligned}
\Delta_n U_1 &= \frac{1}{h}\left(S_{n+1} - S^0 - S^0 \ln\left(\frac{S_{n+1}}{S^0}\right) - S_n + S^0 + S^0 \ln\left(\frac{S_n}{S^0}\right)\right)\\
&= \frac{1}{h}\left(S_{n+1} - S_n - S^0 \ln\left(\frac{S_{n+1}}{S_n}\right)\right)
\end{aligned}$$

Employing that $\ln(1-x) \leq -x$ for all $x < 1$ yields

$$-\ln\left(\frac{S_{n+1}}{S_n}\right) = \ln\left(1 - \left(1 - \frac{S_n}{S_{n+1}}\right)\right) \leq -\left(\frac{S_{n+1} - S_n}{S_{n+1}}\right). \tag{5.9}$$

From (5.9) and (5.1), it follows that

$$\begin{aligned}
\Delta_n U_1 &\leq \frac{1}{h}\left(S_{n+1} - S_n - S^0\left(\frac{S_{n+1} - S_n}{S_{n+1}}\right)\right)\\
&= \frac{1}{h}\left(S_{n+1} - S_n\right)\left(1 - \frac{S^0}{S_{n+1}}\right)\\
&= \left(\Lambda - G_n S_{n+1} - \mu_1 S_{n+1} + \gamma R_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right)\\
&= \left(\Lambda - \mu_1 S_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right) + \gamma R_{n+1}\left(1 - \frac{S^0}{S_{n+1}}\right) - G_n S_{n+1} + G_n S^0\\
&= \mu_1\left(S^0 - S_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right) + \gamma R_{n+1}\left(1 - \frac{S^0}{S_{n+1}}\right) - G_n S_{n+1} + G_n S^0.
\end{aligned} \tag{5.10}$$

Computing $\Delta_n U_2 := U_2(I_{n+1}) - U_2(I_n)$, we have

$$\begin{aligned}
\Delta_n U_2 &= \frac{1}{h}(I_{n+1} - I_n) + (\mu_2 + k)\left(\sum_{j=0}^{m} p_j \sum_{l=n+1-j}^{n+1} I_l - \sum_{j=0}^{m} p_j \sum_{l=n-j}^{n} I_l\right) \\
&= \frac{1}{h}\left(I_{n+1} - I_n\right) + (\mu_2 + k)\sum_{j=0}^{m} p_j (I_{n+1} - I_{n-j}) \\
&= G_n S_{n+1} - (\mu_2 + k) I_{n+1} + (\mu_2 + k) I_{n+1} - (\mu_2 + k)\sum_{j=0}^{m} p_j I_{n-j} \\
&= G_n S_{n+1} - (\mu_2 + k)\sum_{j=0}^{m} p_j I_{n-j}.
\end{aligned} \tag{5.11}$$

Computing $\Delta_n U_3 := U_3(R_{n+1}) - U_3(R_n)$ gives

$$\begin{aligned}
\Delta_n U_3 &= \frac{1}{2h}(R_{n+1}^2 - R_n^2) \\
&= \frac{1}{2h}(R_{n+1} - R_n)(2R_{n+1} + R_n - R_{n+1}) \\
&\leq \frac{1}{2h}(R_{n+1} - R_n)(2R_{n+1}) \\
&= (kI_{n+1} - (\mu_3 + \gamma)R_{n+1})R_{n+1}.
\end{aligned} \tag{5.12}$$

Finally, computing $\Delta_n U_4 := U_4(N_{n+1}, R_{n+1}) - U_3(N_n, R_n)$, we get

$$\begin{aligned}
\Delta_n U_4 &= \frac{1}{2h}\left[(N_{n+1} - S^0 + \eta R_{n+1})^2 - (N_n - S^0 + \eta R_n)^2\right] \\
&= \frac{1}{2h}\left(N_{n+1} + N_n - 2S^0 + \eta(R_{n+1} + R_n)\right)\left(N_{n+1} - N_n + \eta(R_{n+1} - R_n)\right) \\
&= \frac{1}{2h}\left(2N_{n+1} + N_n - N_{n+1} - 2S^0 + \eta(2R_{n+1} + R_n - R_{n+1})\right) \\
&\quad \times \left(N_{n+1} - N_n + \eta(R_{n+1} - R_n)\right) \\
&= \frac{1}{h}\left(N_{n+1} - S^0 + \eta R_{n+1}\right)\left(N_{n+1} - N_n + \eta(R_{n+1} - R_n)\right) \\
&\quad + \frac{1}{2h}\left(N_n - N_{n+1} + \eta(R_n - R_{n+1})\right)\left(N_{n+1} - N_n + \eta(R_{n+1} - R_n)\right) \\
&= \frac{1}{h}\left(N_{n+1} - S^0 + \eta R_{n+1}\right)\left(N_{n+1} - N_n + \eta(R_{n+1} - R_n)\right) \\
&\quad - \frac{1}{2h}\left(N_n - N_{n+1} + \eta(R_n - R_{n+1})\right)\left(N_n - N_{n+1} + \eta(R_n - R_{n+1})\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}\left(N_{n+1} - S^0 + \eta R_{n+1}\right)\left(N_{n+1} - N_n + \eta(R_{n+1} - R_n)\right) \\
&\quad - \frac{1}{2h}\left(N_n - N_{n+1} + \eta(R_n - R_{n+1})\right)^2 \\
&\leq \frac{1}{h}\left(N_{n+1} - S^0 + \eta R_{n+1}\right)\left(N_{n+1} - N_n + \eta(R_{n+1} - R_n)\right).
\end{aligned}
$$

By (5.1) and that $N_{n+1} - N_n = h\left(\Lambda - \mu_1 S_{n+1} - \mu_2 I_{n+1} - \mu_3 R_{n+1}\right)$, we obtain

$$
\begin{aligned}
\Delta_n U_4 &\leq \left[\Lambda - \mu_1 S_{n+1} - \mu_2 I_{n+1} - \mu_3 R_{n+1} + \eta\left(kI_{n+1} - (\mu_3 + \gamma)R_{n+1}\right)\right] \\
&\quad \times \left(N_{n+1} - S^0 + \eta R_{n+1}\right) \\
&= \left[-\mu_1\left(S_{n+1} - S^0 + I_{n+1}\right) - \left(\mu_3 + (\mu_3 + \gamma)\eta\right) R_{n+1}\right] \\
&\quad \times \left(S_{n+1} + I_{n+1} + R_{n+1} - S^0 + \eta R_{n+1}\right) \qquad (5.13) \\
&= -\mu_1\left(S_{n+1} - S^0 + I_{n+1}\right)^2 - (1+\eta)\left(\mu_3 + (\mu_3 + \gamma)\eta\right) R_{n+1}^2 \\
&\quad - \left(\mu_3 + (\mu_3 + \gamma)\eta + \mu_1(1+\eta)\right) I_{n+1} R_{n+1} \\
&\quad - \left(\mu_3 + (\mu_3 + \gamma)\eta + \mu_1(1+\eta)\right)\left(S_{n+1} - S^0\right) R_{n+1}.
\end{aligned}
$$

We define the following candidate for a Lyapunov function:

$$
U(S_n, I_n, R_n) = U_1(S_n) + U_2(I_n) + \rho_1 U_3(R_n) + \rho_2 U_4(N_n, R_n),
$$

where ρ_1 and ρ_2 are positive constants that will be selected in later. Calculating

$$
\Delta_n U := U(S_{n+1}, I_{n+1}, R_{n+1}) - U(S_n, I_n, R_n),
$$

we get by (5.10) and (5.11) that

$$
\begin{aligned}
\Delta_n U &= \Delta_n U_1 + \Delta_n U_2 + \rho_1 \Delta_n U_3 + \rho_2 \Delta_n U_4 \\
&\leq \mu_1\left(S^0 - S_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right) + \gamma R_{n+1}\left(1 - \frac{S^0}{S_{n+1}}\right) - G_n S_{n+1} + G_n S^0 \\
&\quad + G_n S_{n+1} - (\mu_2 + k)\sum_{j=0}^{m} p_j I_{n-j} + \rho_1 \Delta U_3(n) + \rho_2 \Delta U_4(n)
\end{aligned}
$$

$$= \mu_1 \left(S^0 - S_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right) + \gamma R_{n+1}\left(1 - \frac{S^0}{S_{n+1}}\right) + \rho_1 \Delta U_3(n) + \rho_2 \Delta U_4(n)$$
$$- (\mu_2 + k)\sum_{j=0}^{m} p_j I_{n-j}\left(1 - \frac{f_2(I_{n-j})S^0}{\mu_2 + k}\right).$$

Using (5.12), (5.13), the hypothesis and that $f_2(I) \leq f_2^M$ for all $I \geq 0$, we have

$$\begin{aligned}
\Delta_n U &\leq \mu_1 \left(S^0 - S_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right) + \gamma R_{n+1}\left(1 - \frac{S^0}{S_{n+1}}\right) + \rho_1 \Delta U_3(n) + \rho_2 \Delta U_4(n) \\
&\leq \mu_1 \left(S^0 - S_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right) + \gamma R_{n+1}\left(1 - \frac{S^0}{S_{n+1}}\right) + \rho_1 k I_{n+1} R_{n+1} \\
&\quad - \rho_1(\mu_3 + \gamma)R_{n+1}^2 - \rho_2 \mu_1 \left(S_{n+1} - S^0 + I_{n+1}\right)^2 \\
&\quad - \rho_2 \left(1 + \eta\right)\left(\mu_3 + (\mu_3 + \gamma)\eta\right) R_{n+1}^2 \\
&\quad - \rho_2 \left(\mu_3 + (\mu_3 + \gamma)\eta + \mu_1 \left(1 + \eta\right)\right) I_{n+1} R_{n+1} \\
&\quad - \rho_2 \left(\mu_3 + (\mu_3 + \gamma)\eta + \mu_1 \left(1 + \eta\right)\right)\left(S_{n+1} - S^0\right) R_{n+1}.
\end{aligned}$$

By choosing the positive constants ρ_1 and ρ_2 as follows:

$$\rho_1 = \frac{\gamma}{kS^0} \quad \text{and} \quad \rho_2 = \frac{\gamma}{S^0}\left(\mu_3 + (\mu_3 + \gamma)\eta + \mu_1 \left(1 + \eta\right)\right)^{-1},$$

we have

$$\begin{aligned}
\Delta_n U &\leq \mu_1 \left(S^0 - S_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right) + \gamma \left(1 - \frac{S^0}{S_{n+1}}\right) R_{n+1} + \frac{\gamma}{S^0} I_{n+1} R_{n+1} \\
&\quad - \rho_1(\mu_3 + \gamma)R_{n+1}^2 - \rho_2 \mu_1 \left(S_{n+1} - S^0 + I_{n+1}\right)^2 \\
&\quad - \rho_2 \left(1 + \eta\right)\left(\mu_3 + (\mu_3 + \gamma)\eta\right) R_{n+1}^2 - \frac{\gamma}{S^0} I_{n+1} R_{n+1} - \frac{\gamma}{S^0}\left(S_{n+1} - S^0\right) R_{n+1} \\
&= \mu_1 \left(S^0 - S_{n+1}\right)\left(1 - \frac{S^0}{S_{n+1}}\right) + \gamma \left(1 - \frac{S^0}{S_{n+1}}\right) R_{n+1} \\
&\quad - \rho_1(\mu_3 + \gamma)R_{n+1}^2 - \rho_2 \mu_1 \left(S_{n+1} - S^0 + I_{n+1}\right)^2 \\
&\quad - \rho_2 \left(1 + \eta\right)\left(\mu_3 + (\mu_3 + \gamma)\eta\right) R_{n+1}^2 - \frac{\gamma}{S^0}\left(S_{n+1} - S^0\right) R_{n+1}.
\end{aligned}$$

It is clearly that $\Delta_n U \leq 0$ whenever $S_{n+1} = S^0$. Applying $S_n \neq S^0$ for all $n > 0$

yields

$$\begin{aligned}
&\mu_1\left(S^0-S_{n+1}\right)\left(1-\frac{S^0}{S_{n+1}}\right)+\gamma\left(1-\frac{S^0}{S_{n+1}}\right)R_{n+1}+\frac{\gamma}{S^0}\left(S^0-S_{n+1}\right)R_{n+1}\\
&=\left(S^0-S_{n+1}\right)\left[\mu_1\left(1-\frac{S^0}{S_{n+1}}\right)+\frac{\gamma}{S^0}R_{n+1}\right]+\gamma\left(1-\frac{S^0}{S_{n+1}}\right)R_{n+1}\\
&=\left(S^0-S_{n+1}\right)\left(1-\frac{S^0}{S_{n+1}}\right)\left(\frac{1}{S^0}\right)\left[\Lambda+\left(\left(1-\frac{S^0}{S_{n+1}}\right)^{-1}+\frac{S^0}{S^0-S_{n+1}}\right)\gamma R_{n+1}\right]\\
&=-\frac{1}{S^0S_{n+1}}(S_{n+1}-S^0)^2(\Lambda+\gamma R_{n+1}).
\end{aligned}$$

This implies that

$$\begin{aligned}
\Delta_nU\leq&-\rho_1(\mu_3+\gamma)R_{n+1}^2-\rho_2\mu_1\left(S_{n+1}-S^0+I_{n+1}\right)^2\\
&-\rho_2\left(1+\eta\right)\left(\mu_3+(\mu_3+\gamma)\eta\right)R_{n+1}^2-\frac{1}{S^0S_{n+1}}(S_{n+1}-S^0)^2(\Lambda+\gamma R_{n+1}).
\end{aligned}$$

Finally, we have $\Delta_nU\leq 0$. Moreover, $\Delta_nU=0$ implies that $S_{n+1}=S^0$, $R_{n+1}=0$ and then $I_{n+1}=0$ for all $n\geq 0$. Substituting them into model (5.1), we obtain that $S_n=S^0$, $I_n=0$ and $R_n=0$ for all $n\geq 0$. In addition, $\{E^0\}$ is the largest invariant set where $\Delta_nU=0$. By using the LaSalle invariance principle of stability theory of the difference equations, we directly obtain that E^0 is globally attractive. This complete the proof. □

5.2 Numerical Simulations

For the model (5.1), Theorem (5.1) implies that the disease goes to extinction if $\mathcal{R}_{0,2}\leq 1$. In this section, we give numerical simulations to verify theoretical results

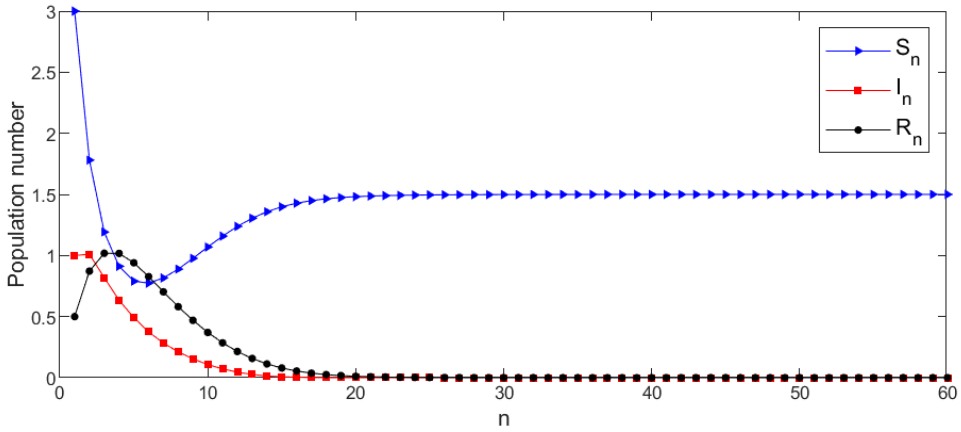

Figure 5.1: Numerical solution (S_n, I_n, R_n) of model (5.14) with $\mathcal{R}_{0,2} = 0.8571$

obtained in the previous section. Consider the following SIRS epidemic model:

$$\begin{cases} S_{n+1} = \Lambda - \sum_{j=0}^{2} p_j \dfrac{\delta I_{n-j}^2 S_{n+1}}{1 + \beta I_{n-j} + \alpha I_{n-j}^2} - \mu_1 S_{n+1} + \gamma R_{n+1} + S_n, \\ I_{n+1} = \sum_{j=0}^{2} p_j \dfrac{\delta I_{n-j}^2 S_{n+1}}{1 + \beta I_{n-j} + \alpha I_{n-j}^2} - (\mu_2 + k) I_{n+1} + I_n, \\ R_{n+1} = k I_{n+1} - (\mu_3 + \gamma) R_{n+1} + R_n, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{5.14}$$

For simplicity, some parameters are fixed: $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.5$, $k = 0.8$, $\gamma = 0.3$, $\delta = 2$. Now, we present the examples and the numerical simulations for model (5.14) with different parameters.

Example 5.4. We choose $\Lambda = 0.3$, $\beta = 0.5$, $\alpha = 1$, $\mu_1 = \mu_3 = 0.2$, $\mu_2 = 0.6$. Assuming the following initial conditions: $S_j = 3$, $I_j = 1$, $R_j = 0.5\,(j = -2, -1, 0)$, by calculation, we have that $\mathcal{R}_{0,2} = 0.8571$ and the disease-free equilibrium $E^0 = (0, 0, 1.5)$. According to Theorem 4.3, the disease-free equilibrium E^0 is globally stable, which is shown in Figure 5.1.

Example 5.5. We choose $\Lambda = 0.3$, $\beta = -0.5$, $\alpha = 2$, $\mu_1 = \mu_3 = 0.2$, $\mu_2 = 0.7$. Assuming the following initial conditions: $S_j = 3$, $I_j = 1$, $R_j = 0.5\,(j = -2, -1, 0)$, by calculation, we have that $\mathcal{R}_{0,2} = 0.8589$ and the disease-free equilibrium $E^0 = (0, 0, 1.5)$. According to Theorem 4.3, the disease-free equilibrium E^0 is globally stable as shown in Figure 5.2.

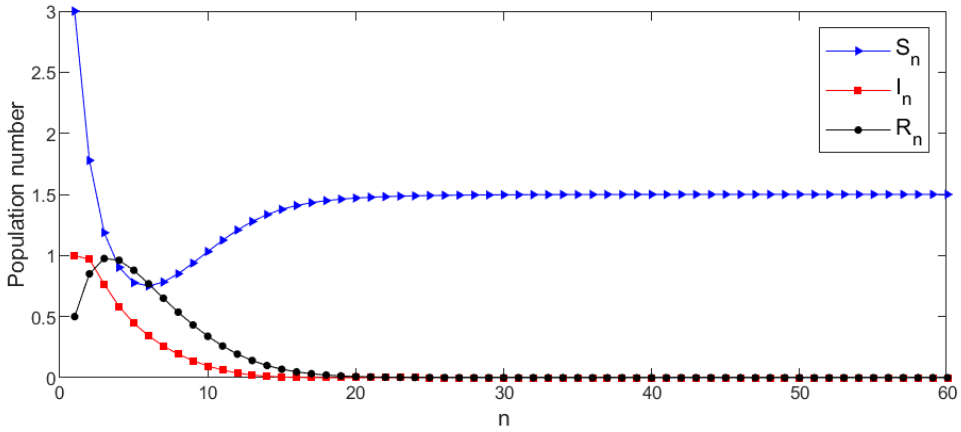

Figure 5.2: Numerical solution (S_n, I_n, R_n) of model (5.14) with $\mathcal{R}_{0,2} = 0.8589$

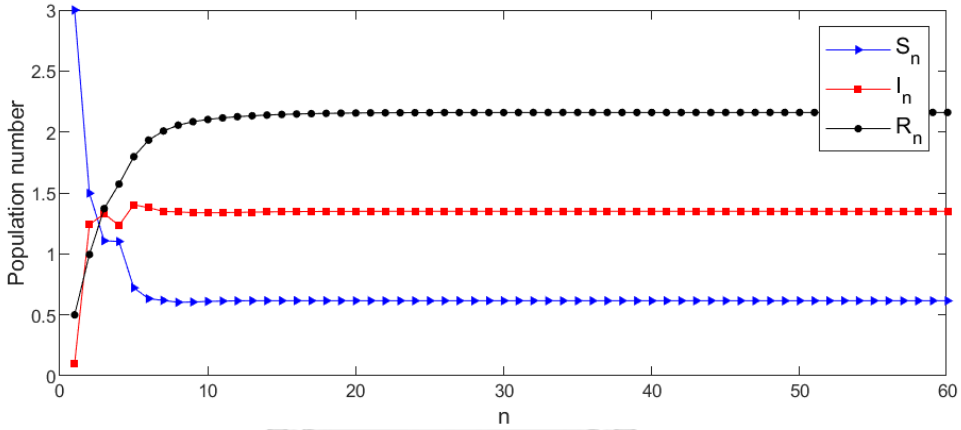

Figure 5.3: Numerical solution (S_n, I_n, R_n) of model (5.14) with $\mathcal{R}_{0,2} = 12.6765$

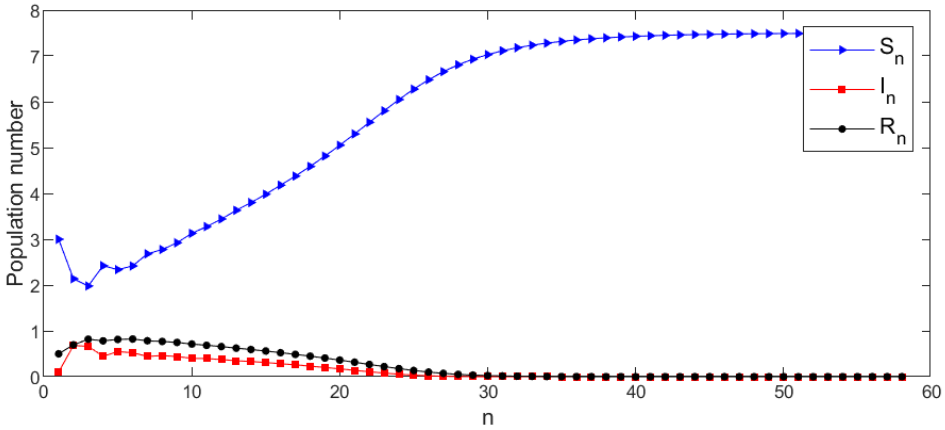

Figure 5.4: Numerical solution (S_n, I_n, R_n) of model (5.14) with $\mathcal{R}_{0,2} = 4.1335$

The following examples indicate that $\mathcal{R}_{0,2} > 1$ is not the sufficient condition for the permanence of the model (5.14).

Example 5.6. We choose $\Lambda = 1.5$, $\beta = -1$, $\alpha = 2$, $\mu_1 = \mu_3 = 0.2$, $\mu_2 = 0.7$. Assuming the following initial conditions: $S_j = 3$, $R_j = 0.5\,(j = -2, -1, 0)$, $I_{-2} = I_{-1} = 1$ and $I_0 = 0.1$, by calculation, we have that $\mathcal{R}_{0,2} = 12.6765$. Figure 5.3 demonstrates that the infection persist indefinitely.

Example 5.7. We choose $\Lambda = 1.5$, $\beta = -1$, $\alpha = 0.8$, $\mu_1 = \mu_3 = 0.2$, $\mu_2 = 1.5$. Assuming the following initial conditions: $S_j = 3$, $R_j = 0.5\,(j = -2, -1, 0)$, $I_{-2} = I_{-1} = 1$ and $I_0 = 0.1$, by calculation, we get $\mathcal{R}_{0,2} = 4.1335$. Figure 5.4 indicates that the outbreak will extinct.

CHAPTER VI
FURTHER RESEARCH

In this thesis, we have concentrated on the SIRS epidemic model with the incidence rates

$$\frac{\delta I}{1+\beta I+\alpha I^2} \quad \text{or} \quad \frac{\delta I^2}{1+\beta I+\alpha I^2}.$$

The distributed time-delay is also included in such model, so the dynamic behavior of the model at time t depends on the whole period prior to time t. Actually, the model can be extended to be more realistic. For a further research, we have the following ideas for the modeling:

1) Epidemic models with a general nonlinear incidence rate. From our analysis presented in this thesis, sufficient conditions for the global attractivity of disease-free equilibrium are obtained. Although many numerical simulations exhibit interesting behaviors of solutions of the model, one question still unanswered is whether the endemic equilibrium of the model is a global attractor. However, the techniques used in this work can be extended to study the SIRS epidemic model with a more general nonlinear incidence rate.

2) Non-autonomous epidemic models. In this work, we assume that the parameters in the epidemic model such as the recruitment rate, death rates, transmission of the disease, and recovery rate etc., are all constant. Actually, in the real environment, those parameters will usually change with time. Therefore, non-autonomous epidemic models with time-dependent parameters, are more suitable and more realistic to model the dynamics of disease. There has been some research work on the non-autonomous epidemic models (see [32, 33]).

3) Epidemic models with vaccination. The model (1.1) can be extended by including a class for vaccinated individuals.

$$\begin{cases} \dot{S}(t) = \Lambda - \beta_s S(t)I(t) - (\mu_s + k_v)S(t) + \gamma_r R(t) + \gamma_v V(t), \\ \dot{I}(t) = \beta_s S(t)I(t) + \beta_v V(t)I(t) - (\mu_i + k_r)I(t), \\ \dot{R}(t) = k_r I(t) - (\mu_r + \gamma_r)R(t), \\ \dot{V}(t) = k_v S(t) - (\mu_v + \gamma_v)V(t) - \beta_v V(t)I(t), \end{cases}$$

where β_s and β_v are the transmission coefficients of the S- and V- classes respectively; $\mu_s, \mu_i, \mu_r, \mu_v$ are, respectively, the death rates of S-, I-, R-, and V-classes; k_r and k_v are the recovery and the vaccination rates respectively; γ_r and γ_v are the loss of immunity rates of R- and V-classes respectively. To the best of our knowledge, there is limited literature on discrete epidemic models including vaccinated class.

4) Epidemic models with age-structure. Age-structure in epidemic models has been studied in both discrete and continuous approaches. See for instance [4,7,27]. In our model, we assume that the diseases exhibit temporary immunity of recovered individuals from infection. To study the role of age-dependent immunity, the age of recovery should be characterized in the model. Therefore, the third equation of (1.1) may be replaced by the equation of the form

$$\frac{\partial r(a,t)}{\partial a} + \frac{\partial r(a,t)}{\partial t} = -(\mu_3 + \gamma(a))r(a,t),$$
$$r(a,t) = kI(t), \quad r(a,0) = r_0(a),$$

where $r(a,t)$ is the density of recovered individuals with respect to the age of recovery a at time t, and $\gamma(a)$ is the rate that the recovered individual with the recovery-age a loses immunity and returns to be susceptible. In addition, for the models with vaccination, we may take into account the effect of age of vaccination as well.

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